Research Article

On Asymptotic Behaviour of Solutions to $n$-Dimensional Systems of Neutral Differential Equations

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This paper presents the properties and behaviour of solutions to a class of $n$-dimensional functional differential systems of neutral type. Sufficient conditions for solutions to be either oscillatory, or \( \lim_{t \to \infty} y_i(t) = 0 \), or \( \lim_{t \to \infty} |y_i(t)| = \infty \), \( i = 1, 2, \ldots, n \), are established. One example is given.

1. Introduction

The authors have investigated some properties of solutions to \( n \)-dimensional functional differential systems

\[
\begin{align*}
[y_1(t) - a(t)y_1(g(t))]' &= p_1(t)y_2(t), \\
y'_i(t) &= p_i(t)y_{i+1}(t), & i = 2, 3, \ldots, n - 1, \\
y'_n(t) &= \sigma p_n(t)f(y_1(h(t))), & t \geq t_0,
\end{align*}
\]

in [1]. We studied the properties of solutions presupposing that both functions \( a(t) \) and \( y_1(t) \) were bounded and there were presented theorems where sufficient conditions to every solution with the first component of the solution \( y_1(t) \) to be either oscillatory, or \( \lim_{t \to \infty} y_i(t) = 0 \) for \( i = 1, 2, \ldots, n \).
The goal of this paper is to enquire about the behaviour of the solution to \( n \)-dimensional functional differential system of neutral type (1.1) under no restriction to \( a(t) \) and to the first component \( y_1(t) \) of solution \( y(t) \). Results are given in theorems where sufficient conditions are stated to every solution to have the next properties: a solution to be either oscillatory, or \( \lim_{t \to \infty} y_i(t) = 0 \), or \( \lim_{t \to \infty} |y_i(t)| = \infty \), \( i = 1, 2, \ldots, n \).

The system (1.1) is investigated under the assumptions \( \sigma \in \{-1, 1\}, n \geq 3 \), and throughout this paper, the next conditions are considered:

(a) \( a : [t_0, \infty) \to (0, \infty) \) is a continuous function;

(b) \( g : [t_0, \infty) \to \mathbb{R} \) is a continuous and increasing function, \( \lim_{t \to \infty} g(t) = \infty \);

(c) \( p_i : [t_0, \infty) \to [0, \infty), \ i = 1, 2, \ldots, n, \) are continuous functions; \( p_n \) not identically equal to zero in any neighbourhood of infinity, \( \int_{t_0}^{\infty} p_j(t) \, dt = \infty, \ j = 1, 2, \ldots, n - 1 \);

(d) \( h : [t_0, \infty) \to \mathbb{R} \) is a continuous and increasing function, \( \lim_{t \to \infty} h(t) = \infty \);

(e) \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function; moreover, for \( u \neq 0, uf(u) > 0 \) and \( |f(u)| \geq K |u| \) hold, where \( K \) is a positive constant.

For a function \( y_1(t) \),

\[
z_1(t) = y_1(t) - a(t)y_1(g(t))
\]

(1.2)

is defined, and for \( t_1 \geq t_0 \), we introduce

\[
\tilde{t}_1 = \min\{t_1, g(t_1), h(t_1)\}.
\]

(1.3)

A vector function \( y = (y_1, \ldots, y_n) \) is a solution to the system (1.1) if there is a \( t_1 \geq t_0 \) such that \( y \) is continuous on \( [\tilde{t}_1, \infty) \); functions \( z_1(t), y_i(t), i = 2, 3, \ldots, n \) are continuously differentiable on \( [t_1, \infty) \) and \( y \) satisfies (1.1) on \( [t_1, \infty) \).

\( W \) denotes the set of all solutions \( y = (y_1, \ldots, y_n) \) to the system (1.1) that exist on some interval \( [T_y, \infty) \subset [t_0, \infty) \) and satisfy the condition

\[
\sup \left\{ \sum_{i=1}^{n} |y_i(t)| : t \geq T \right\} > 0 \quad \text{for any } T \geq T_y.
\]

(1.4)

A solution \( y \in W \) is considered nonoscillatory if there exists a \( T_y \geq t_0 \) such that every component is different from zero for \( t \geq T_y \). Otherwise a solution \( y \in W \) is said to be oscillatory.

Properties of solutions to similar differential equations and systems like system (1.1) have been studied in [1–6] and in the references cited therein. Problems of existence of solutions to neutral differential systems were analysed, for example, in [7, 8].
Abstract and Applied Analysis

It will be useful to define two types of recursion formulae. Let $i_k \in \{1, 2, \ldots, n\}$, $k = 1, 2, \ldots, n$, and $t, u \in [t_0, \infty)$. One has

\begin{equation}
I_0(u, t) \equiv 1,
\end{equation}

\begin{equation}
I_k(u, t; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = \int_t^u p_{i_1}(x)I_{k-1}(x, t; p_{i_2}, p_{i_3}, \ldots, p_{i_k})dx,
\end{equation}

\begin{equation}
J_0(u, t) \equiv 1,
\end{equation}

\begin{equation}
J_k(u, t; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = \int_t^u p_{i_1}(x)J_{k-1}(u, x; p_{i_2}, p_{i_3}, \ldots, p_{i_k})dx.
\end{equation}

It is easy to prove that the following identities hold:

\begin{equation}
I_k(u, t; p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = J_k(u, t; p_{i_1}, p_{i_2}, \ldots, p_{i_k})
\end{equation}

for $k = 1, 2, \ldots, n$.

Functions $g^{-1}(t)$, $h^{-1}(t)$ denote the inverse functions to $g(t)$, $h(t)$.

2. Preliminaries

Lemma 2.1 (see [9, Lemma 1]). Let $y \in W$ be a solution of (1.1) with $y_1(t) \neq 0$ on $[t_1, \infty)$, $t_1 \geq t_0$. Then $y$ is nonoscillatory and $z_1(t), y_2(t), \ldots, y_n(t)$ are monotone on some ray $[T, \infty)$, $T \geq t_1$.

Let $y \in W$ be a non-oscillatory solution of (1.1). By (1.1) and (c), it follows that the function $z_1(t)$ from (1.2) has to be eventually of constant sign, so that either

\begin{equation}
y_1(t)z_1(t) > 0
\end{equation}

or

\begin{equation}
y_1(t)z_1(t) < 0
\end{equation}

for sufficiently large $t$.

We mention for the comfort of proofs a classification of non-oscillatory solutions of the system (1.1) which was introduced by the authors in [1].

Assume first that (2.1) holds.

By [9, Lemma 4], the statement in Lemma 2.2 follows.

Lemma 2.2. Let $y = (y_1, y_2, \ldots, y_n) \in W$ be a non-oscillatory solution to (1.1) on $[t_1, \infty)$, and assume that (2.1) holds. Then there exists an integer $l \in \{1, 2, \ldots, n\}$ such that $\sigma \cdot (-1)^{n+l} = 1$ or $l = n$, and $t_2 \geq t_1$ such that for $t \geq t_2$

\begin{equation}
y_i(t)z_1(t) > 0, \quad i = 1, 2, \ldots, l,
\end{equation}

\begin{equation}
(-1)^{i+l}y_i(t)z_1(t) > 0, \quad i = l + 1, \ldots, n.
\end{equation}
Denote by \( N_1^+ \) the set of non-oscillatory solutions to (1.1) satisfying (2.3). Now assume that (2.2) holds.

By the aid of Kiguradze’s lemma, it is easy to prove Lemma 2.3.

**Lemma 2.3.** Let \( y = (y_1, y_2, \ldots, y_n) \in W \) be a non-oscillatory solution to (1.1) on \([t_1, \infty)\), and assume that (2.2) holds. Then there exists an integer \( l \in \{1, 2, \ldots, n\} \) and \( \sigma \cdot (-1)^{l+1} = 1 \) or \( l = n \), and \( t_2 \geq t_1 \) such that for \( t \geq t_2 \) either

\[
y_1(t)z_1(t) < 0,
\]

(2.4)

\[
(-1)^l y_i(t)z_1(t) < 0, \quad i = 2, \ldots, n,
\]

or

\[
y_1(t)z_1(t) < 0,
y_i(t)z_1(t) > 0, \quad i = 2, 3, \ldots, l,
\]

(2.5)

\[
(-1)^{l+i} y_i(t)z_1(t) > 0, \quad i = l + 1, \ldots, n.
\]

Denote by \( N_1^- \) the set of non-oscillatory solutions to (1.1) satisfying (2.4), and by \( N_1^+ \) the set of non-oscillatory solutions to (1.1) satisfying (2.5). Denote by \( N \) the set of all non-oscillatory solutions to the system (1.1). Obviously by Lemmas 2.2 and 2.3, we have the classification of non-oscillatory solutions to the system (1.1):

**n odd, \( \sigma = 1 \):**

\[
N = N_2^+ \cup N_4^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ \cup N_1^- \cup N_3^- \cup \cdots \cup N_n^-,
\]

(2.6)

**n odd, \( \sigma = -1 \):**

\[
N = N_1^+ \cup N_3^+ \cup \cdots \cup N_n^+ \cup N_2^- \cup N_4^- \cup \cdots \cup N_{n-1}^- \cup N_n^-,
\]

(2.7)

**n even, \( \sigma = 1 \):**

\[
N = N_1^+ \cup N_3^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ \cup N_1^- \cup N_3^- \cup \cdots \cup N_n^-,
\]

(2.8)

**n even, \( \sigma = -1 \):**

\[
N = N_2^+ \cup N_4^+ \cup \cdots \cup N_n^+ \cup N_1^- \cup N_3^- \cup \cdots \cup N_{n-1}^- \cup N_n^-.
\]

(2.9)

The next lemma can be proved similarly as Lemma 2 in [9].
Abstract and Applied Analysis

**Theorem 3.1.** Suppose that

\[ \text{Let } y = (y_1, y_2, \ldots, y_n) \in W \text{ be a non-oscillatory solution to (1.1) on } [t_1, \infty), \ t_1 \geq t_0, \]  
and let \( \lim_{t \to \infty} |z_1(t)| = L_1, \ \lim_{t \to \infty} |y_k(t)| = L_k, \ k = 2, \ldots, n. \) Then

\[
k \geq 2, \quad L_k > 0 \implies L_i = \infty, \quad i = 1, \ldots, k - 1, \\
1 \leq k < n, \quad L_k < \infty \implies L_i = 0, \quad i = k + 1, \ldots, n.
\] (2.10)

**Remark 2.5.** If \( g(t) < t, \) and \( 0 < a(t) \leq \lambda^* < 1, \) (\( \lambda^* \) is a constant), then from [9], we have \( N_1 = \emptyset, \ k \in \{2, 3, \ldots, n\}. \)

**Lemma 2.4.** Let \( y = (y_1, y_2, \ldots, y_n) \in \) be a non-oscillatory solution to (1.1) on \([t_1, \infty), \ t_1 \geq t_0, \) and let \( \lim_{t \to \infty} |z_1(t)| = L_1, \ \lim_{t \to \infty} |y_k(t)| = L_k, \ k = 2, \ldots, n. \) Then

\[
|y(t)| \text{ for } t \geq t_0.
\] (2.11)

Let \( y_1(t) \) be a continuous non-oscillatory solution to the functional inequality

\[
y_1(t) [y_1(t) - a(t) y_1(g(t))] > 0
\] (2.12)
defined in a neighbourhood of infinity. Suppose that \( g(t) > t \) for \( t \geq t_0. \) Then \( y_1(t) \) is bounded. If, moreover,

\[
1 < \lambda_* \leq a(t), \quad t \geq t_0
\] (2.13)

for some positive constant \( \lambda_*), \) then \( \lim_{t \to \infty} y_1(t) = 0. \)

**3. Main Results**

**Theorem 3.1.** Suppose that

\[ 0 < a(t) \leq \lambda^* < 1, \quad \text{for some constant } \lambda^*, \quad t \geq t_0, \] (3.1)

\[ g(t) < h(t) < t, \quad \text{for } t \geq t_0, \] (3.2)

\[ \alpha : [t_0, \infty) \to \mathbb{R} \text{ is a continuous function, } \alpha(t) < t, \quad \lim_{t \to \infty} \alpha(t) = \infty, \] (3.3)

\[
\int_{\infty}^{\infty} p_1(x_1) \int_{\infty}^{\infty} p_2(x_2) \int_{\infty}^{\infty} p_3(x_3) \cdots \int_{\infty}^{\infty} p_{n-1}(x_{n-1}) \int_{\infty}^{\infty} p_n(x_n) dx_n \cdots dx_1 = \infty,
\] (3.4)

\[
\lim_{l \to \infty} \text{sup}_{l \to \infty} K I_{l-2} (t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{l-1} (t, \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}))
\times \int_{h^{l-1}(t)}^{\infty} p_n(x_n) dx_n > 1
\] (3.5)

for \( l = 3, 5, \ldots, n - 2, \)

\[
\lim_{l \to \infty} \text{sup}_{l \to \infty} K I_{l-1} (t, \alpha(t); p_1, p_2, \ldots, p_{n-1}) \int_{h^{l-1}(t)}^{\infty} p_n(x_n) dx_n > 1.
\] (3.6)
If \( n \) is odd and \( \sigma = -1 \), then every solution \( y \in W \) to (1.1) is oscillatory or \( \lim_{t \to \infty} y_i(t) = 0, \forall i = 1,2,\ldots,n \).

**Proof.** Let \( y \in W \) be a non-oscillatory solution to (1.1). The Expression (2.7) holds. Taking into account Remark 2.5, one may write

\[
N = N_1^+ \cup N_3^+ \cup \cdots \cup N_n^+.
\]  

(7.3)

Without loss of generality we may suppose that \( y_1(t) \) is positive for \( t \geq t_2 \).

(I) Let \( y \in N_1^+ \) on \([t_2, \infty)\). In this case, we can write for \( t \geq t_2 \)

\[
y_1(t) > 0, z_1(t) > 0, y_2(t) < 0, y_3(t) > 0, \ldots, y_n(t) > 0,
\]  

(3.8)

and \( \lim_{t \to \infty} z_1(t) = L_1 \geq 0 \). We claim that \( L_1 = 0 \). Otherwise \( L_1 > 0 \). Then

\[
L_1 \leq z_1(h(t)) \leq y_1(h(t)) \quad \text{for } t \geq t_3,
\]  

(3.9)

where \( t_3 \geq t_2 \) is sufficiently large.

Integrating the last equation of (1.1) from \( x_{n-1} \) to \( x_{n-1}^* \), we get for \( x_{n-1} \geq t_3 \)

\[
y_n(x_{n-1}) - y_n(x_{n-1}^*) = \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)f(y_1(h(x_n)))dx_n.
\]  

(3.10)

From (3.10) with regard to (e), (3.8), and (3.9), we have for \( x_{n-1}^* \to \infty \)

\[
y_n(x_{n-1}) \geq KL_1 \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)dx_n, \quad x_{n-1} \geq t_3.
\]  

(3.11)

Multiplying (3.11) by \( p_{n-1}(x_{n-1}) \) and then using the \((n - 1)\)th equation of the system (1.1), we get for \( x_{n-1} \geq t_3 \)

\[
y'_{n-1}(x_{n-1}) \geq KL_1 p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)dx_n.
\]  

(3.12)

Integrating (3.12) from \( x_{n-2} \) to \( x_{n-2}^* \to \infty \), and then using (3.8), we get for \( x_{n-2} \geq t_3 \)

\[
-y_{n-1}(x_{n-2}) \geq KL_1 \int_{x_{n-2}}^{x_{n-2}^*} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)dx_n dx_{n-1}.
\]  

(3.13)

Multiplying (3.13) by \( p_{n-2}(x_{n-2}) \) and then using the \((n - 2)\)th equation of the system (1.1), and the new inequality we integrate from \( x_{n-3} \) to \( x_{n-3}^* \to \infty \) we employ (3.8) and for \( x_{n-3} \geq t_3 \)

\[
y_{n-2}(x_{n-3}) \geq KL_1 \int_{x_{n-3}}^{x_{n-3}^*} p_{n-2}(x_{n-2}) \int_{x_{n-2}}^{x_{n-2}^*} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n)dx_n dx_{n-1} dx_{n-2}.
\]  

(3.14)
Similarly for \( x_1 \geq t_3 \), we have

\[
-z_1'(t) \geq KL_1 p_1(x_1) \int_{x_1}^{\infty} p_2(x_2) \int_{x_2}^{\infty} p_3(x_3) \cdots p_{n-1}(x_{n-1}) \cdot \int_{x_{n-1}}^{\infty} p_n(x_n) dx_n \cdot dx_{n-1} \cdots dx_2. \tag{3.15}
\]

Integrating (3.15) from \( T \) to \( T^* \to \infty \) and then using (3.8), we get for \( T \geq t_3 \)

\[
z_1(T) \geq KL_1 \int_T^{\infty} p_1(x_1) \int_{x_1}^{\infty} p_2(x_2) \cdots p_{n-1}(x_{n-1}) \cdot \int_{x_{n-1}}^{\infty} p_n(x_n) dx_n \cdot dx_{n-1} \cdots dx_1, \tag{3.16}
\]

which a contradiction to (3.4). Hence \( \lim_{t \to \infty} z_1(t) = 0 \).

Then \( z_1(t) \leq 1, \quad t \geq t_4, \) where \( t_4 \geq t_3 \) is sufficiently large and

\[
y_1(t) \leq a(t) y_1(g(t)) + 1 \leq \lambda^* y_1(g(t)) + 1, \quad t \geq t_4. \tag{3.17}
\]

We prove that \( y_1(t) \) is bounded indirectly. Let \( y_1(t) \) be unbounded. Then there exists a sequence \( \{\tilde{t}_n\}_{n=1}^{\infty}, \tilde{t}_n \geq t_4, \) where \( n = 1, 2, \ldots, \tilde{t}_n \to \infty \) as \( n \to \infty, \)

\[
\lim_{n \to \infty} y_1(\tilde{t}_n) = \infty, \quad y_1(\tilde{t}_n) = \max_{t \leq \tilde{t}_n} y_1(s). \tag{3.18}
\]

It follows from (3.1), (3.2), and (3.17),

\[
y_1(\tilde{t}_n) \leq \lambda^* y_1(g(\tilde{t}_n)) + 1 \leq \lambda^* y_1(\tilde{t}_n) + 1,
\]

\[
y_1(\tilde{t}_n) \leq \frac{1}{1 - \lambda^*}, \quad n = 1, 2, \ldots. \tag{3.19}
\]

That is a contradiction to \( \lim_{n \to \infty} y_1(\tilde{t}_n) = \infty \), and the function \( y_1(t) \) is bounded. We claim that \( \lim_{t \to \infty} y_1(t) = 0 \) and prove it indirectly. Let \( \limsup_{t \to \infty} y_1(t) = s > 0 \). Let \( \{t_n^*\}_{n=1}^{\infty}, t_n^* \geq t_4, \) \( n = 1, 2, \ldots \), be such a kind of sequence, that \( t_n^* \to \infty \) as \( n \to \infty \), and \( \limsup_{n \to \infty} y_1(t_n^*) = s \). Then \( \limsup_{n \to \infty} y_1(t_n^*) \leq s \). From (1.2) and (3.1),

\[
z_1(t_n^*) \geq y_1(t_n^*) - \lambda^* y_1(g(t_n^*)), \quad n = 1, 2, \ldots,
\]

\[
y_1(g(t_n^*)) \geq \frac{y_1(t_n^*) - z_1(t_n^*)}{\lambda^*}, \quad n = 1, 2, \ldots \tag{3.20}
\]

follow.

From the last inequality, we have

\[
s \geq \frac{s}{\lambda^*}, \quad \lambda^* \geq 1. \tag{3.21}
\]
That is a contradiction to condition (3.1) and \( \lim_{t \to \infty} y_1(t) = 0 = \lim_{t \to \infty} y_1(t) \). Since \( \lim_{t \to \infty} z_1(t) = L_1 = 0 \) and from Lemma 2.4, imply \( \lim_{t \to \infty} y_i(t) = 0, \ i = 2, 3, \ldots, n. \)

(II) Let \( y \in N^*_t \), for some \( l = 3, 5, \ldots, n - 2 \), on \( [t_2, \infty) \). In this case, one can consider for \( t \geq t_2 \)

\[
y_1(t) > 0, z_1(t) > 0, y_2(t) > 0, \ldots, y_l(t) > 0, y_{l+1}(t) < 0, \ldots, y_n(t) > 0.
\]  

Integrating the first equation of the system (1.1) from \( a(t) \) to \( t \) and using (3.22) above, we get

\[
z_1(t) \geq \int_{a(t)}^{t} p_1(x_1) y_2(x_1) \, dx_1, \quad t \geq t_3,
\]  

where \( t_3 \geq t_2 \) is sufficiently large. Integrating step by step 2nd, 3rd, \ldots, \((l - 1)\)th equations of the system (1.1) and subsequently substituting into (3.23) for \( t \geq t_3 \), we obtain

\[
z_1(t) \geq \int_{a(t)}^{t} p_1(x_1) \int_{a(t)}^{x_1} p_2(x_2) \cdots \int_{a(t)}^{x_{l-2}} p_{l-1}(x_{l-1}) y_l(x_l) \, dx_{l-1} \, dx_{l-2} \cdots dx_1.
\]  

Integrating \( l \)th, \((l + 1)\)th, \ldots, \((n - 1)\)th equation of the system (1.1) and using (3.22), we have

\[
y_l(x_{l-1}) \geq -\int_{x_{l-1}}^{x_l} p_l(x_l) y_{l+1}(x_l) \, dx_l,
\]

\[
-y_{l+1}(x_l) \geq \int_{x_{l-1}}^{x_l} p_{l+1}(x_{l+1}) y_{l+2}(x_{l+1}) \, dx_{l+1},
\]

\[
y_{l+2}(x_{l+1}) \geq -\int_{x_{l+1}}^{x_l} p_{l+2}(x_{l+2}) y_{l+3}(x_{l+2}) \, dx_{l+2},
\]

\[\vdots\]

\[
-y_{n-1}(x_{n-2}) \geq \int_{x_{n-2}}^{x_l} p_{n-1}(x_{n-1}) y_n(x_{n-1}) \, dx_{n-1}.
\]

Combining expressions (3.24) and (3.25) and using (3.22), we get for \( t \geq t_3 \)

\[
z_1(t) \geq y_n(t) \int_{a(t)}^{t} p_1(x_1) \int_{a(t)}^{x_1} p_2(x_2) \cdots \int_{a(t)}^{x_{l-2}} p_{l-1}(x_{l-1}) \int_{x_{l-1}}^{x_l} p_l(x_l) \]

\[
\times \int_{x_{l+1}}^{x_l} p_{l+1}(x_{l+1}) \cdots \int_{x_{n-2}}^{x_{n-1}} p_{n-1}(x_{n-1}) \, dx_{n-1} \, dx_{n-2} \cdots dx_1.
\]  

The formula above may be rewritten by (1.5) and (1.6) for \( t \geq t_3 \) to

\[
z_1(t) \geq y_n(t) L_{l-2}(t, a(t); p_1, p_2, \ldots, p_{l-1}(*)) \times J_{n-l+1}((*), a(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}),
\]  

where \( x_1, x_2, \ldots, x_{l-1} \) are the \( (n - l + 1) \) points which satisfy conditions (1.5) and (1.6) depending on \( p_1, p_2, \ldots, p_{l-1}(\cdot) \) and \( a(t) \) in the interval \( [t_2, \infty) \).
Integrating the last equation of (1.1) from $t \to t^* \to \infty$ and using (e), (1.2), and (3.22), we obtain for $t \geq t_4$ where $t_4 \geq t_3$ is sufficiently large,

$$y_n(t) \geq K \int_t^\infty p_n(x_n)z_1(h(x_n))dx_n.$$  \hspace{1cm} (3.28)

From (3.2), (3.27), and (3.28) and the monotonicity of $z_1(h)$, we have

$$z_1(t) \geq KI_{t-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}((*) , \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}))$$

$$\times \int_t^\infty p_n(x_n)z_1(h(x_n))dx_n$$

$$\geq z_1(t)KI_{t-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}((*) , \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}))$$

$$\times \int_{h^{-1}(t)}^\infty p_n(x_n)dx_n,$$  \hspace{1cm} (3.29)

$$1 \geq KI_{t-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}((*) , \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}))$$

$$\times \int_{h^{-1}(t)}^\infty p_n(x_n)dx_n$$

for $t \geq t_4$, which is a contradiction to (3.5), and it gives

$$N_3^+ \cup N_5^+ \cup \cdots \cup N_{n-2}^+ = \emptyset.$$  \hspace{1cm} (3.30)

(III) Let $y \in N_n^+$ on $[t_2, \infty)$. In this case we consider for the components of solution $y(t)$ and for function $z_1$

$$z_1(t) > 0, \quad y_i(t) > 0, \quad i = 1, 2, \ldots, n, \quad t \geq t_2.$$  \hspace{1cm} (3.31)

Analogically as in the previous part of the proof,

$$z_1(t) \geq y_n(t)I_{n-1}(t, \alpha(t); p_1, p_2, \ldots, p_{n-1}), \quad t \geq t_3,$$  \hspace{1cm} (3.32)

holds and also (3.28), and for $t \geq t_3$

$$1 \geq KI_{n-1}(t, \alpha(t); p_1, p_2, \ldots, p_{n-1}) \int_{h^{-1}(t)}^\infty p_n(x_n)dx_n,$$  \hspace{1cm} (3.33)

which is a contradiction to (3.6) and $N_n^+ = \emptyset$. \hfill \Box
**Theorem 3.2.** Suppose that (3.1)–(3.4) are employed and (3.5) holds for \( l = 3, 5, \ldots, n - 1 \) and

\[
\int_{s}^{\infty} p_{n}(x_{n}) \int_{h(s)}^{h(x_{n})} p_{1}(x_{1}) \int_{h(s)}^{x_{1}} p_{2}(x_{2}) \cdots \int_{h(s)}^{x_{n-2}} p_{n-1}(x_{n-1}) dx_{n-1} \cdots dx_{2} dx_{1} dx_{n} = \infty \tag{3.34}
\]

for \( s \) sufficiently large.

If \( n \) is even and \( \sigma = 1 \), then every solution \( y \in W \) to the system (1.1) is either oscillatory, or \( \lim_{t \to \infty} y_{i}(t) = 0, i = 1, 2, \ldots, n \), or \( \lim_{t \to \infty} |y_{i}(t)| = \infty, i = 1, 2, \ldots, n \).

**Proof.** Let \( y \in W \) be a non-oscillatory solution to (1.1). Expression (2.8) holds. Taking into account Remark 2.5,

\[
N = N_{1}^{\ast} \cup N_{3}^{\ast} \cup \cdots \cup N_{n-1}^{\ast} \cup N_{n}^{\ast}. \tag{3.35}
\]

Without loss of generality we may suppose that \( y_{1}(t) \) is positive for \( t \geq t_{2} \).

(I) Let \( y \in N_{1}^{\ast} \) on \([t_{2}, \infty)\). In this case, for \( t \geq t_{2} \)

\[
y_{1}(t) > 0, \ y_{2}(t) < 0, \ y_{3}(t) > 0, \ y_{4}(t) < 0, \ldots, \ y_{n}(t) < 0. \tag{3.36}
\]

We may choose analogical approach as in Theorem 3.1 part (I). Equation (3.9) holds and we replace (3.11) by inequality

\[
-y_{n}(x_{n-1}) \geq KL_{1} \int_{x_{n-1}}^{\infty} p_{n}(x_{n}) dx_{n}, \quad x_{n-1} \geq t_{3}. \tag{3.37}
\]

Moreover (3.15) holds and similarly as in the proof of Theorem 3.1 case (I). We prove that \( \lim_{t \to \infty} y_{i}(t) = 0, i = 1, 2, \ldots, n \).

(II) Let \( y \in N_{1}^{\ast} \) on \([t_{2}, \infty)\), for some \( l = 3, 5, \ldots, n - 1 \). In this case, for \( t \geq t_{2} \),

\[
y_{1}(t) > 0, \ y_{2}(t) > 0, \ldots, \ y_{l}(t) > 0, \ y_{l+1}(t) < 0, \ldots, \ y_{n}(t) < 0. \tag{3.38}
\]

The analogical approach as in Theorem 3.1 part (II) follows out. Instead of inequality (3.27), we get for \( t \geq t_{3} \)

\[
z_{1}(t) \geq -y_{n}(t)I_{l-2}(t, \alpha(t); p_{1}, p_{2}, \ldots, p_{l-2}(\ast) \times J_{n-1}(\ast, \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1})) \tag{3.39}
\]

and instead of (3.28) for \( t \geq t_{4} \)

\[
-y_{n}(t) \geq K \int_{t}^{\infty} p_{n}(x_{n}) z_{1}(h(x_{n})) dx_{n}, \tag{3.40}
\]

and in the end we gain the contradiction to (3.5).
(III) Let \( y \in N^*_1 \) on \( [t_2, \infty) \). In this case (3.31) holds. Integrating the last equation of the system (1.1) and on the basis of (3.31), (3.2), (e), and (1.2), we have
\[
y_n(t) \geq K \int_s^t p_n(x_n) z_1(h(x_n)) dx_n, \quad t \geq s \geq t_3,
\]
(3.41)
where \( t_3 \geq t_2 \) is sufficiently large.

Integrating the first equation of the system (1.1) from \( h(s) \) to \( h(x_n) \) and employing (3.31), we obtain
\[
z_1(h(x_n)) \geq \int_{h(s)}^{h(x_n)} p_1(x_1) y_2(x_1) dx_1, \quad s \geq t_3.
\]
(3.42)
Combining (3.41) and (3.42), we have for \( t \geq s \geq t_3 \)
\[
y_n(t) \geq K \int_s^t p_n(x_n) \int_{h(s)}^{h(t)} p_1(x_1) y_2(x_1) dx_1 dx_n.
\]
(3.43)
Further consequently integrating the 2nd, 3rd, ..., (\( l - 1 \))th equations of the system (1.1) and step by step substituting into (3.43), we get for \( t \geq s \geq t_3 \)
\[
y_n(t) \geq K \int_s^t p_n(x_n) \int_{h(s)}^{h(x_n)} \int_{h(s)}^{x_1} p_1(x_1) \int_{h(s)}^{x_2} p_2(x_2) \cdots \int_{h(s)}^{x_{n-2}} p_{n-1}(x_{n-1}) y_n(x_{n-1}) dx_{n-1} dx_{n-2} \cdots dx_2 dx_1 dx_n.
\]
(3.44)
On basis of (3.31), for \( x_{n-1} \geq t_3 \)
\[
y_n(x_{n-1}) \geq C, \quad 0 < C = \text{const.}, \quad \text{for } x_{n-1} \geq t_3,
\]
(3.45)
hold.
Combining (3.44) and (3.45) for \( t \geq s \geq t_3 \), we have
\[
y_n(t) \geq KC \int_s^t p_n(x_n) \int_{h(s)}^{h(x_n)} \int_{h(s)}^{x_1} p_1(x_1) \int_{h(s)}^{x_2} p_2(x_2) \cdots \int_{h(s)}^{x_{n-2}} p_{n-1}(x_{n-1}) dx_{n-1} dx_{n-2} \cdots dx_2 dx_1 dx_n.
\]
(3.46)
From the inequality above and relation (3.34), we obtain \( \lim_{t \to \infty} y_n(t) = \infty \). Lemma 2.4 implies \( \lim_{t \to \infty} z_1(t) = \infty \) and \( \lim_{t \to \infty} y_i(t) = \infty, i = 2, 3, \ldots, n - 1. \) Since \( z_1(t) < y_1(t) \) for \( t \geq t_2 \), so \( \lim_{t \to \infty} y_1(t) = \infty \) and the final conclusion is \( \lim_{t \to \infty} |y_i(t)| = \infty, i = 1, 2, \ldots, n. \) \( \Box \)
Theorem 3.3. Suppose that (3.3) holds and

\[ 1 < \lambda^* \leq a(t) \quad \text{for some constant } \lambda^*, \quad t \geq t_0, \]

\[ t < g(t) < h(t) \quad \text{for } t \geq t_0, \]

\[ \int_{x_1}^{\infty} p_1(x_1) \int_{x_2}^{\infty} p_2(x_2) \int_{x_3}^{\infty} p_3(x_3) \cdots \int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1}) \]

\[ \times \int_{x_{n-1}}^{\infty} \frac{p_n(x_n)dx_n}{a(g^{-1}(h(x_n)))} = \infty, \]

\[ \limsup_{t \to \infty} KI_{-2}(t, (\alpha(t), p_1, p_2, \ldots, p_{n-2}(\ast) \times J_{n-1}((\ast), \alpha(t), p_{n-1}, p_{n-2}, \ldots, p_{n-1})) \]

\[ \times \int_{t}^{\infty} \frac{p_n(x_n)dx_n}{a(g^{-1}(h(x_n)))} > 1, \]

for \( l = 3, 5, \ldots, n - 2, \)

\[ \limsup_{t \to \infty} KI_{n-1}(t, (\alpha(t), p_1, p_2, \ldots, p_{n-1})) \int_{t}^{\infty} \frac{p_n(x_n)dx_n}{a(g^{-1}(h(x_n)))} > 1. \]

If \( n \) is odd and \( \sigma = 1 \) then every solution \( y \in W \) to (1.1) is either oscillatory, or \( \lim_{t \to \infty} y_i(t) = 0, \)

\( i = 1, 2, \ldots, n. \)

Proof. Let \( y \in W \) be a non-oscillatory solution to (1.1). Expression (2.6) holds. Without loss of generality we may suppose that \( y_1(t) \) is positive for \( t \geq t_2. \)

(I) Let \( y \in N_2^+ \cup N_4^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ \) on \([t_2, \infty)\). Lemma 2.6 implies \( \lim_{t \to \infty} y_1(t) = 0. \)

In this case, for \( t \geq t_2, \)

\[ 0 < z_1(t) < y_1(t), \]

and so \( \lim_{t \to \infty} z_1(t) = 0 \) which is a contradiction to the fact that the \( z_1(t) \) is positive and a nondecreasing function on the interval \([t_2, \infty)\) and

\[ N_2^+ \cup N_4^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ = \emptyset. \]

(II) Let \( y \in N_1^- \) on \([t_2, \infty)\). In this case, we can write for \( t \geq t_2 \)

\[ y_1(t) > 0, z_1(t) < 0, \quad y_2(t) > 0, \quad y_3(t) < 0, \ldots, \quad y_n(t) < 0. \]

We indirectly prove \( \lim_{t \to \infty} z_1(t) = 0. \)

Since \( z_1(t) \) is nondecreasing \( \lim_{t \to \infty} z_1(t) = -L_1, \quad L_1 > 0, \quad L_1 = \text{const.}, \) and

\[ z_1(t) \leq -L_1 \quad \text{for } t \geq t_2. \]
Because $z_1(t) > -a(t)y_1(g(t))$,

$$
z_1\left(g^{-1}(h(t))\right) > -a\left(g^{-1}(h(t))\right)y_1(h(t)),
$$

(3.56)

$$
y_1(h(t)) < \frac{z_1\left(g^{-1}(h(t))\right)}{a\left(g^{-1}(h(t))\right)} , \quad t \geq t_2
$$

(3.57)

follows.

From (3.55) and (3.57), we get

$$
-L_1 \geq z_1\left(g^{-1}(h(x_n))\right) \geq -a\left(g^{-1}(h(x_n))\right)y_1(h(x_n)), \quad x_n > t_2.
$$

(3.58)

By (c), (e), the last equation of (1.1), and (3.58), we get for $x_n > t_2$

$$
\frac{KL_1p_n(x_n)}{a\left(g^{-1}(h(x_n))\right)} \leq Kp_n(x_n)y_1(h(x_n)) \leq p_n(x_n)f\left(y_1(h(x_n))\right) = y'_n(x_n).
$$

(3.59)

Integrating (3.59) from $x_{n-1}$ to $x_{n-1}^* \to \infty$, we get

$$
KL_1\int_{x_{n-1}}^{\infty} \frac{p_n(x_n)dx_n}{a\left(g^{-1}(h(x_n))\right)} \leq -y_{n-1}(x_{n-1}) \quad \text{for} \quad x_{n-1} \geq t_2.
$$

(3.60)

Multiplying (3.60) by $p_{n-1}(x_{n-1})$ and then using the $(n-1)$th equation of system (1.1), we get for $x_{n-1} \geq t_2$

$$
KL_1p_{n-1}(x_{n-1})\int_{x_{n-1}}^{\infty} \frac{p_n(x_n)dx_n}{a\left(g^{-1}(h(x_n))\right)} \leq -y_{n-1}(x_{n-1}).
$$

(3.61)

Integrating (3.61) from $x_{n-2}$ to $x_{n-2}^* \to \infty$, we get for $x_{n-2} \geq t_2$

$$
KL_1\int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1})\int_{x_{n-1}}^{\infty} \frac{p_n(x_n)dx_ndx_{n-1}}{a\left(g^{-1}(h(x_n))\right)} \leq y_{n-1}(x_{n-2}).
$$

(3.62)

Similarly we continue by the same way until we derive for $x_1 \geq t_2$

$$
KL_1p_1(x_1)\int_{x_1}^{\infty} p_2(x_2)\int_{x_2}^{\infty} \cdots \int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1})
\times \int_{x_{n-1}}^{\infty} \frac{p_n(x_n)dx_ndx_{n-1} \cdots dx_2}{a\left(g^{-1}(h(x_n))\right)} \leq z_1(x_1).
$$

(3.63)
Integrating (3.63) from $T$ to $T^* \to \infty$, we get for $T \geq t_2$

$$KL_1 \int_T^\infty p_1(x_1) \int_{x_1}^\infty p_2(x_2) \int_{x_2}^\infty \cdots \int_{x_{n-2}}^\infty p_{n-1}(x_{n-1}) \times \int_{x_{n-1}}^\infty \frac{p_n(x_n)dx_n dx_{n-1} \cdots dx_2 dx_1}{a(g^{-1}(h(x_n)))} \leq -z_1(T). \quad (3.64)$$

That contradicts (3.49), and consequently $\lim_{t \to \infty} z_1(t) = 0$ holds.

We prove that $y_1(t)$ is bounded and $\lim_{t \to \infty} y_1(t) = 0$. There is some positive constant $B > 0$, $z_1(t) \geq -B$ for $t \geq t_2$, and by (1.2) and (3.47), one has for $t \geq t_2$

$$y_1(t) = a(t)y_1(g(t)) + z_1(t) \geq a(t)y_1(g(t)) - B \geq \lambda^* y_1(g(t)) - B. \quad (3.65)$$

We prove indirectly that $y_1(t)$ is bounded. Let us suppose that $y_1(t)$ is unbounded. Then $y_1(g(t))$ is unbounded, and there is a sequence

$$\{i_n\}_{n=1}^\infty \quad i_n \geq t_2, \quad n = 1, 2, \ldots, \quad i_n \to \infty \quad \text{as} \quad n \to \infty,$$

$$\lim_{n \to \infty} y_1(i_n) = \infty, \quad y_1(g(i_n)) = \max_{t_2 \leq s \leq g(i_n)} y_1(s). \quad (3.66)$$

By (3.65)

$$\lambda^* y_1(g(i_n)) \leq y_1(i_n) + B \leq y_1(g(i_n)) + B,$$

$$y_1(g(i_n)) \leq \frac{B}{\lambda^* - 1}, \quad n = 1, 2, \ldots. \quad (3.67)$$

That is a contradiction to $\lim_{n \to \infty} y_1(g(i_n)) = \infty$, and the function $y_1(t)$ is bounded. We claim that $\lim_{n \to \infty} y_1(t) = 0$, and we will prove it indirectly.

Let $\lim \sup_{t \to \infty} y_1(g(t)) = s$, $0 < s$, $s = \text{const}$. Then $\lim \sup_{t \to \infty} y_1(t) = s$.

Let $\{i_n\}_{n=1}^\infty$, $i_n \geq t_2$, $n = 1, 2, \ldots$, be such a kind of sequence that $\lim_{n \to \infty} i_n = \infty$ and $\lim \sup_{n \to \infty} y_1(g(i_n)) = s$.

Then, $\lim \sup_{n \to \infty} y_1(i_n) \leq s$.

By (1.2) and (3.47),

$$z_1(i_n) \leq y_1(i_n) - \lambda^* y_1(g(i_n)), \quad n = 1, 2, \ldots,$$

$$y_1(g(i_n)) \leq \frac{y_1(i_n) - z_1(i_n)}{\lambda^*}, \quad n = 1, 2, \ldots. \quad (3.68)$$

follows.

By the last inequality, we have

$$s = \lim \sup_{t \to \infty} y_1(g(i_n)) \leq \frac{\lim \sup_{t \to \infty} y_1(i_n)}{\lambda^*} \leq \frac{s}{\lambda^*}. \quad (3.69)$$
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1 ≥ λ* holds. That is a contradiction to (3.47). It means lim sup_{t→∞} y_1(g(t)) = 0 and also lim sup_{t→∞} y_1(t) = 0. Moreover, y_1(t) > 0 holds, so lim inf_{t→∞} lim_{t→∞} y_1(t) = 0 and this leads to lim_{t→∞} y_1(t) = 0.

By Lemma 2.4 it follows that

$$\lim_{t \to \infty} y_i(t) = 0, \quad i = 2, 3, \ldots, n. \quad (3.70)$$

(III) Let \( y \in N^{-}_l, l = 3, 5, \ldots, n-2 \), on \([t_2, \infty)\). In this case for, \( t \geq t_2 \),

$$y_1(t) > 0, \quad z_1(t) < 0, \quad y_2(t) < 0, \ldots, y_l(t) < 0, \quad y_{l+1}(t) > 0, \ldots, y_n(t) < 0. \quad (3.71)$$

Integrating the first equation of (1.1) from \( \alpha(t) \) to \( t \) and using (3.71), we get

$$z_1(t) \geq \int_{\alpha(t)}^{t} p_1(x_1) y_2(x_1) \, dx_1, \quad t \geq t_3, \quad (3.72)$$

where \( t_3 \geq t_2 \) is sufficiently large.

Integrating the 2nd, 3rd, \ldots, \((l-1)\)th equations of the system (1.1), and substituting into (3.72), we get for \( t \geq t_3 \)

$$z_1(t) \leq \int_{\alpha(t)}^{t} \int_{\alpha(t)}^{x_1} \int_{\alpha(t)}^{x_2} \cdots \int_{\alpha(t)}^{x_{l-2}} p_1(x_1) p_2(x_2) \cdots p_{l-1}(x_{l-1}) y_l(x_{l-1}) \, dx_{l-1} \, dx_{l-2} \cdots dx_1. \quad (3.73)$$

Integrating \( l \)th, \((l+1)\)th, \ldots, \((n-1)\)th equations of the system (1.1) we gain the system

$$y_{l}(x_{l-1}) \leq - \int_{x_{l-1}}^{x_{l-2}} p_{l}(x_{l}) y_{l+1}(x_{l}) \, dx_{l},$$

$$-y_{l+1}(x_{l}) \leq \int_{x_{l}}^{x_{l-2}} p_{l+1}(x_{l+1}) y_{l+2}(x_{l+1}) \, dx_{l+1},$$

$$y_{l+2}(x_{l+1}) \leq - \int_{x_{l+1}}^{x_{l-2}} p_{l+2}(x_{l+2}) y_{l+3}(x_{l+2}) \, dx_{l+2},$$

$$\vdots$$

$$-y_{n-1}(x_{n-2}) \leq \int_{x_{n-2}}^{x_{n-3}} p_{n-1}(x_{n-1}) y_{n}(x_{n-1}) \, dx_{n-1}. \quad (3.74)$$
We combine the formulae (3.73) and (3.74), and with regard to (3.71), we get for \( t \geq t_3 \)
\[
z_1(t) \leq y_n(t) \int_{a(t)}^{t} p_1(x_1) \int_{a(t)}^{x_1} p_2(x_2) \cdots \int_{a(t)}^{x_{i-1}} p_{i-1}(x_{i-1}) \int_{a(t)}^{x_{i-2}} p_1(x_i) \times \int_{x_i}^{x_{i-1}} p_{i+1}(x_{i+1}) \cdots \int_{x_{n-2}}^{x_{n-1}} p_{n-1}(x_{n-1}) \, dx_{n-1} \, dx_{n-2} \cdots \, dx_1. \tag{3.75}
\]
Employing (1.5) and (1.6) the equation above may be rewritten to
\[
z_1(t) \leq y_n(t) I_{i-2}(t, \alpha; p_1, p_2, \ldots, p_{i-2}(\ast) \times J_{n-i+1}(\ast, \alpha; p_{n-1}, \ldots, p_{l-1})) \tag{3.76}
\]
for \( t \geq t_3 \).

Integrating the last equation of (1.1) from \( t \) to \( t^* \rightarrow \infty \) and using (e) and (3.71),
\[
y_n(t) \leq -K \int_{t}^{\infty} p_n(x_n) y_t(h(x_n)) \, dx_n, \quad t \geq t_3. \tag{3.77}
\]
From (3.2), (3.57) in regard to (3.76), (3.77) and monotonicity of \( z_1(g^{-1}(h)) \), we get for \( t \geq t_3 \)
\[
z_1(t) \leq K I_{i-2}(t, \alpha; p_1, p_2, \ldots, p_{i-2}(\ast) \times J_{n-i+1}(\ast, \alpha; p_{n-1}, \ldots, p_{l-1}))
\times \int_{t}^{\infty} \frac{p_n(x_n) z_1(g^{-1}(h(x_n))) \, dx_n}{a(g^{-1}(h(x_n)))}
\leq z_1(t) K I_{i-2}(t, \alpha; p_1, p_2, \ldots, p_{i-2}(\ast) \times J_{n-i+1}(\ast, \alpha; p_{n-1}, \ldots, p_{l-1}))
\times \int_{t}^{\infty} \frac{p_n(x_n) \, dx_n}{a(g^{-1}(h(x_n)))} \tag{3.78}
\]
which means for \( t \geq t_3 \)
\[
1 \geq K I_{i-2}(t, \alpha; p_1, p_2, \ldots, p_{i-2}(\ast) \times J_{n-i+1}(\ast, \alpha; p_{n-1}, \ldots, p_{l-1}))
\times \int_{t}^{\infty} \frac{p_n(x_n) \, dx_n}{a(g^{-1}(h(x_n)))}. \tag{3.79}
\]
This is a contradiction to (3.50) and
\[
N_{3}^- \cup N_{5}^- \cup \cdots \cup N_{n-2}^- = \emptyset. \tag{3.80}
\]
(IV) Let \( y \in N_{i}^- \), on \([t_2, \infty)\).
In this case, we can write for \( t \geq t_2 \)
\[
y_1(t) > 0, \quad z_1(t) < 0, \quad y_i(t) < 0, \quad i = 2, 3, \ldots, n. \tag{3.81}
\]
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We may lead the proof analogically as in the previous part of the proof and we will prove that (3.77), (3.57), and

\[ z_1(t) \leq y_n(t) I_{n-1}(t, \alpha(t); p_1, p_2, \ldots, p_{n-1}) \]  

(3.82)

hold and also

\[ 1 \geq K I_{n-1}(t, \alpha(t); p_1, p_2, \ldots, p_{n-1}) \int_{t}^{\infty} \frac{p_n(x_n) dx_n}{a(g^{-1}(h(x_n)))}, \quad t \geq t_3 \]  

(3.83)

which is a contradiction to (3.51) and \( N_n^\sigma = \emptyset \).

\[ \square \]

**Theorem 3.4.** Suppose that (3.3), (3.47)–(3.49) hold and condition (3.50) is fulfilled for \( l = 3, 5, \ldots, n - 1 \), and

\[ \int_{s}^{\infty} \frac{p_n(x_n)}{a(g^{-1}(h(x_n)))} \int_{g^{-1}(h(s))}^{x_1} p_1(x_1) \int_{g^{-1}(h(s))}^{x_2} p_2(x_2) \cdots \int_{g^{-1}(h(s))}^{x_{n-2}} p_{n-1}(x_{n-1}) dx_{n-1} dx_{n-2} \cdots dx_{1} dx_{n} = \infty \]

(3.84)

for \( s \geq t_0 \).

If \( n \) is even and \( \sigma = -1 \), then every solution \( y \in W \) to (1.1) is either oscillatory, or \( \lim_{t \to \infty} y_i(t) = 0 \), \( i = 1, 2, \ldots, n \), or \( \lim_{t \to \infty} |y_i(t)| = \infty \) and \( \lim_{t \to \infty} |y_i(t)| = \infty \), \( i = 2, \ldots, n \).

**Proof.** Let \( y \in W \) be a non-oscillatory solution to (1.1). Expression (2.9) holds.

(I) Let \( y \in N_2^+ \cup N_4^+ \cup \cdots \cup N_n^+ \). Analogically as in the proof of Theorem 3.3 (I), we prove that

\[ N_2^+ \cup N_4^+ \cup \cdots \cup N_n^+ = \emptyset. \]  

(3.85)

(II) Let \( y \in N_1^+ \) on \([t_2, \infty)\). Similarly to the proof of Theorem 3.3 (II), we prove \( \lim_{t \to \infty} y_i(t) = 0 \), \( i = 1, 2, \ldots, n \).

(III) Let \( y \in N_i^- \), for some \( i = 3, 5, \ldots, n - 1 \), for \( t \in [t_2, \infty) \). Likewise as proof of Theorem 3.3 (III), for sets \( N_i^- \) we prove that \( N_3^- \cup N_5^- \cup \cdots \cup N_{n-1}^- = \emptyset \).

(IV) Let \( y \in N_n^- \) for \( t \in [t_2, \infty) \). Analogically to the proof of case (III) of Theorem 3.2, we claim \( \lim_{t \to \infty} |z_1(t)| = \infty \), \( \lim_{t \to \infty} |y_i(t)| = \infty \), \( i = 2, \ldots, n \).

\[ \square \]
Example 3.5. We consider system (1.1) as follows:

\[
\left( y_1(t) - \frac{1}{2} y_1\left( \frac{t}{4} \right) \right)' = e^t y_2(t),
\]
\[
y_2'(t) = \frac{1}{2} e^{\frac{t}{4}} y_3(t),
\]
\[
y_3'(t) = \frac{1}{2} e^{\frac{t}{8}} y_4(t),
\]
\[
y_4'(t) = \frac{1}{16} \left( e^{-\frac{3t}{8}} + \frac{5}{8} e^{-\frac{9t}{8}} \right) y_1\left( \frac{t}{2} \right), \quad t \geq 1.
\]

All assumptions of Theorem 3.2 are satisfied, and every solution \( y \in W \) to (3.86) is either oscillatory or

\[
\lim_{t \to \infty} y_i(t) = 0, \quad i = 1, 2, 3, 4, \quad \text{or} \quad \lim_{t \to \infty} \left| y_i(t) \right| = \infty, \quad i = 1, 2, 3, 4.
\]

One of the solutions has particular components as follows:

\[
y_1(t) = e^t, \quad y_2(t) = e^{t/2} - \frac{1}{8} e^{-t/4},
\]
\[
y_3(t) = e^{t/4} + \frac{1}{16} e^{-t/2}, \quad y_4(t) = \frac{1}{2} \left( e^{t/8} - \frac{1}{8} e^{-5t/8} \right), \quad t \geq 1,
\]

and in this case

\[
\lim_{t \to \infty} y_i(t) = \infty, \quad i = 1, 2, 3, 4.
\]

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References


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