Research Article

Multiple Positive Solutions for Semilinear Elliptic Equations with Sign-Changing Weight Functions in $\mathbb{R}^N$

Tsing-San Hsu

Center for General Education, Chang Gung University, Kwei-Shan, Tao-Yuan 333, Taiwan

Correspondence should be addressed to Tsing-San Hsu, tshsu@mail.cgu.edu.tw

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Existence and multiplicity of positive solutions for the following semilinear elliptic equation:

$$-\Delta u + u = a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N,$$

$$u > 0 \quad \text{in} \quad \mathbb{R}^N,$$

$$u \in H^1(\mathbb{R}^N),$$

where $\lambda > 0, 1 < q < 2 < p < 2^* \quad (2^* = 2N/(N-2) \quad \text{if} \quad N \geq 3, \quad 2^* = \infty \quad \text{if} \quad N = 1, 2)$, $a$, $b$ satisfy suitable conditions, and $b$ maybe changes sign in $\mathbb{R}^N$. The study is based on the extraction of the Palais-Smale sequences in the Nehari manifold.

1. Introduction

In this paper, we deal with the multiplicity of positive solutions for the following semilinear elliptic equation:

$$-\Delta u + u = a(x)|u|^{p-1} + \lambda b(x)|u|^{q-1} \quad \text{in} \quad \mathbb{R}^N,$$

$$u > 0 \quad \text{in} \quad \mathbb{R}^N,$$

$$u \in H^1(\mathbb{R}^N),$$

where $\lambda > 0, 1 < q < 2 < p < 2^* \quad (2^* = 2N/(N-2) \quad \text{if} \quad N \geq 3, \quad 2^* = \infty \quad \text{if} \quad N = 1, 2)$ and $a$, $b$ are measurable functions and satisfy the following conditions:

(a1) $0 < a \in L^{\infty}(\mathbb{R}^N)$, where $\lim_{|x| \to \infty} a(x) = 1$, and there exist $C_0 > 0$ and $\delta_0 > 0$ such that

$$a(x) \geq 1 - C_0 e^{-\delta_0 |x|} \quad \forall x \in \mathbb{R}^N.$$ (1.1)
(b1) \( b \in L^q(\mathbb{R}^N) \) \((q^* = p/(p-q))\), \( b^+ = \max\{b,0\} \neq 0 \), \( b^- = \max\{-b,0\} \) is bounded and 
\( b^- \) has a compact support \( K \) in \( \mathbb{R}^N \).

(b2) There exist \( C_1 > 0 \), \( 0 < \delta_1 < \min\{\delta_0, q\} \) and \( R_0 > 0 \) such that

\[ b^+(x) - b(x) \geq C_1 e^{-\delta_1 |x|} \quad \forall |x| \geq R_0, \quad (1.2) \]

Semilinear elliptic equations with concave-convex nonlinearities in bounded domains are widely studied. For example, Ambrosetti et al. [1] considered the following equation:

\[-\Delta u = u^{p-1} + \lambda u^{\delta-1} \quad \text{in} \ \Omega, \]

\[ u > 0 \quad \text{in} \ \Omega, \]

\[ u = 0 \quad \text{on} \ \partial \Omega, \quad (E_1) \]

where \( \lambda > 0 \), \( 1 < q < 2 < p < 2^* \). They proved that there exists \( \lambda_0 > 0 \) such that \((E_1)\) admits at least two positive solutions for all \( \lambda \in (0, \lambda_0) \), has one positive solution for \( \lambda = \lambda_0 \) and no positive solution for \( \lambda > \lambda_0 \). Actually, Adimurthi et al. [2], Damascelli et al. [3], Korman [4], Ouyang and Shi [5], and Tang [6] proved that there exists \( \lambda_0 > 0 \) such that \((E_1)\) in the unit ball \( B^N(0;1) \) has exactly two positive solutions for \( \lambda \in (0, \lambda_0) \), has exactly one positive solution for \( \lambda = \lambda_0 \) and no positive solution exists for \( \lambda > \lambda_0 \). For more general results of \((E_1)\) (involving sign-changing weights) in bounded domains; see, the work of Ambrosetti et al. in [7], of Garcia Azorero et al. in [8], of Brown and Wu in [9], of Brown and Zhang in [10], of Cao and Zhong in [11], of de Figueiredo et al. in [12], and their references.

However, little has been done for this type of problem in \( \mathbb{R}^N \). We are only aware of the works [13–17] which studied the existence of solutions for some related concave-convex elliptic problems (not involving sign-changing weights). Furthermore, we do not know of any results for concave-convex elliptic problems involving sign-changing weight functions except [18, 19]. Wu in [18] have studied the multiplicity of positive solutions for the following equation involving sign-changing weights:

\[-\Delta u + u = f_\lambda(x) u^{\delta-1} + g_\mu(x) u^{p-1} \quad \text{in} \ \mathbb{R}^N, \]

\[ u > 0 \quad \text{in} \ \mathbb{R}^N, \]

\[ u \in H^1(\mathbb{R}^N), \quad (E_{f_\lambda,g_\mu}) \]

where \( 1 < q < 2 < p < 2^* \) the parameters \( \lambda, \mu \geq 0 \). He also assumed that \( f_\lambda(x) = \lambda f_1(x) + f_2(x) \) is sign changing and \( g_\mu(x) = a(x) + \mu b(x) \), where \( a \) and \( b \) satisfy suitable conditions and proved that \((E_{f_\lambda,g_\mu})\) has at least four positive solutions.

In a recent work [19], Hsu and Lin have studied \((E_{a,b})\) in \( \mathbb{R}^N \) with a sign-changing weight function. They proved there exists \( \lambda_0 > 0 \) such that \((E_{a,b})\) has at least two positive solutions for all \( \lambda \in (0, \lambda_0) \) provided that \( a, b \) satisfy suitable conditions and \( b \) maybe changes sign in \( \mathbb{R}^N \).

Continuing our previous work [19], we consider \((E_{a,b})\) in \( \mathbb{R}^N \) involving a sign-changing weight function with suitable assumptions which are different from the assumptions in [19].
In order to describe our main result, we need to define

\[ \Lambda_0 = \left( \frac{2 - q}{(p - q)\|a\|_{L^\infty}} \right)^{(2-q)/(p-2)} \left( \frac{p - 2}{(p - q)\|b^*\|_{L^{p^*}}} \right) S_p^{(2-q)/(2(p-2)+q/2)} > 0, \tag{1.3} \]

where \( \|a\|_{L^\infty} = \sup_{x \in \mathbb{R}^N} a(x), \|b^*\|_{L^{p^*}} = (\int_{\mathbb{R}^N} |b^+(x)|^q \, dx)^{1/q} \) and \( S_p \) is the best Sobolev constant for the imbedding of \( H^1(\mathbb{R}^N) \) into \( L^p(\mathbb{R}^N) \).

**Theorem 1.1.** Assume that (a1), (b1)-(b2) hold. If \( \lambda \in (0, (q/2)\Lambda_0) \), \((E_{a,\lambda b})\) admits at least two positive solutions in \( H^1(\mathbb{R}^N) \).

This paper is organized as follows. In Section 2, we give some notations and preliminary results. In Section 3, we establish the existence of a local minimum. In Section 4, we prove the existence of a second solution of \((E_{a,\lambda b})\).

At the end of this section, we explain some notations employed. In the following discussions, we will consider \( H = H^1(\mathbb{R}^N) \) with the norm \( \|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx)^{1/2}. \) We denote by \( S_p \) the best constant which is given by

\[ S_p = \inf_{u \in H^1(\mathbb{R}^N)} \frac{\|u\|^2}{(\int_{\mathbb{R}^N} |u|^p \, dx)^{2/p}}. \tag{1.4} \]

The dual space of \( H \) will be denoted by \( H^* \). \( \langle \cdot, \cdot \rangle \) denote the dual pair between \( H^* \) and \( H \). We denote the norm in \( L^s(\mathbb{R}^N) \) by \( \| \cdot \|_{L^s} \) for \( 1 \leq s \leq \infty \). \( B_N(x; r) \) is a ball in \( \mathbb{R}^N \) centered at \( x \) with radius \( r \). \( o_n(1) \) denotes \( o_n(1) \to 0 \) as \( n \to \infty \). \( C, C_i \) will denote various positive constants, the exact values of which are not important.

## 2. Preliminary Results

Associated with (1.3), the energy functional \( J_\lambda : H \to \mathbb{R}^N \) defined by

\[ J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} a(x)|u|^p \, dx - \frac{1}{q} \int_{\mathbb{R}^N} b(x)|u|^q \, dx, \tag{2.1} \]

for all \( u \in H \) is considered. It is well-known that \( J_\lambda \in C^1(H, \mathbb{R}) \) and the solutions of \((E_{a,\lambda b})\) are the critical points of \( J_\lambda \).

Since \( J_\lambda \) is not bounded from below on \( H \), we will work on the Nehari manifold. For \( \lambda > 0 \) we define

\[ \mathcal{N}_\lambda = \{ u \in H \setminus \{0\} : \langle J'_\lambda(u), u \rangle = 0 \}. \tag{2.2} \]

Note that \( \mathcal{N}_\lambda \) contains all nonzero solutions of \((E_{a,\lambda b})\) and \( u \in \mathcal{N}_\lambda \) if and only if

\[ \langle J'_\lambda(u), u \rangle = \|u\|^2 - \int_{\mathbb{R}^N} a(x)|u|^p \, dx - \lambda \int_{\mathbb{R}^N} b(x)|u|^q \, dx = 0. \tag{2.3} \]

**Lemma 2.1.** \( J_\lambda \) is coercive and bounded from below on \( \mathcal{N}_\lambda \).
Proof. If \( u \in \mathcal{A}_\lambda \), then by (b1), (2.3), and the Hölder and Sobolev inequalities, one has

\[
J_\lambda(u) = \frac{p-2}{2p} \|u\|^2 - \lambda \left( \frac{p-q}{pq} \right) \int_{\mathbb{R}^N} b(x) |u|^q dx \\
\geq \frac{p-2}{2p} \|u\|^2 - \lambda \left( \frac{p-q}{pq} \right) s_p^{q/2} \|u\|_{L^p}^q. 
\]

(2.4)

Since \( q < 2 < p \), it follows that \( J_\lambda \) is coercive and bounded from below on \( \mathcal{A}_\lambda \).

The Nehari manifold is closely linked to the behavior of the function of the form \( q_u : t \to J_\lambda(tu) \) for \( t > 0 \). Such maps are known as fibering maps and were introduced by Drábek and Pohozaev in [20] and are also discussed by Brown and Zhang in [10]. If \( u \in H \), we have

\[
q_u(t) = \frac{t^2}{2} \|u\|^2 - \frac{tp}{p} \int_{\mathbb{R}^N} a(x) |u|^p dx - \frac{tq}{q} \lambda \int_{\mathbb{R}^N} b(x) |u|^q dx, \\
q_u'(t) = t \|u\|^2 - (p-1) \frac{tp}{p} \int_{\mathbb{R}^N} a(x) |u|^p dx - \lambda \int_{\mathbb{R}^N} b(x) |u|^q dx, \\
q_u''(t) = \|u\|^2 - (p-1) \frac{tp}{p} \int_{\mathbb{R}^N} a(x) |u|^p dx - \lambda \int_{\mathbb{R}^N} b(x) |u|^q dx. 
\]

(2.6)

It is easy to see that

\[
\lambda p q_u'(t) = \|tu\|^2 - \int_{\mathbb{R}^N} a(x) |tu|^p dx - \lambda \int_{\mathbb{R}^N} b(x) |tu|^q dx, 
\]

(2.7)

and so, for \( u \in H \setminus \{0\} \) and \( t > 0 \), \( q_u'(t) = 0 \) if and only if \( tu \in \mathcal{A}_\lambda \) that is, the critical points of \( q_u \) correspond to the points on the Nehari manifold. In particular, \( q_u''(1) = 0 \) if and only if \( u \in \mathcal{A}_\lambda \). Thus, it is natural to split \( \mathcal{A}_\lambda \) into three parts corresponding to local minima, local maxima, and points of inflection. Accordingly, we define

\[
\mathcal{A}^+_\lambda = \{ u \in \mathcal{A}_\lambda : q_u''(1) > 0 \}, \\
\mathcal{A}^0_\lambda = \{ u \in \mathcal{A}_\lambda : q_u''(1) = 0 \}, \\
\mathcal{A}^-_\lambda = \{ u \in \mathcal{A}_\lambda : q_u''(1) < 0 \}. 
\]

(2.8)

and note that if \( u \in \mathcal{A}_\lambda \), then \( q_u'(1) = 0 \), then

\[
q_u''(1) = (2-q) \|u\|^2 - (p-q) \int_{\mathbb{R}^N} a(x) |u|^p dx, \\
= (2-p) \|u\|^2 - (q-p) \lambda \int_{\mathbb{R}^N} b(x) |u|^q dx. 
\]

(2.9)

(2.10)

We now derive some basic properties of \( \mathcal{A}^+_\lambda, \mathcal{A}^0_\lambda \) and \( \mathcal{A}^-_\lambda \).
Lemma 2.2. Suppose that \( u_0 \) is a local minimizer for \( J_\lambda \) on \( \mathcal{N}_\lambda \) and \( u_0 \notin \mathcal{N}_\lambda^0 \), then \( J_\lambda'(u_0) = 0 \) in \( H^* \).

Proof. See the work of Brown and Zhang in [10, Theorem 2.3]. \( \square \)

Lemma 2.3. If \( \lambda \in (0, \Lambda_0) \), then \( \mathcal{N}_\lambda^0 = \emptyset \).

Proof. We argue by contradiction. Suppose that there exists \( \lambda \in (0, \Lambda_0) \) such that \( \mathcal{N}_\lambda^0 \neq \emptyset \). Then for \( u \in \mathcal{N}_\lambda^0 \) by (2.9) and the Sobolev inequality, we have

\[
\frac{2 - q}{p - q} \|u\|^2 = \int_{\mathbb{R}^N} a(x)|u|^p dx \leq \|a\|_{L^p} S_p^{-p/2} \|u\|^p,
\]

and so

\[
\|u\| \geq \left( \frac{2 - q}{(p - q)\|a\|_{L^p}} \right)^{1/(p-2)} S_p^{p/2(p-2)}.
\]

Similarly, using (2.10), Hölder and Sobolev inequalities, we have

\[
\|u\|^2 = \lambda^{p - q} \parallel u \parallel_{L^q}^q \leq \lambda^{p - q} \|b^+\|_{L^q} S_p^{-q/2} \|u\|^q
\]

which implies

\[
\|u\| \leq \left( \frac{\lambda^{p - q} \|b^+\|_{L^q}}{p - 2} \right)^{1/(2-q)} S_p^{(2-q)/(2(p-2))^2}.
\]

Hence, we must have

\[
\lambda \geq \left( \frac{2 - q}{(p - q)\|a\|_{L^p}} \right)^{(2-q)/(p-2)} \left( \frac{p - 2}{(p - q)\|b^+\|_{L^q}} \right) S_p^{2(p-2)/(2(p-2)+q/2) = \Lambda_0}
\]

which is a contradiction. \( \square \)

In order to get a better understanding of the Nehari manifold and fibering maps, we consider the function \( q_u : \mathbb{R}^+ \to \mathbb{R} \) defined by

\[
q_u(t) = t^{2-q}\|u\|^2 - t^{p-q} \int_{\mathbb{R}^N} a(x)|u|^p dx \text{ for } t > 0.
\]

Clearly, \( tu \in \mathcal{N}_\lambda \) if and only if \( q_u(t) = \lambda \int_{\mathbb{R}^N} b(x)|u|^q dx \). Moreover,

\[
q_u'(t) = (2 - q)t^{1-q}\|u\|^2 - (p - q)t^{p-q-1} \int_{\mathbb{R}^N} a(x)|u|^p dx \text{ for } t > 0,
\]

and so

\[
h \text{ for } t > 0.
\]
and so it is easy to see that if $tu \in \mathcal{N}_\lambda$, then $t^{-1}q_u'(t) = q_u''(t)$. Hence, $tu \in \mathcal{N}_\lambda^+$ (or $tu \in \mathcal{N}_\lambda^-$) if and only if $q_u'(t) > 0$ (or $q_u'(t) < 0$).

Let $u \in H \setminus \{0\}$. Then, by (2.17), $q_u$ has a unique critical point at $t = t_{\text{max}}(u)$, where

$$
  t_{\text{max}}(u) = \left( \frac{(2 - q)\|u\|^2}{(p - q) \int_{\mathbb{R}^N} a(x)|u|^p\,dx} \right)^{1/(p-2)} > 0,
$$

and clearly $q_u$ is strictly increasing on $(0, t_{\text{max}}(u))$ and strictly decreasing on $(t_{\text{max}}(u), \infty)$ with $\lim_{u \to \infty}q_u(t) = -\infty$. Moreover, if $\lambda \in (0, \Lambda_0)$, then

$$
  q_u(t_{\text{max}}(u)) = \left[ \left( \frac{2 - q}{p - q} \right)^{(2-q)/(p-2)} - \left( \frac{2 - q}{p - q} \right)^{(p-q)/(p-2)} \right] \frac{\|u\|^{2(p-q)/(p-2)}}{\int_{\mathbb{R}^N} a(x)|u|^p\,dx} \geq \lambda \|b^+\|_{L^r}^q S_p^{2(q^2)/(2p-2)} \|u\|^q.
$$

Therefore, we have the following lemma.

**Lemma 2.4.** Let $\lambda \in (0, \Lambda_0)$ and $u \in H \setminus \{0\}$.

(i) If $\lambda \int_{\mathbb{R}^N} b(x)|u|^q\,dx \leq 0$, then there exists a unique $t^- = t^-(u) > t_{\text{max}}(u)$ such that $t^-u \in \mathcal{N}_\lambda^-$, $q_u$ is increasing on $(0, t^-)$ and decreasing on $(t^-, \infty)$. Moreover,

$$
  J_1(t^-u) = \sup_{t \geq 0} J_1(tu).
$$

(ii) If $\lambda \int_{\mathbb{R}^N} b(x)|u|^q\,dx > 0$, then there exist unique $0 < t^+ = t^+(u) < t_{\text{max}}(u) < t^- = t^-(u)$ such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$, $q_u$ is decreasing on $(0, t^+)$, increasing on $(t^+, t^-)$ and decreasing on $(t^-, \infty)$

$$
  J_1(t^+u) = \inf_{0 \leq t \leq t_{\text{max}}(u)} J_1(tu), \quad J_1(t^-u) = \sup_{t \geq t^-} J_1(tu).
$$

(iii) $\mathcal{N}_{\lambda}^- = \{u \in H \setminus \{0\} : t^-(u) = (1/\|u\|)t^-(u/\|u\|) = 1\}$.

(iv) There exists a continuous bijection between $\mathcal{U} = \{u \in H \setminus \{0\} : \|u\| = 1\}$ and $\mathcal{N}_{\lambda}^-$. In particular, $t^-$ is a continuous function for $u \in H \setminus \{0\}$.

**Proof.** See the work of Hsu and Lin in [19, Lemma 2.5].
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We remark that it follows Lemma 2.4, $\mathcal{M}_1 = \mathcal{M}_1^+ \cup \mathcal{M}_1^-$ for all $\lambda \in (0, \Lambda_0)$. Furthermore, by Lemma 2.4 it follows that $\mathcal{M}_1^+$ and $\mathcal{M}_1^-$ are non-empty and by Lemma 2.1 we may define

$$\alpha_\lambda = \inf_{u \in \mathcal{M}_1} J_\lambda(u), \quad \alpha_\lambda^+ = \inf_{u \in \mathcal{M}_1^+} J_\lambda(u), \quad \alpha_\lambda^- = \inf_{u \in \mathcal{M}_1^-} J_\lambda(u).$$  \hspace{1cm} (2.22)

**Theorem 2.5.** (i) If $\lambda \in (0, \Lambda_0)$, then we have $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

(ii) If $\lambda \in (0, (q/2)\Lambda_0)$, then $\alpha_\lambda^+ > d_0$ for some $d_0 > 0$.

In particular, for each $\lambda \in (0, (q/2)\Lambda_0)$, we have $\alpha_\lambda^+ = \alpha_\lambda < 0 < \alpha_\lambda^-$.

**Proof.** See the work of Hsu and Lin in [19, Theorem 3.1].  \hspace{1cm} \Box

**Remark 2.6.** (i) If $\lambda \in (0, \Lambda_0)$, then by (2.9), H"older and Sobolev inequalities, for each $u \in \mathcal{M}_1^+$ we have

$$\|u\|^2 \leq \frac{p-q}{p-2} \frac{l}{R^\lambda} |b(x)||u|^q \lambda d x$$

$$\leq \frac{p-q}{p-2} \frac{l}{R^\lambda} \|b\|_{L^p} S_p^{-q/2} \|u\|^q$$

$$\leq \frac{p-q}{p-2} \Lambda_0 \|b\|_{L^p} S_p^{-q/2} \|u\|^q,$$

and so

$$\|u\| \leq \left( \frac{p-q}{p-2} \Lambda_0 \|b\|_{L^p} S_p^{-q/2} \right)^{1/(2-q)} \quad \forall u \in \mathcal{M}_1^+. \hspace{1cm} (2.24)$$

(ii) If $\lambda \in (0, (q/2)\Lambda_0)$, then by Lemma 2.4(i), (ii) and Theorem 2.5(ii), for each $u \in \mathcal{M}_1^-$ we have

$$J_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu) \geq \alpha_\lambda^- > 0.$$  \hspace{1cm} (2.25)

**3. Existence of a Positive Solution**

First, we define the Palais-Smale (simply by (PS)) sequences, (PS)-values, and (PS)-conditions in $H$ for $J_\lambda$ as follows.

**Definition 3.1.** (i) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a (PS)$_c$-sequence in $H$ for $J_\lambda$ if $J_\lambda(u_n) = c + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$ strongly in $H^*$ as $n \to \infty$.

(ii) $c \in \mathbb{R}$ is a (PS)-value in $H$ for $J_\lambda$ if there exists a (PS)$_c$-sequence in $H$ for $J_\lambda$.

(iii) $J_\lambda$ satisfies the (PS)$_c$-condition in $H$ if any (PS)$_c$-sequence $\{u_n\}$ in $H$ for $J_\lambda$ contains a convergent subsequence.

Now we will ensure that there are (PS)$_{\alpha_\lambda^-}$-sequence and (PS)$_{\alpha_\lambda^+}$-sequence in $\mathcal{M}_1$ and $\mathcal{M}_1^-$, respectively, for the functional $J_\lambda$. 

Proposition 3.2. If $\lambda \in (0, (q/2)\Lambda_0)$, then

(i) there exists a $(PS)_{\alpha_1^*}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda$ in $H$ for $J_\lambda$.

(ii) there exists a $(PS)_{\alpha_1^*}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^*$ in $H$ for $J_\lambda$.

Proof. See Wu [21, Proposition 9].

Now, we establish the existence of a local minimum for $J_\lambda$ on $\mathcal{N}_\lambda^*$.

Theorem 3.3. Assume (a1) and (b1) hold. If $\lambda \in (0, (q/2)\Lambda_0)$, then there exists $u_\lambda \in \mathcal{N}_\lambda^*$ such that

(i) $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_1^* < 0$,

(ii) $u_\lambda$ is a positive solution of $(E_{a,\lambda b})$,

(iii) $\|u_\lambda\| \to 0$ as $\lambda \to 0^+$.

Proof. From Proposition 3.2(i) it follows that there exists $\{u_n\} \subset \mathcal{N}_\lambda$ satisfying

$$J_\lambda(u_n) = \alpha_\lambda + o_n(1) = \alpha_1^* + o_n(1), \quad J'_\lambda(u_n) = o_n(1) \quad \text{in } H^*.$$  \hspace{1cm} (3.1)

By Lemma 2.1 we infer that $\{u_n\}$ is bounded on $H$. Passing to a subsequence (Still denoted by $\{u_n\}$), there exists $u_\lambda \in H$ such that as $n \to \infty$

$$u_n \rightharpoonup u_\lambda \quad \text{weakly in } H,$$

$$u_n \to u_\lambda \quad \text{almost everywhere in } \mathbb{R}^N,$$

$$u_n \to u_\lambda \quad \text{strongly in } L^s_{\text{loc}}(\mathbb{R}^N) \forall 1 \leq s < 2^*.$$  \hspace{1cm} (3.2)

By (b1), Egorov theorem and Hölder inequality, we have

$$\lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx = \lambda \int_{\mathbb{R}^N} b(x)|u_\lambda|^q dx + o_n(1) \quad \text{as } n \to \infty.$$  \hspace{1cm} (3.3)

By (3.1) and (3.2), it is easy to see that $u_\lambda$ is a solution of $(E_{a,\lambda b})$. From $u_n \in \mathcal{N}_\lambda$ and (2.4), we deduce that

$$\lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx = \frac{q(p-2)}{2(p-q)}\|u_n\|^2 - \frac{pq}{p-q} J_\lambda(u_n).$$  \hspace{1cm} (3.4)

Let $n \to \infty$ in (3.4). By (3.1), (3.3) and $\alpha_1 < 0$, we get

$$\lambda \int_{\mathbb{R}^N} b(x)|u_\lambda|^q dx \geq - \frac{pq}{p-q} \alpha_1 > 0.$$  \hspace{1cm} (3.5)

Thus, $u_\lambda \in \mathcal{N}_\lambda$ is a nonzero solution of $(E_{a,\lambda b})$. 
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Next, we prove that \( u_n \to u_1 \) strongly in \( H \) and \( J_1(u_1) = \alpha_1 \). From the fact \( u_n, u_1 \in \mathcal{N}_1 \) and applying Fatou’s lemma, we get

\[
\alpha_1 \leq J_1(u_1) = \frac{p-2}{2p} \| u_1 \|^2 - \frac{p-q}{pq} \int_{\mathbb{R}^N} b(x) |u_1|^q \, dx
\]

\[
\leq \liminf_{n \to \infty} \left( \frac{p-2}{2p} \| u_n \|^2 - \frac{p-q}{pq} \int_{\mathbb{R}^N} b(x) |u_n|^q \, dx \right)
\]

\[
\leq \liminf_{n \to \infty} J_1(u_n) = \alpha_1.
\]

This implies that \( J_1(u_1) = \alpha_1 \) and \( \lim_{n \to \infty} \| u_n \|^2 = \| u_1 \|^2 \). Standard argument shows that \( u_n \to u_1 \) strongly in \( H \). By Theorem 2.5, for all \( \lambda \in (0, (q/2) \Lambda_0) \) we have that \( u_1 \in \mathcal{N}_1 \) and \( J_1(u_1) = \alpha_1^* < \alpha_1^* \) which implies \( u_1 \in \mathcal{N}_1^* \). Since \( J_1(u_1) = J_1(|u_1|) \) and \( |u_1| \in \mathcal{N}_1^* \), by Lemma 2.2 we may assume that \( u_1 \) is a nonzero nonnegative solution of \( (E_{a, b}) \). By Harnack inequality [22] we deduce that \( u_1 > 0 \) in \( \mathbb{R}^N \). Finally, by (2.10), Hölder and Sobolev inequalities,

\[
\| u_1 \|^{2-q} < \lambda \frac{p-q}{p-2} \| b^+ \|_{L^p} \Omega^{q/2},
\]

and thus we conclude the proof.

\[\square\]

4. Second Positive Solution

In this section, we will establish the existence of the second positive solution of \( (E_{a, b}) \) by proving that \( J_1 \) satisfies the \( (PS)_{\alpha_1^*} \)-condition.

Lemma 4.1. Assume that (a1) and (b1) hold. If \( \{u_n\} \subset H \) is a \( (PS)_c \)-sequence for \( J_1 \), then \( \{u_n\} \) is bounded in \( H \).

Proof. See the work of Hsu and Lin in [19, Lemma 4.1].

\[\square\]

Let us introduce the problem at infinity associated with \( (E_{a, b}) \):

\[-\Delta u + u = u^{p-1} \quad \text{in} \ \mathbb{R}^N, \ \ u \in H, \ \ u > 0 \ \text{in} \ \mathbb{R}^N. \quad (E^\infty)\]

We state some known results for problem \( (E^\infty) \). First of all, we recall that by Lions [23] has studied the following minimization problem closely related to problem \( (E^\infty) \):

\[
S^\infty = \inf \{ J^\infty(u) : u \in H, \ u \neq 0, \ (J^\infty)'(u) = 0 \} > 0,
\]

where \( J^\infty(u) = (1/2)\| u \|^2 - (1/p) \int_{\mathbb{R}^N} |u|^p \, dx \). Note that a minimum exists and is attained by a ground state \( w_0 > 0 \) in \( \mathbb{R}^N \) such that

\[
S^\infty = J^\infty(w_0) = \sup_{t \geq 0} J^\infty(tw_0) = \left( \frac{1}{2} - \frac{1}{p} \right) S_p^{p/(p-2)},
\]

(4.2)
exist positive constants $C_\varepsilon, C_2$ such that for all $x \in \mathbb{R}^N$,

$$C_\varepsilon \exp(-(1 + \varepsilon)|x|) \leq w_0(x) \leq C_2 \exp(-|x|).$$

We define

$$w_n(x) = w_0(x - ne), \quad \text{where } e = (0, 0, \ldots, 0, 1) \text{ is a unit vector in } \mathbb{R}^N. \quad (4.4)$$

Clearly, $w_n(x) \in H$.

**Lemma 4.2.** Let $\Omega$ be a domain in $\mathbb{R}^N$. If $f : \Omega \to \mathbb{R}$ satisfies

$$\int_{\Omega} \left| f(x)e^{\sigma|x|} \right| dx < \infty \quad \text{for some } \sigma > 0, \quad (4.5)$$

then

$$\left( \int_{\Omega} f(x)e^{-\sigma|x-\bar{x}|} dx \right)e^{\sigma|\bar{x}|} = \int_{\Omega} f(x)e^{\sigma(x,\bar{x})/|\bar{x}|}dx + o(1) \quad \text{as } |\bar{x}| \to \infty. \quad (4.6)$$

**Proof.** We know $\sigma|\bar{x}| \leq \sigma|x| + \sigma|x - \bar{x}|$. Then,

$$\left| f(x)e^{-\sigma|x-\bar{x}|}e^{\sigma|\bar{x}|} \right| \leq \left| f(x)e^{\sigma|x|} \right|. \quad (4.7)$$

Since $-\sigma|x - \bar{x}| + \sigma|\bar{x}| = \sigma(x,\bar{x})/|\bar{x}| + o(1)$ as $|\bar{x}| \to \infty$, then the lemma follows from the Lebesgue dominated convergence theorem. \qed

**Lemma 4.3.** Under the assumptions (a1), (b1)-(b2) and $\lambda \in (0, \Lambda_0)$. Then there exists a number $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\sup_{t \geq 0} J_1(tw_n) < S^\infty. \quad (4.8)$$

In particular, $a^*_1 < S^\infty$ for all $\lambda \in (0, \Lambda_0)$.

**Proof.** (i) First, since $\|w_n\| = \|w_0\|$ for all $n \in \mathbb{N}$ and $J_1$ is continuous in $H$ and $J_1(0) = 0$, we infer that there exists $t_1 > 0$ such that

$$J_1(tw_n) < S^\infty \quad \forall n \in \mathbb{N}, \quad t \in [0, t_1]. \quad (4.9)$$
(ii) Since \( \lim_{|x| \to \infty} a(x) = 1 \), there exists \( n_1 \in \mathbb{N} \) such that if \( n \geq n_1 \), we get \( a(x) \geq 1/2 \) for \( x \in B^N(ne;1) \). Then, for \( n \geq n_1 \)

\[
J_1(tw_n) = \frac{t^2}{2} ||w_n||^2 - \frac{tp}{p} \int_{\mathbb{R}^N} a(x)|w_n|^p dx - \frac{tq}{q} \int_{\mathbb{R}^N} \lambda b(x)|w_n|^q dx
\]

\[
\leq \frac{t^2}{2} ||w_n||^2 - \frac{tp}{p} \int_{B^N(0;1)} a(x+ne)|w_0|^p dx + \frac{tq}{q} \lambda \|b^-\|_{L^\infty} \int_{\mathbb{R}^N} |w_0|^q dx
\]

\[
\leq \frac{t^2}{2} ||w_0||^2 - \frac{tp}{2p} \int_{B^N(0;1)} |w_0|^p dx + \frac{tq}{q} \lambda \|b^-\|_{L^\infty} \int_{\mathbb{R}^N} |w_0|^q dx
\]

\[
\longrightarrow -\infty \quad \text{as} \quad t \longrightarrow \infty.
\]

Thus, there exists \( t_2 > 0 \) such that for any \( t > t_2 \) and \( n > n_1 \) we get

\[
J_1(tw_n) < 0. \tag{4.11}
\]

(iii) By (i) and (ii), we need to show that there exists \( n_0 \) such that for \( n \geq n_0 \)

\[
\sup_{t_1 \leq t \leq t_2} J_1(tw_n) < S^\infty. \tag{4.12}
\]

We know that \( \sup_{t \geq 0} J^\infty(tw_0) = S^\infty \). Then, \( t_1 \leq t \leq t_2 \), we have

\[
J_1(tw_n) = \frac{1}{2} ||tw_n||^2 - \frac{1}{p} \int_{\mathbb{R}^N} a(x)(tw_n)^p dx - \frac{1}{q} \int_{\mathbb{R}^N} \lambda b(x)(tw_n)^q dx
\]

\[
\leq \frac{t^2}{2} ||w_0||^2 - \frac{tp}{p} \int_{\mathbb{R}^N} w^p_n dx + \frac{tq}{q} \int_{\mathbb{R}^N} (1 - a(x))w^p_n dx - \frac{tq}{q} \int_{\mathbb{R}^N} \lambda b(x)w^q_n dx
\]

\[
\leq S^\infty + \frac{tp}{p} \int_{\mathbb{R}^N} (1 - a)^+(x)w^p_n dx + \frac{tq}{q} \int_{\mathbb{R}^N} \lambda b^+(x)w^q_n dx
\]

Suppose \( a \) satisfies (a1), we get \( (1 - a)^+(x) \leq C_0 e^{-\delta_0 |x|} \) for all \( x \in \mathbb{R}^N \) and some positive constant \( \delta_0 \). By (4.3) and Lemma 4.3, there exists \( n_2 > n_1 \) such that for any \( n \geq n_2 \)

\[
\int_{\mathbb{R}^N} (1 - a)^+(x)w^p_n dx \leq C_3 e^{-\min(\delta_0,n)} \tag{4.14}
\]

By (b1) and (4.3), we get

\[
\int_{\mathbb{R}^N} \lambda b^+(x)w^q_n dx \leq \lambda \|b^-\|_{L^\infty} C_2 \int_K e^{-q|x-ne|} dx
\]

\[
\leq \lambda C_3 e^{-qn}. \tag{4.15}
\]
By (b2), (4.3) and Lemma 4.3, we have
\[
\int_{\mathbb{R}^N} \lambda b^*(x) w_n^d dx \geq \lambda \mathcal{C}_1 \mathcal{C}_\epsilon \int_{|x| \geq R_0} e^{-\delta_1|x|} e^{-q(1+\epsilon)|x-n_0|} dx
\geq \lambda \mathcal{C}_1 e^{-\delta_1 n_0}. \tag{4.16}
\]

Since \(0 < \delta_1 < \min\{\delta_0, q\} \leq \min\{\delta_0, p\}\) and \(\lambda \in (0, \Lambda_0)\) and using (4.13)–(4.16), we have there exists \(n_0 > n_2\) such that for all \(n \geq n_0\), then
\[
\sup_{t \leq t_0} J_1(tw_n) < S^\infty, \quad \lambda \int_{\mathbb{R}^N} b(x)|w_n|^d dx > 0. \tag{4.17}
\]
This implies that if \(\lambda \in (0, \Lambda_0)\), then for all \(n \geq n_0\) we get
\[
\sup_{t \geq 0} J_1(tw_n) < S^\infty. \tag{4.18}
\]
From \(a(x) > 0\) for all \(x \in \mathbb{R}^N\) and (4.17), we have
\[
\int_{\mathbb{R}^N} a(x)|w_n|^p dx > 0, \quad \int_{\mathbb{R}^N} b(x)|w_n|^q dx > 0. \tag{4.19}
\]
Combining this with Lemma 2.4(ii), from the definition of \(\alpha_\lambda^*\) and \(\sup_{t \geq 0} J_1(tw_n) < S^\infty\), for all \(\lambda \in (0, \Lambda_0)\), we obtain that there exists \(t_0 > 0\) such that \(t_0 w_n \in \mathcal{M}_\lambda^*\) and
\[
\alpha_\lambda^* \leq J_1(t_0 w_n) \leq \sup_{t \geq 0} J_1(tw_n) < S^\infty. \tag{4.20}
\]

**Lemma 4.4.** Assume that (a1) and (b1) hold. If \(\{u_n\} \subset H\) is a \((PS)_c\)-sequence for \(J_1\) with \(c \in (0, S^\infty)\), then there exists a subsequence of \(\{u_n\}\) converging weakly to a nonzero solution of \((E_{a,\lambda b})\) in \(\mathbb{R}^N\).

**Proof.** Let \(\{u_n\} \subset H\) be a \((PS)_c\)-sequence for \(J_1\) with \(c \in (0, S^\infty)\). We know from Lemma 4.1 that \(\{u_n\}\) is bounded in \(H\), and then there exist a subsequence of \(\{u_n\}\) (still denoted by \(\{u_n\}\)) and \(u_0 \in H\) such that
\[
u_n \rightharpoonup u_0 \quad \text{weakly in } H,
\]
\[
u_n \rightarrow u_0 \quad \text{almost everywhere in } \mathbb{R}^N,
\]
\[
u_n \rightarrow u_0 \quad \text{strongly in } L^s_{\text{loc}}(\mathbb{R}^N), \forall 1 \leq s < 2^*.
\]
It is easy to see that \(J_1'(u_0) = 0\) and by (b1), Egorov theorem and Hölder inequality, we have
\[
\lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx = \lambda \int_{\mathbb{R}^N} b(x)|u_0|^q dx + o_n(1). \tag{4.22}
\]
Next we verify that \( u_0 \neq 0 \). Arguing by contradiction, we assume \( u_0 \equiv 0 \). By (a1), for any \( \varepsilon > 0 \), there exists \( R_0 > 0 \) such that \( |a(x) - 1| < \varepsilon \) for all \( x \in \mathbb{R}^N \). Since \( u_n \to 0 \) strongly in \( L^s_{\text{loc}}(\mathbb{R}^N) \) for \( 1 \leq s < 2^* \), \( \{u_n\} \) is a bounded sequence in \( H \), therefore \( \int_{\mathbb{R}^N} (a(x) - 1)|u_n|^p \leq C \int_{B(0; R_0)} |u_n|^p + \varepsilon C \). Setting \( n \to \infty \), then \( \varepsilon \to 0 \), we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} a(x)|u_n|^p \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p \, dx. \tag{4.23}
\]

We set

\[
l = \lim_{n \to \infty} \int_{\mathbb{R}^N} a(x)|u_n|^p \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p \, dx. \tag{4.24}
\]

Since \( f_{\lambda}'(u_n) = o_n(1) \) and \( \{u_n\} \) is bounded, then by (4.22), we can deduce that

\[
0 = \lim_{n \to \infty} \langle f_{\lambda}'(u_n), u_n \rangle = \lim_{n \to \infty} \left( \|u_n\|^2 - \int_{\mathbb{R}^N} a(x)|u_n|^p \, dx \right) = \lim_{n \to \infty} \|u_n\|^2 - l, \tag{4.25}
\]

that is,

\[
\lim_{n \to \infty} \|u_n\|^2 = l. \tag{4.26}
\]

If \( l = 0 \), then we get \( c = \lim_{n \to \infty} f_{\lambda}(u_n) = 0 \), which contradicts to \( c > 0 \). Thus we conclude that \( l > 0 \). Furthermore, by the definition of \( S_p \) we obtain

\[
\|u_n\|^2 \geq S_p \left( \int_{\mathbb{R}^N} |u_n|^p \, dx \right)^{2/p}. \tag{4.27}
\]

Then, as \( n \to \infty \), we have

\[
l = \lim_{n \to \infty} \|u_n\|^2 \geq S_p l^{2/p}, \tag{4.28}
\]

which implies that

\[
l \geq S_p^{p/(p-2)}. \tag{4.29}
\]
Hence, from (4.2) and (4.22)–(4.29), we get

\[
c = \lim_{n \to \infty} J_1(u_n)
= \frac{1}{2} \lim_{n \to \infty} \|u_n\|^2 - \frac{1}{p} \lim_{n \to \infty} \int_{\mathbb{R}^N} a(x)|u_n|^p \, dx - \frac{1}{q} \lim_{n \to \infty} \int_{\mathbb{R}^N} b(x)|u_n|^q \, dx
= \left( \frac{1}{2} - \frac{1}{p} \right) l
\geq \frac{p-2}{2p} S_p/(p-2) = S^\infty.
\]

(4.30)

This is a contradiction to \( c < S^\infty \). Therefore, \( u_0 \) is a nonzero solution of \((E_{a,lb})\). \(\square\)

Now, we establish the existence of a local minimum of \( J_1 \) on \( \mathcal{N}_1 \).

**Theorem 4.5.** Assume that (a1) and (b1)-(b2) hold. If \( \lambda \in (0, (q/2)\Lambda_0) \), then there exists \( U_1 \in \mathcal{N}_1 \) such that

(i) \( J_1(U_1) = \alpha_1^\infty \),

(ii) \( U_1 \) is a positive solution of \((E_{a,lb})\).

**Proof.** If \( \lambda \in (0, (q/2)\Lambda_0) \), then by Theorem 2.5(ii), Proposition 3.2(ii) and Lemma 4.3(ii), there exists a \((PS)_{\alpha_1}\)-sequence \( \{u_n\} \subset \mathcal{N}_1^\infty \) in \( H \) for \( J_1 \) with \( \alpha_1^\infty \in (0, S^\infty) \). From Lemma 4.4, there exist a subsequence still denoted by \( \{u_n\} \) and a nonzero solution \( U_1 \in H \) of \((E_{a,lb})\) such that \( u_n \rightharpoonup U_1 \) weakly in \( H \).

First, we prove that \( U_1 \in \mathcal{N}_1^\infty \). On the contrary, if \( U_1 \notin \mathcal{N}_1^\infty \), then by \( \mathcal{N}_1^\infty \) is closed in \( H \), we have \( \|U_1\|^2 < \inf_{n \to \infty} \|u_n\|^2 \). From (2.9) and \( a(x) > 0 \) for all \( x \in \mathbb{R}^N \), we get

\[
\int_{\mathbb{R}^N} b(x)|U_1|^q \, dx > 0, \quad \int_{\mathbb{R}^N} a(x)|U_1|^p \, dx > 0.
\]

(4.31)

By Lemma 2.4(ii), there exists a unique \( t_1^\infty \) such that \( t_1^\infty U_1 \in \mathcal{N}_1^\infty \). If \( u \in \mathcal{N}_1 \), then it is easy to see that

\[
J_1(u) = \frac{p-2}{2p} \|u\|^2 - \frac{p-q}{pq} \int_{\mathbb{R}^N} b(x)|u|^q \, dx.
\]

(4.32)

From (3.1), \( u_n \in \mathcal{N}_1^\infty \) and (4.32), we can deduce that

\[
\alpha_1^\infty = J_1(t_1^\infty U_1) < \lim_{n \to \infty} J_1(t_1^\infty u_n) \leq \lim_{n \to \infty} J_1(u_n) = \alpha_1
\]

(4.33)

which is a contradiction. Thus, \( U_1 \in \mathcal{N}_1^\infty \).

Next, by the same argument as that in Theorem 3.3, we get that \( u_n \rightharpoonup U_1 \) strongly in \( H \) and \( J_1(U_1) = \alpha_1 > 0 \) for all \( \lambda \in (0, (q/2)\Lambda_0) \). Since \( J_1(U_1) = J_1(|U_1|) \) and \( |U_1| \in \mathcal{N}_1^\infty \) by Lemma 2.2 we may assume that \( U_1 \) is a nonzero nonnegative solution of \((E_{a,lb})\). Finally, by the Harnack inequality [22] we deduce that \( U_1 > 0 \) in \( \mathbb{R}^N \). \(\square\)
Now, we complete the proof of Theorem 1.1. By Theorems 3.3, 4.5, we obtain \((E_{a,bb})\) has two positive solutions \(u_1\) and \(U_1\) such that \(u_1 \in \mathcal{M}_f, U_1 \in \mathcal{M}_r\). Since \(\mathcal{M}_f \cap \mathcal{M}_r = \emptyset\), this implies that \(u_1\) and \(U_1\) are distinct. It completes the proof of Theorem 1.1.

References


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