Research Article

Conjugacy of Self-Adjoint Difference Equations of Even Order

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We study oscillation properties of 2n-order Sturm-Liouville difference equations. For these equations, we show a conjugacy criterion using the p-criticality of the existence of linear dependent recessive solutions at $\infty$ and $-\infty$. We also show the equivalent condition of p-criticality for one term 2n-order equations.

1. Introduction

In this paper, we deal with 2n-order Sturm-Liouville difference equations and operators

$$L(y)_k = \sum_{\nu=0}^{n} (-\Delta)^{\nu} (r^{[\nu]}_k \Delta^{\nu} y_{k+n-\nu}) = 0, \quad r^{[n]}_k > 0, \quad k \in \mathbb{Z}, \quad (1.1)$$

where $\Delta$ is the forward difference operator, that is, $\Delta y_k = y_{k+1} - y_k$, and $r^{[\nu]}$, $\nu = 0, \ldots, n$, are real-valued sequences. The main result is the conjugacy criterion which we formulate for the equation $L(y)_k + q_k y_{k+n} = 0$, that is viewed as a perturbation of (1.1), and we suppose that (1.1) is at least p-critical for some $p \in \{1, \ldots, n\}$. The concept of p-criticality (a disconjugate equation is said to be p-critical if and only if it possesses p solutions that are recessive both at $\infty$ and $-\infty$, see Section 3) was introduced for second-order difference equations in [1], and later in [2] for (1.1). For the continuous counterpart of the used techniques, see [3–5] from where we get an inspiration for our research.

The paper is organized as follows. In Section 2, we recall necessary preliminaries. In Section 3, we recall the concept of p-criticality, as introduced in [2], and show the first
result—the equivalent condition of $p$-criticality for the one term difference equation

$$\Delta^n(r_k \Delta^n y_k) = 0 \quad (1.2)$$

(Theorem 3.4). In Section 4 we show the conjugacy criterion for equation

$$(-\Delta)^n(r_k \Delta^n y_k) + q_k y_{k+n} = 0, \quad (1.3)$$

and Section 5 is devoted to the generalization of this criterion to the equation with the middle terms

$$\sum_{\nu=0}^{n} (-\Delta)^{\nu}(r_k^{[\nu]} \Delta^n y_{k+n-\nu}) + q_k y_{k+n} = 0. \quad (1.4)$$

2. Preliminaries

The proof of our main result is based on equivalency of (1.1) and the linear Hamiltonian difference systems

$$\Delta x_k = A x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A^T u_k, \quad (2.1)$$

where $A, B_k$, and $C_k$ are $n \times n$ matrices of which $B_k$ and $C_k$ are symmetric. Therefore, we start this section recalling the properties of (2.1), which we will need later. For more details, see the papers [6–11] and the books [12, 13].

The substitution

$$x_k^{[y]} = \begin{pmatrix} y_{k+n-1} \\ \Delta y_{k+n-2} \\ \vdots \\ \Delta^{n-1} y_k \end{pmatrix}, \quad u_k^{[y]} = \begin{pmatrix} \sum_{\nu=1}^{n} (-\Delta)^{\nu-1}(r_k^{[\nu]} \Delta^n y_{k+n-\nu}) \\ \vdots \\ -\Delta (r_k^{[n]} \Delta^n y_k) + r_k^{[n-1]} \Delta^{n-1} y_{k+1} \\ r_k^{[n]} \Delta^n y_k \end{pmatrix} \quad (2.2)$$

transforms (1.1) to linear Hamiltonian system (2.1) with the $n \times n$ matrices $A, B_k$, and $C_k$ given by

$$A = (a_{ij})_{ij=1}^{n}, \quad a_{ij} = \begin{cases} 1, & \text{if } j = i+1, \ i = 1, \ldots, n-1, \\ 0, & \text{elsewhere,} \end{cases} \quad (2.3)$$

$$B_k = \text{diag}\left\{0, \ldots, 0, \frac{1}{r_k^{[n]}}\right\}, \quad C_k = \text{diag}\left\{r_k^{[0]}, \ldots, r_k^{[n-1]}\right\}.$$ 

Then, we say that the solution $(x,u)$ of (2.1) is generated by the solution $y$ of (1.1).
Let us consider, together with system (2.1), the matrix linear Hamiltonian system

\[
\Delta X_k = AX_{k+1} + B_k U_k, \quad \Delta U_k = C_k X_{k+1} - A^T U_k,
\]

where the matrices \(A, B_k,\) and \(C_k\) are also given by (2.3). We say that the matrix solution \((X, U)\) of (2.4) is generated by the solutions \(y_{[1]}, \ldots, y^{[n]}\) of (1.1) if and only if its columns are generated by \(y_{[1]}, \ldots, y^{[n]}\), respectively, that is, \((X, U) = (x_{[1]}, \ldots, x^{[n]}; u_{[1]}, \ldots, u^{[n]})\).

Reversely, if we have the solution \((X, U)\) of (2.4), the elements from the first line of the matrix \(X\) are exactly the solutions \(y_{[1]}, \ldots, y^{[n]}\) of (1.1). Now, we can define the oscillatory properties of (1.1) via the corresponding properties of the associated Hamiltonian system (2.1) with matrices \(A, B_k,\) and \(C_k\) given by (2.3), for example, (1.1) is disconjugate if and only if the associated system (2.1) is disconjugate, the system of solutions \(y_{[1]}, \ldots, y^{[n]}\) is said to be recessive if and only if it generates the recessive solution \(X\) of (2.4), and so forth. Therefore, we define the following properties just for linear Hamiltonian systems.

For system (2.4), we have an analog of the continuous Wronskian identity. Let \((X, U)\) and \((\tilde{X}, \tilde{U})\) be two solutions of (2.4). Then,

\[
X_k^T \tilde{U}_k - U_k^T \tilde{X}_k = W
\]

holds with a constant matrix \(W\). We say that the solution \((X, U)\) of (2.4) is a conjoined basis, if

\[
X_k^T U_k \equiv U_k^T X_k, \quad \text{rank} \begin{pmatrix} X \\ U \end{pmatrix} = n. \quad (2.6)
\]

Two conjoined bases \((X, U), (\tilde{X}, \tilde{U})\) of (2.4) are called normalized conjoined bases of (2.4) if \(W = I\) in (2.5) (where \(I\) denotes the identity operator).

System (2.1) is said to be right disconjugate in a discrete interval \([l, m]\), \(l, m \in \mathbb{Z}\), if the solution \((\tilde{X}, \tilde{U})\) of (2.4) given by the initial condition \(X_l = 0, U_l = I\) satisfies

\[
\text{ker} X_{k+1} \subseteq \text{ker} X_k, \quad X_k X_{k+1} \dagger (I - A)^{-1} B_k \geq 0, \quad (2.7)
\]

for \(k = l, \ldots, m - 1\), see [6]. Here \(\text{ker}, \dagger, \geq\) stand for kernel, Moore-Penrose generalized inverse, and nonnegative definiteness of the matrix indicated, respectively. Similarly, (2.1) is said to be left disconjugate on \([l, m]\), if the solution given by the initial condition \(X_m = 0, U_m = -I\) satisfies

\[
\text{ker} X_k \subseteq \text{ker} X_{k+1}, \quad X_{k+1} X_k \dagger B_k (I - A)^{T-1} \geq 0, \quad k = l, \ldots, m - 1. \quad (2.8)
\]

System (2.1) is disconjugate on \(\mathbb{Z}\), if it is right disconjugate, which is the same as left disconjugate, see [14, Theorem 1], on \([l, m]\) for every \(l, m \in \mathbb{Z}, l < m\). System (2.1) is said to be nonoscillatory at \(\infty\) (nonoscillatory at \(-\infty\)), if there exists \(l \in \mathbb{Z}\) such that it is right disconjugate on \([l, m]\) for every \(m > l\) (there exists \(m \in \mathbb{Z}\) such that (2.1) is left disconjugate on \([l, m]\) for every \(l < m\)).
We call a conjoined basis \( \left( \tilde{X}, \tilde{U} \right) \) of \( \eqref{2.4} \) the recessive solution at \( \infty \), if the matrices \( \tilde{X}_k \) are nonsingular, \( \tilde{X}_k \tilde{X}_{k+1}^{-1} (I - A_k)^{-1} B_k \geq 0 \) (both for large \( k \)), and for any other conjoined basis \( \left( X, U \right) \), for which the (constant) matrix \( X^T U - U^T \tilde{X} \) is nonsingular, we have

\[
\lim_{k \to \infty} X_k^T \tilde{X}_k = 0. \tag{2.9}
\]

The solution \( (X, U) \) is called the dominant solution at \( \infty \). The recessive solution at \( \infty \) is determined uniquely up to a right multiple by a nonsingular constant matrix and exists whenever \( \eqref{2.4} \) is nonoscillatory and eventually controllable. (System is said to be eventually controllable if there exist \( N, \kappa \in \mathbb{N} \) such that for any \( m \geq N \) the trivial solution \( (\tilde{x}, \tilde{u}) = (0, 0) \) of \( \eqref{2.1} \) is the only solution for which \( x_m = x_{m+1} = \cdots = x_{m+\kappa} = 0 \).) The equivalent characterization of the recessive solution \( \left( \tilde{X}, \tilde{U} \right) \) of eventually controllable Hamiltonian difference systems \( \eqref{2.1} \) is

\[
\lim_{k \to \infty} \left( \sum_{j=1}^{k} k \tilde{X}_{j+1}^{-1} (I - A)^{-1} B_j \tilde{X}_{j}^{-1} \right)^{-1} = 0, \tag{2.10}
\]

see [12]. Similarly, we can introduce the recessive and the dominant solutions at \( -\infty \). For related notions and results for second-order dynamic equations, see, for example, [15, 16].

We say that a pair \((x, u)\) is admissible for system \( \eqref{2.1} \) if and only if the first equation in \( \eqref{2.1} \) holds.

The energy functional of \( \eqref{1.1} \) is given by

\[
\mathcal{F}(y) := \sum_{k=-\infty}^{\infty} \sum_{v=0}^{n} r_k^{[v]} (\Delta^v y_{k+n-v})^2. \tag{2.11}
\]

Then, for admissible \((x, u)\), we have

\[
\mathcal{F}(y) = \sum_{k=-\infty}^{\infty} \sum_{v=0}^{n} r_k^{[v]} (\Delta^v y_{k+n-v})^2
\]

\[
= \sum_{k=-\infty}^{\infty} \left[ \sum_{v=0}^{n-1} r_k^{[v]} (\Delta^v y_{k+n-v})^2 + \frac{1}{r_k^{[n]}} (r_k^{[n]} \Delta^n y_k)^2 \right] \tag{2.12}
\]

\[
= \sum_{k=-\infty}^{\infty} \left[ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right] =: \mathcal{F}(x, u).
\]

To prove our main result, we use a variational approach, that is, the equivalency of disconjugacy of \( \eqref{1.1} \) and positivity of \( \mathcal{F}(x, u) \); see [6].

Now, we formulate some auxiliary results, which are used in the proofs of Theorems 3.4 and 4.1. The following Lemma describes the structure of the solution space of

\[
\Delta^n (r_k \Delta^n y_k) = 0, \quad r_k > 0. \tag{2.13}
\]
Lemma 2.1 (see [17, Section 2]). Equation (2.13) is disconjugate on $\mathbb{Z}$ and possesses a system of solutions $y^{[i]}, \tilde{y}^{[i]}, j = 1, \ldots, n$, such that
\begin{equation}
y^{[1]} < \cdots < y^{[n]} < \tilde{y}^{[1]} < \cdots < \tilde{y}^{[n]} \tag{2.14}\end{equation}
as $k \to \infty$, where $f < g$ as $k \to \infty$ for a pair of sequences $f, g$ means that $\lim_{k \to \infty} (f_k/g_k) = 0$. If (2.14) holds, the solutions $y^{[i]}$ form the recessive system of solutions at $\infty$, while $\tilde{y}^{[i]}$ form the dominant system, $j = 1, \ldots, n$. The analogous statement holds for the ordered system of solutions as $k \to -\infty$.

Now, we recall the transformation lemma.

Lemma 2.2 (see [14, Theorem 4]). Let $h_k > 0$, $L(y) = \sum_{\nu=0}^{n} (-\Delta)^{\nu}(r_k^{[\nu]} \Delta^{\nu} y_{k+n-\nu})$ and consider the transformation $y_k = h_k z_k$. Then, one has
\begin{equation}h_{k+n} L(y) = \sum_{\nu=0}^{n} (-\Delta)^{\nu}(R_k^{[\nu]} \Delta^{\nu} z_{k+n-\nu}), \tag{2.15}\end{equation}
where
\begin{equation}R_k^{[n]} = h_{k+n} h_k r_k^{[n]}, \quad R_k^{[0]} = h_{k+n} L(h), \tag{2.16}\end{equation}
that is, $y$ solves $L(y) = 0$ if and only if $z$ solves the equation
\begin{equation}\sum_{\nu=0}^{n} (-\Delta)^{\nu}(R_k^{[\nu]} \Delta^{\nu} z_{k+n-\nu}) = 0. \tag{2.17}\end{equation}
The next lemma is usually called the second mean value theorem of summation calculus.

Lemma 2.3 (see [17, Lemma 3.2]). Let $n \in \mathbb{N}$ and the sequence $a_k$ be monotonic for $k \in [K + n - 1, L + n - 1]$ (i.e., $\Delta a_k$ does not change its sign for $k \in [K + n - 1, L + n - 2]$). Then, for any sequence $b_k$ there exist $n_1, n_2 \in [K, L - 1]$ such that
\begin{equation}\sum_{j=K}^{L-1} a_{n+j} b_j \leq a_{K+n-1} \sum_{i=K}^{n_1-1} b_i + a_{L+n-1} \sum_{i=n_1}^{L-1} b_i, \tag{2.18}\end{equation}
\begin{equation}\sum_{j=K}^{L-1} a_{n+j} b_j \geq a_{K+n-1} \sum_{i=K}^{n_2-1} b_i + a_{L+n-1} \sum_{i=n_2}^{L-1} b_i. \tag{2.19}\end{equation}

Now, let us consider the linear difference equation
\begin{equation}y_{k+n} + a_k^{[n-1]} y_{k+n-1} + \cdots + a_k^{[0]} y_k = 0, \tag{2.19}\end{equation}
where $k \geq n_0$ for some $n_0 \in \mathbb{N}$ and $a_k^{[0]} \neq 0$, and let us recall the main ideas of [18] and [19, Chapter IX].
An integer $m > n_0$ is said to be a generalized zero of multiplicity $k$ of a nontrivial solution $y$ of (2.19) if $y_{m-1} \neq 0$, $y_{m} = y_{m+1} = \cdots = y_{m+k-2} = 0$, and $(-1)^k y_{m+k-1} \geq 0$. Equation (2.19) is said to be eventually disconjugate if there exists $N \in \mathbb{N}$ such that no non-trivial solution of this equation has $n$ or more generalized zeros (counting multiplicity) on $[N, \infty)$.

A system of sequences $u_k^{[1]}, \ldots, u_k^{[n]}$ is said to form the D-Markov system of sequences for $k \in [N, \infty)$ if Casoratians

$$C(u_1^{[1]}, \ldots, u_1^{[j]}) = \begin{vmatrix} u_k^{[1]} & \cdots & u_k^{[j]} \\ u_{k+1}^{[1]} & \cdots & u_{k+1}^{[j]} \\ \vdots & & \vdots \\ u_{k+j-1}^{[1]} & \cdots & u_{k+j-1}^{[j]} \end{vmatrix}, \quad j = 1, \ldots, n$$

(2.20)

are positive on $(N + j, \infty)$.

**Lemma 2.4** (see [19, Theorem 9.4.1]). Equation (2.19) is eventually disconjugate if and only if there exist $N \in \mathbb{N}$ and solutions $y_1^{[1]}, \ldots, y_1^{[n]}$ of (2.19) which form a D-Markov system of solutions on $(N, \infty)$. Moreover, this system can be chosen in such a way that it satisfies the additional condition

$$\lim_{k \to \infty} \frac{y_k^{[i]}}{y_k^{[i+1]}} = 0, \quad i = 1, \ldots, n - 1.$$  

(2.21)

### 3. Criticality of One-Term Equation

Suppose that (1.1) is disconjugate on $\mathbb{Z}$ and let $\tilde{y}_1^{[i]}$ and $\tilde{y}_1^{[i]}$, $i = 1, \ldots, n$, be the recessive systems of solutions of $L(y) = 0$ at $-\infty$ and $\infty$, respectively. We introduce the linear space

$$\mathcal{K} = \operatorname{Lin}\{\tilde{y}_1^{[1]}, \ldots, \tilde{y}_1^{[n]}\} \cap \operatorname{Lin}\{\tilde{y}_1^{[1]}, \ldots, \tilde{y}_1^{[n]}\}.$$  

(3.1)

**Definition 3.1** (see [2]). Let (1.1) be disconjugate on $\mathbb{Z}$ and let $\dim \mathcal{K} = p \in \{1, \ldots, n\}$. Then, we say that the operator $L$ (or (1.1)) is $p$-critical on $\mathbb{Z}$. If $\dim \mathcal{K} = 0$, we say that $L$ is subcritical on $\mathbb{Z}$. If (1.1) is not disconjugate on $\mathbb{Z}$, that is, $L \not\equiv 0$, we say that $L$ is supercritical on $\mathbb{Z}$.

To prove the result in this section, we need the following statements, where we use the generalized power function

$$k^{(0)} = 1, \quad k^{(i)} = k(k - 1) \cdots (k - i + 1), \quad i \in \mathbb{N}.$$  

(3.2)

For reader’s convenience, the first statement in the following lemma is slightly more general than the corresponding one used in [2] (it can be verified directly or by induction).
Lemma 3.2 (see [2]). The following statements hold.

(i) Let $z_k$ be any sequence, $m \in \{0, \ldots, n\}$, and

$$y_k := \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} z_j,$$

then

$$\Delta^m y_k = \begin{cases} (n-1)^{(m)} \sum_{j=0}^{k-1} (k-j-1)^{(n-1-m)} z_j, & m \leq n-1, \\ (n-1)! z_k, & m = n. \end{cases}$$

(ii) The generalized power function has the binomial expansion

$$(k-j)^{(n)} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} k^{(n-i)} (j+i-1)^{(i)}.$$  

We distinguish two types of solutions of (2.13). The polynomial solutions $k^{(i)}$, $i = 0, \ldots, n-1$, for which $\Delta^n y_k = 0$, and nonpolynomial solutions

$$\sum_{j=0}^{k-1} (k-j-1)^{(n-1)} j^{(i)} r_j^{-1}, \quad i = 0, \ldots, n-1,$$

for which $\Delta^n y_k \neq 0$. (Using Lemma 3.2(i) we obtain $\Delta^n y_k = (n-1)! k^{(i)} r_k^{-1}$.)

Now, we formulate one of the results of [20].

Proposition 3.3 (see [20, Theorem 4]). If for some $m \in \{0, \ldots, n-1\}$

$$\sum_{k=-\infty}^{0} \left[ k^{(n-m-1)} r_k^{-1} \right]^2 = \infty = \sum_{k=0}^{\infty} \left[ k^{(n-m-1)} r_k^{-1} \right]^2,$$

then

$$\text{Lin}\left\{ 1, \ldots, k^{(m)} \right\} \subseteq \mathcal{H},$$

that is, (2.13) is at least $(m+1)$-critical on $\mathbb{Z}$.

Now, we show that (3.7) is also sufficient for (2.13) to be at least $(m+1)$-critical.

Theorem 3.4. Let $m \in \{0, \ldots, n-1\}$. Equation (2.13) is at least $(m+1)$-critical if and only if (3.7) holds.
Proof. Let $U^+$ and $U^-$ denote the subspaces of the solution space of (2.13) generated by the recessive system of solutions at $\infty$ and $-\infty$, respectively. Necessity of (3.7) follows directly from Proposition 3.3. To prove sufficiency, it suffices to show that if one of the sums in (3.7) is convergent, then $\{1, \ldots, k^{(m)}\} \notin U^+ \cap U^-$. We show this statement for the sum $\sum_{k=0}^\infty$. The other case is proved similarly, so it will be omitted. Particularly, we show

$$\sum_{k=0}^\infty k^{(n-m-1)} \left( r_k \right)^2 < \infty \implies k^{(m)} \notin U^+. \quad (3.9)$$

Let us denote $p := n - m - 1$, and let us consider the following nonpolynomial solutions of (2.13):

$$y_k^{[\ell]} = \sum_{j=0}^{k-1} (k - j - 1)^{(n-1)} j^{(p+\ell-1)} r_j^{-1} - \sum_{i=0}^p \left( -1 \right)^i \frac{(n-1-i)!}{(n-m-1-i)!} (k-1)^{(n-m-1-i)} \sum_{j=0}^\infty j^{(p)} (j + i - 1)^{(i)} r_j^{-1}, \quad (3.10)$$

where $\ell = 1 - p, \ldots, m + 1$. By Stolz-Cesàro theorem, since (using Lemma 3.2(i)) $\Delta^n y_k^{[\ell]} = (n-1)!k^{(p+\ell-1)} r_k^{-1}$, these solutions are ordered, that is, $y_k^{[i]} < y_k^{[i+1]}$, $i = 1 - p, \ldots, m$, as well as the polynomial solutions, that is, $k^{(i)} < k^{(i+1)}$, $i = 0, \ldots, n - 2$.

By some simple calculation and by Lemma 3.2 (at first, we use (i), and at the end, we use (ii)), we have

$$\Delta^m y_k^{[1]}$$

$$= \frac{(n-1)!}{(n-m-1)!} \sum_{j=0}^{k-1} (k - j - 1)^{(n-m-1)} j^{(p)} r_j^{-1}$$

$$- \sum_{i=0}^p \left( -1 \right)^i \frac{(n-1-i)!}{(n-m-1-i)!} (k-1)^{(n-m-1-i)} \sum_{j=0}^\infty j^{(p)} (j + i - 1)^{(i)} r_j^{-1}$$

$$= \frac{(n-1)!}{p!} \sum_{j=0}^{k-1} (k - j - 1)^{(p)} j^{(p)} r_j^{-1}$$

$$- \sum_{i=0}^p \left( -1 \right)^i \frac{(n-1-i)!}{(n-1-i)!} \frac{(n-1)!}{(p-i)!} (k-1)^{(p-i)} \sum_{j=0}^\infty j^{(p)} (j + i - 1)^{(i)} r_j^{-1}$$

$$= \frac{(n-1)!}{p!} \left\{ \sum_{j=0}^{k-1} (k - j - 1)^{(p)} j^{(p)} r_j^{-1} - \sum_{i=0}^p \left( -1 \right)^i \frac{(p)}{i} (k-1)^{(p-i)} \sum_{j=0}^\infty j^{(p)} (j + i - 1)^{(i)} r_j^{-1} \right\}$$

$$= \frac{(n-1)!}{p!} \left[ \sum_{j=0}^{k-1} (k - j - 1)^{(p)} j^{(p)} r_j^{-1} - \sum_{j=0}^\infty (k - j - 1)^{(p)} j^{(p)} r_j^{-1} \right]$$
In this section, we show the conjugacy criterion for two-term equation.

4. Conjugacy of Two-Term Equation

In this section, we show the conjugacy criterion for two-term equation.

Theorem 4.1. Let \( n > 1, q_k \) be a real-valued sequence, and let there exist an integer \( m \in \{0, \ldots, n-1\} \) and real constants \( c_0, \ldots, c_m \) such that (2.13) is at least \((m+1)\)-critical and the sequence \( h_k := c_0 + c_1 k + \cdots + c_m k^{(m)} \) satisfies

\[
\limsup_{k \to \infty} \sum_{l=0}^{L} q_k h_{k+n}^2 \leq 0. \tag{4.1}
\]

If \( q \neq 0 \), then

\[
(-\Delta)^n (r_k \Delta^n y_k) + q_k y_{k+n} = 0 \tag{4.2}
\]

is conjugate on \( \mathbb{Z} \).

Proof. We prove this theorem using the variational principle; that is, we find a sequence \( y \in \ell^2_c(\mathbb{Z}) \) such that the energy functional \( F(y) = \sum_{k=-\infty}^{\infty} [r_k (\Delta^n y_k)^2 + q_k y_{k+n}^2] < 0 \).

At first, we estimate the first term of \( F(y) \). To do this, we use the fact that this term is an energy functional of (2.13). Let us denote it by \( \tilde{F} \) that is,

\[
\tilde{F}(y) = \sum_{k=-\infty}^{\infty} r_k (\Delta^n y_k)^2. \tag{4.3}
\]
Using the substitution (2.2), we find out that (2.13) is equivalent to the linear Hamiltonian system (2.1) with the matrix $C_k \equiv 0$; that is,

$$
\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = -A^T u_k,
$$

(4.4)

and to the matrix system

$$
\Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = -A^T U_k.
$$

(4.5)

Now, let us denote the recessive solutions of (4.5) at $-\infty$ and $\infty$ by $(X^-, U^-)$ and $(X^+, U^+)$, respectively, such that the first $m + 1$ columns of $X^+$ and $X^-$ are generated by the sequences $1, k, \ldots, k^{(m)}$. Let $K, L, M$, and $N$ be arbitrary integers such that $N - M > 2n, M - L > 2n$, and $L - K > 2n$ (some additional assumptions on the choice of $K, L, M, N$ will be specified later), and let $(x^{[f]}, u^{[f]})$ and $(x^{[g]}, u^{[g]})$ be the solutions of (4.4) given by the formulas

$$
x^{[f]}_k = X^-_k \left( \sum_{j=k}^{k-1} B^-_j \right) \left( \sum_{j=k}^{L-1} B^-_j \right)^{-1} (X^-_L)^{-1} x^{[h]}_L,
$$

$$
u^{[f]}_k = U^-_k \left( \sum_{j=k}^{K-1} B^-_j \right) \left( \sum_{j=k}^{L-1} B^-_j \right)^{-1} (X^-_L)^{-1} x^{[h]}_L + (X^-_k)^{-1} \left( \sum_{j=K}^{L-1} B^-_j \right)^{-1} (X^-_L)^{-1} x^{[h]}_L,
$$

(4.6)

$$
x^{[g]}_k = X^+_k \left( \sum_{j=k}^{M-1} B^+_j \right) \left( \sum_{j=M}^{N-1} B^+_j \right)^{-1} (X^+_M)^{-1} x^{[h]}_M,
$$

$$
u^{[g]}_k = U^+_k \left( \sum_{j=k}^{N-1} B^+_j \right) \left( \sum_{j=M}^{N-1} B^+_j \right)^{-1} (X^+_M)^{-1} x^{[h]}_M - (X^+_k)^{-1} \left( \sum_{j=M}^{N-1} B^+_j \right)^{-1} (X^+_M)^{-1} x^{[h]}_M,
$$

where

$$
B^-_k = (X^-_{k+1})^{-1} (I - A)^{-1} B_k (X^-_k)^{-1},
$$

$$
B^+_k = (X^+_{k+1})^{-1} (I - A)^{-1} B_k (X^+_k)^{-1},
$$

(4.7)

and $(x^{[h]}, u^{[h]})$ is the solution of (4.4) generated by $h$. By a direct substitution, and using the convention that $\sum_{k=0}^{k-1} = 0$, we obtain

$$
x^{[f]}_L = x^{[f]}_L, \quad x^{[g]}_M = x^{[g]}_M, \quad x^{[f]}_N = x^{[g]}_N = 0.
$$

(4.8)

Now, from (4.1), together with the assumption $q \not\equiv 0$, we have that there exist $\tilde{k} \in \mathbb{Z}$ and $\varepsilon > 0$ such that $q_{\tilde{k}} \leq -\varepsilon$. Because the numbers $K, L, M,$ and $N$ have been “almost free” so far, we may choose them such that $L < \tilde{k} < M - n - 1$. 
Let us introduce the test sequence

\[ y_k := \begin{cases} 
0, & k \in (-\infty, K-1], \\
 f_k, & k \in [K, L-1], \\
 h_k(1 + D_k), & k \in [L, M-1], \\
g_k, & k \in [M, N-1], \\
0, & k \in [N, \infty),
\end{cases} \quad (4.9) \]

where

\[ D_k = \begin{cases} 
\delta > 0, & k = \tilde{k} + n, \\
0, & \text{otherwise.}
\end{cases} \quad (4.10) \]

To finish the first part of the proof, we use (4.4) to estimate the contribution of the term

\[ \bar{F}(y) = \sum_{k=\infty}^{\infty} r_k (\Delta^2 y_k)^2 = \sum_{k=\infty}^{\infty} u_k^{[y]^T} B_k u_k^{[y]} = \sum_{k=\infty}^{N-1} u_k^{[y]^T} B_k u_k^{[y]}. \quad (4.11) \]

Using the definition of the test sequence \( y \), we can split \( \bar{F} \) into three terms. Now, we estimate two of them as follows. Using (4.4), we obtain

\[
\sum_{k=K}^{L-1} u_k^{[f]^T} B_k u_k^{[f]} = \sum_{k=K}^{L-1} u_k^{[f]^T} (\Delta x_k^{[f]} - A x_{k+1}^{[f]}) = \sum_{k=K}^{L-1} u_k^{[f]^T} \Delta x_k^{[f]} - \sum_{k=K}^{L-1} u_k^{[f]^T} A x_{k+1}^{[f]}
\]

\[
= \sum_{k=K}^{L-1} \Delta \left( u_k^{[f]^T} x_k^{[f]} \right) - \sum_{k=K}^{L-1} u_k^{[f]^T} \left( \Delta u_k^{[f]} + A^T u_k^{[f]} \right) = \sum_{k=K}^{L-1} \left( x_k^{[f]} \right)^T A \left( x_k^{[f]} \right)
\]

\[
= x_k^{[h]^T} \left[ U_L^{-1} X_L^{[h]} + (X_L^{-1} \sum_{j=K}^{L-1} B_j^{-1})^{-1} (X_L^{-1})^{-1} x_k^{[h]} \right]
\]

\[
= x_k^{[h]^T} (X_L^{-1})^{-1} \left( \sum_{j=K}^{L-1} B_j^{-1} \right)^{-1} (X_L^{-1})^{-1} x_k^{[h]} \quad =: G_k,
\]

\[
\text{where}
\]

The remaining part of the proof is analogous to the previous cases.
where we used the fact that $x_L^{[h]^T}U_L^{-1}(X_L)^{-1}x_L^{[h]} \equiv 0$ (recall that the last $n - m - 1$ entries of $x_L^{[h]}$ are zeros and that the first $m + 1$ columns of $X$ and $U$ are generated by the solutions $1, \ldots, k^{(m)}$). Similarly,

$$\sum_{k=M}^{N-1} u_k^{[g]^T} B_k u_k^{[g]} = -x_M^{[g]^T} U_M^{-1} x_M^{[g]} = x_M^{[h]^T} (X_M^{-1}) \left( \sum_{j=M}^{N-1} B_j^+ \right)^{-1} (X_M^+)^{-1} x_M^{[h]} =: \mathcal{E}. \quad (4.13)$$

Using property (2.10) of recessive solutions of the linear Hamiltonian difference systems, we can see that $\mathcal{G} \to 0$ as $K \to -\infty$ and $\mathcal{E} \to 0$ as $N \to \infty$. We postpone the estimation of the middle term of $\tilde{F}$ to the end of the proof.

To estimate the second term of $F(y)$, we estimate at first its terms

$$\sum_{k=K}^{L-1} q_k f_k^2 \sum_{k=M}^{N-1} q_k g_k^2 \quad (4.14)$$

For this estimation, we use Lemma 2.3. To do this, we have to show the monotonicity of the sequences

$$f_k \quad \text{for} \quad k \in [K + n - 1, L + n - 1],$$

$$g_k \quad \text{for} \quad k \in [M + n - 1, N + n - 1]. \quad (4.15)$$

Let $x^{[1]}, \ldots, x^{[2n]}$ be the ordered system of solutions of (2.13) in the sense of Lemma 2.1. Then, again by Lemma 2.1, there exist real numbers $d_1, \ldots, d_n$ such that $h = d_1 x^{[1]} + \cdots + d_n x^{[n]}$. Because $h \neq 0$, at least one coefficient $d_i$ is nonzero. Therefore, we can denote $p := \max\{i \in [1, n] : d_i \neq 0\}$, and we replace the solution $x^{[p]}$ by $h$. Let us denote this new system again $x^{[1]}, \ldots, x^{[2n]}$ and note that this new system has the same properties as the original one.

Following Lemma 2.2, we transform (2.13) via the transformation $y_k = h_k z_k$, into

$$\sum_{\nu=0}^{n} (-\Delta)^\nu \left( R_k^{[\nu]} \Delta^\nu z_{k+n-\nu} \right) = 0, \quad (4.16)$$

that is,

$$(-\Delta)^n \left( R_k h_k h_{k+n} \Delta^{n-1} w_k \right) + \cdots - \Delta \left( R_k^{[1]} w_{k+n-1} \right) = 0 \quad (4.17)$$
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possesses the fundamental system of solutions

\[ \omega^{[1]} = -\Delta \left( \frac{x^{[1]}}{h} \right), \ldots, \omega^{[p-1]} = -\Delta \left( \frac{x^{[p-1]}}{h} \right), \]
\[ \omega^{[p]} = \Delta \left( \frac{x^{[p+1]}}{h} \right), \ldots, \omega^{[2n-1]} = \Delta \left( \frac{x^{[2n]}}{h} \right). \] (4.18)

Now, let us compute the Casoratians

\[ C(\omega^{[1]}) = \omega^{[1]} = -\Delta \left( \frac{x^{[1]}}{h} \right) = \frac{C(x^{[1]}, h)}{h_k h_{k+1}} > 0, \]
\[ C(\omega^{[1]}, \omega^{[2]}) = \frac{C(x^{[1]}, x^{[2]}, h)}{h_k h_{k+1} h_{k+2}} > 0, \] (4.19)
\[ \vdots \]
\[ C(\omega^{[1]}, \ldots, \omega^{[2n-1]}) = \frac{C(x^{[1]}, \ldots, x^{[p-1]}, x^{[p+1]}, \ldots, x^{[2n]}, h)}{h_k \cdots h_{k+2n-1}} > 0. \]

Hence, \( \omega^{[1]}, \ldots, \omega^{[2n-1]} \) form the D-Markov system of sequences on \([M, \infty)\), for \( M \) sufficiently large. Therefore, by Lemma 2.4, (4.17) is eventually disconjugate; that is, it has at most \( 2n - 2 \) generalized zeros (counting multiplicity) on \([M, \infty)\). The sequence \( \Delta(g/h) \) is a solution of (4.17), and we have that this sequence has generalized zeros of multiplicity \( n - 1 \) both at \( M \) and at \( N \); that is,

\[ \Delta \left( \frac{g_{M+i}}{h_{M+i}} \right) = 0 = \Delta \left( \frac{g_{N+i}}{h_{N+i}} \right), \quad i = 0, \ldots, n - 2. \] (4.20)

Moreover, \( g_M/h_M = 1 \) and \( g_N/h_N = 0 \). Hence, \( \Delta(g_k/h_k) \leq 0, k \in [M, N + n - 1] \). We can proceed similarly for the sequence \( f/h \).

Using Lemma 2.3, we have that there exist integers \( \xi_1 \in [K, L - 1] \) and \( \xi_2 \in [M, N - 1] \) such that

\[ \sum_{k=K}^{L-1} q_k f_{k+n}^2 = \sum_{k=K}^{L-1} \left[ q_k h_{k+n}^2 \left( \frac{f_{k+n}}{h_{k+n}} \right)^2 \right] \leq \sum_{k=\xi_1}^{L-1} q_k h_{k+n}^2, \]
\[ \sum_{k=M}^{N-1} q_k g_{k+n}^2 = \sum_{k=M}^{N-1} \left[ q_k h_{k+n}^2 \left( \frac{g_{k+n}}{h_{k+n}} \right)^2 \right] \leq \sum_{k=M}^{\xi_2-1} q_k h_{k+n}^2. \] (4.21)
Finally, we estimate the remaining term of \( F(y) \). By (4.9), we have

\[
\sum_{k=L}^{M-1} \left[ r_k (\Delta^n y_k)^2 + q_k y_{k+n}^2 \right] = \sum_{k=L}^{M-1} \left\{ r_k [\Delta^n h_k + \Delta^n (h_k D_k)]^2 + q_k (h_{k+n} + h_{k+n} D_{k+n})^2 \right\}
\]

\[
= \sum_{k=L}^{M-1} \left\{ r_k [\Delta^n (h_k D_k)]^2 + q_k h_{k+n}^2 + 2q_k h_{k+n}^2 D_{k+n} + q_k h_{k+n}^2 D_{k+n}^2 \right\}
\]

\[
= \sum_{k=k}^{k+n} \left\{ r_k [\Delta^n (h_k D_k)]^2 \right\} + \sum_{k=L}^{M-1} \left[ q_k h_{k+n}^2 \right] + 2q_k h_{k+n}^2 D_{k+n} + q_k h_{k+n}^2 D_{k+n}^2
\]

\[
\leq \delta^2 h_{k+n}^2 \sum_{k=k}^{k+n} \left[ r_k \left( \frac{n}{k - \bar{k}} \right)^2 \right] + \sum_{k=L}^{M-1} \left[ q_k h_{k+n}^2 \right] - 2\delta \epsilon h_{k+n}^2 + \delta^2 \epsilon h_{k+n}^2
\]

\[
< \delta^2 h_{k+n}^2 \sum_{k=k}^{k+n} \left[ r_k \left( \frac{n}{k - \bar{k}} \right)^2 \right] + \sum_{k=L}^{M-1} \left[ q_k h_{k+n}^2 \right] - 2\delta \epsilon h_{k+n}^2.
\]

Altogether, we have

\[
F(y) < \delta^2 h_{k+n}^2 \sum_{k=k}^{k+n} \left[ r_k \left( \frac{n}{k - \bar{k}} \right)^2 \right] + \sum_{k=L}^{M-1} \left[ q_k h_{k+n}^2 \right] - 2\delta \epsilon h_{k+n}^2 + \mathcal{G} + \mathcal{L} + \sum_{k=M}^{L-1} q_k h_{k+n}^2 + \sum_{k=L}^{L-1} q_k h_{k+n}^2
\]

\[
= \delta^2 h_{k+n}^2 \sum_{k=k}^{k+n} \left[ r_k \left( \frac{n}{k - \bar{k}} \right)^2 \right] - 2\delta \epsilon h_{k+n}^2 + \mathcal{G} + \mathcal{L} + \sum_{k=M}^{L-1} q_k h_{k+n}^2
\]

(4.23)

where for \( K \) sufficiently small is \( \mathcal{G} < \delta^2 / 3 \), for \( N \) sufficiently large is \( \mathcal{L} < \delta^2 / 3 \), and, from (4.1), \( \sum_{k=1}^{L-1} q_k h_{k+n}^2 < \delta^2 / 3 \) for \( \xi_1 < L \) and \( \xi_2 > M \). Therefore,

\[
F(y) < \delta^2 + \delta^2 h_{k+n}^2 \sum_{k=k}^{k+n} \left[ r_k \left( \frac{n}{k - \bar{k}} \right)^2 \right] - 2\delta \epsilon h_{k+n}^2
\]

\[
= \delta \left\{ \delta \left[ 1 + h_{k+n}^2 \sum_{k=k}^{k+n} \left[ r_k \left( \frac{n}{k - \bar{k}} \right)^2 \right] \right] - \epsilon h_{k+n}^2 \right\},
\]

(4.24)

which means that \( F(y) < 0 \) for \( \delta \) sufficiently small, and (4.2) is conjugate on \( \mathbb{Z} \). \( \square \)
5. Equation with the Middle Terms

Under the additional condition $q_k \leq 0$ for large $|k|$, and by combining of the proof of Theorem 4.1 with the proof of [2, Lemma 1], we can establish the following criterion for the full $2n$-order equation.

**Theorem 5.1.** Let $n > 1$, $q_k$ be a real-valued sequence, and let there exist an integer $m \in \{0, \ldots, n-1\}$ and real constants $c_0, \ldots, c_m$ such that (1.1) is at least $(m+1)$-critical and the sequence $h_k := c_0 + c_1 k + \cdots + c_m k^m$ satisfies

$$\limsup_{K \to -\infty, L \to \infty} \sum_{k=K}^{L} q_k h_{k+n}^2 \leq 0. \quad (5.1)$$

If $q_k \leq 0$ for large $|k|$ and $q \equiv 0$, then

$$L(y)_k + q_k y_{k+n} = \sum_{v=0}^{n} (-\Delta)^v \left( r_k^{|v|} \Delta^v y_{k+n-v} \right) + q_k y_{k+n} = 0 \quad (5.2)$$

is conjugate on $\mathbb{Z}$.

**Remark 5.2.** Using Theorem 3.4, we can see that the statement of Theorem 4.1 holds if and only if (3.7) holds. Finding a criterion similar to Theorem 3.4 for (1.1) is still an open question.

**Remark 5.3.** In the view of the matrix operator associated to (1.1) in the sense of [21], we can see that the perturbations in Theorem 4.1 affect the diagonal elements of the associated matrix operator. A description of behavior of (1.1), with regard to perturbations of limited part of the associated matrix operator (but not only of the diagonal elements), is given in [2].

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**References**


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