Research Article

Oscillation Properties for Second-Order Partial Differential Equations with Damping and Functional Arguments

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Using an integral averaging method and generalized Riccati technique, by introducing a parameter $\beta \geq 1$, we derive new oscillation criteria for second-order partial differential equations with damping. The results are of high degree of generality and sharper than most known ones.

1. Introduction

Consider the second-order partial delay differential equation

$$\begin{aligned} \frac{\partial}{\partial t} \left(r(t) \frac{\partial}{\partial t} u(x, t) \right) + p(t) \frac{\partial u(x, t)}{\partial t} = & a(t) \Delta u(x, t) + \sum_{k=1}^s a_k(t) \Delta u(x, t - \rho_k(t)) - q(x, t) f(u(x, t)) \\ & - \sum_{j=1}^m q_j(x, t) f_j(u(x, t - \sigma_j)), \quad (x, t) \in \Omega \times R_+ \equiv G, \end{aligned} \quad (1.1)$$

where Δ is the Laplacian in R^N , $R_+ = [0, \infty)$ and Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$.

Throughout this paper, we assume that

(H₁) $r(t) \in C^1(R_+, (0, \infty))$, $p(t) \in C(R_+, R)$;

(H₂) $q(x, t), q_j(x, t) \in C(\overline{G}, R_+)$, $q(t) = \min_{x \in \overline{G}} q(x, t)$, $q_j(t) = \min_{x \in \overline{G}} q_j(x, t)$, $j \in I_m = \{1, 2, \dots, m\}$;

(H₃) $a(t), a_k(t), \rho_k(t) \in C(R_+, R_+)$, $\lim_{t \rightarrow \infty} (t - \rho_k(t)) = \infty$, and σ_j are nonnegative constants, $j \in I_m, k \in I_s = \{1, 2, \dots, s\}$;

(H₄) $f(u) \in C^1(R, R)$, $f_j(u) \in C(R, R)$ are convex in R_+ with $uf_j(u) > 0$, $uf(u) > 0$, and $f'(u) \geq \mu > 0$, ($u \neq 0$).

We say that a continuous function $H(t, s)$ belongs to the function class ω , denoted by $H \in \omega$, if $H \in C(D, R_+)$, where $D = \{(t, s) : -\infty < s \leq t < +\infty\}$, satisfy

$$H(t, t) = 0, \quad H(t, s) > 0, \quad -\infty < s < t < +\infty. \quad (1.2)$$

Furthermore, the continuous partial derivative $\partial H / \partial s$ exists on D , and there is $h \in L_{\text{loc}}(D, R)$, such that

$$\frac{\partial H}{\partial s} = -h(t, s)\sqrt{H(t, s)}. \quad (1.3)$$

Various results on the oscillation for the partial functional differential equation have been obtained recently. We refer the reader to [1–3] for parabolic equations and to [4–11] for hyperbolic equations.

Recently, Li and Cui [12] studied the equation of the form

$$\begin{aligned} \frac{\partial}{\partial t} \left[p(t) \frac{\partial}{\partial t} \left(u(x, t) + \sum_{i=1}^l \lambda_i(t) u(x, t - \tau_i) \right) \right] &= a(t) \Delta u(x, t) + \sum_{k=1}^s a_k(t) \Delta u(x, t - \rho_k(t)) - q(t) u(x, t) \\ &\quad - \sum_{j=1}^m q_j(x, t) u(x, t - \sigma_j), \quad (x, t) \in \Omega \times R_+ \equiv G \end{aligned} \quad (1.4)$$

with Robin boundary condition

$$\frac{\partial u(x, t)}{\partial \gamma} + g(x, t) u(x, t) = 0, \quad (x, t) \in \partial \Omega \times R_+, \quad (1.5)$$

where γ is the unit exterior vector to $\partial \Omega$ and $g(x, t)$ is a nonnegative continuous function on $\partial \Omega \times R_+$ and obtained the following result.

Theorem A (see [12, Theorem 2.2]). *Suppose that $H \in \omega$, let*

(C₁) $0 < \inf_{s \geq t_0} \{ \liminf_{t \rightarrow \infty} (H(t, s) / H(t, t_0)) \} \leq \infty$, *suppose that there exists some $j_0 \in I_m$ and there exist two functions $\phi \in C^1[t_0, \infty)$, $A \in C[t_0, \infty)$ satisfying,*

(C₂) $\limsup_{t \rightarrow \infty} (1 / H(t, t_0)) \int_{t_0}^t p(s - \sigma_{j_0}) \phi(s) h^2(t, s) ds < \infty$,

(C₃) $\int_{t_0}^\infty (A_+^2(s) / p(s - \sigma_{j_0}) \phi(s)) ds = \infty$, *and for every $t_1 \geq t_0$,*

(C₄) $\limsup_{t \rightarrow \infty} (1 / H(t, t_1)) \int_{t_1}^t [H(t, s) \psi(s) - (1/4) \phi(s) p(s - \sigma_{j_0}) h^2(t, s)] ds \geq A(t_1)$,

where $\phi(s) = \exp\{-2 \int^s \phi(\xi) d\xi\}$, $A_+(s) = \max\{A(s), 0\}$, and $\psi(s) = \phi(s) \{ \alpha_{j_0} q_{j_0}(s) [1 - \sum_{i=1}^l \lambda_i(s - \sigma_{j_0})] + p(s - \sigma_{j_0}) \phi^2(s) - [p(s - \sigma_{j_0}) \phi(s)]' \}$. Then every solution $u(x, t)$ of the problem (1.4), (1.5) is oscillatory in G .

In 2008, Rogovchenko and Tuncay [13] established new oscillation criteria for second-order nonlinear differential equations with damping term

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0, \quad (1.6)$$

without an assumption that has been required in related results reported in the literature over the last two decades. Motivated by the ideas in [12, 13], by introducing a Parameter $\beta \geq 1$, we will further improve Theorems A and derive new interval criteria for oscillation of (1.1). We suggest two different approaches which allow one to remove condition (C_2) in Theorem A. A modified integral averaging technique enables one to simplify essentially the proofs of oscillation criteria.

2. Main Results

Theorem 2.1. *Suppose that there exists a function $y \in C^1[t_0, \infty)$ such that for some $\beta \geq 1$ and for some $H \in \omega$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left(H(t, s)\psi(s) - \frac{\beta}{4\mu} v(s)r(s)h^2(t, s) \right) ds = \infty, \quad t \geq 0, \quad (2.1)$$

where

$$v(t) = \exp \left(-2 \int^t \left(\mu y(s) - \frac{p(s)}{2r(s)} \right) ds \right), \quad (2.2)$$

$$\psi(t) = v(t) \left[q(t) + \mu r(t)y^2(t) - p(t)y(t) - (r(t)y(t))' \right], \quad (2.3)$$

then every solution $u(x, t)$ of the problem (1.1), (1.5) is oscillatory in G .

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x, t)$ of the problem (1.1), (1.5) which has no zero on $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality, we assume that $u(x, t) > 0$, $u(x, t - \rho_k(t)) > 0$ and $u(x, t - \sigma_j) > 0$ in $\Omega \times [t_1, \infty)$, $t_1 \geq t_0$, $k \in I_s$, $j \in I_m$. Integrating (1.1) with respect to x over the domain Ω , we have

$$\begin{aligned} & \frac{d}{dt} \left(r(t) \frac{d}{dt} \int_{\Omega} u(x, t) dx \right) + p(t) \frac{d}{dt} \int_{\Omega} u(x, t) dx \\ &= a(t) \int_{\Omega} \Delta u(x, t) dx + \sum_{k=1}^s a_k(t) \int_{\Omega} \Delta u(x, t - \rho_k(t)) dx \\ & \quad - \int_{\Omega} q(x, t) f(u(x, t)) dx - \sum_{j=1}^m \int_{\Omega} q_j(x, t) f_j(u(x, t - \sigma_j)) dx, \quad t \geq t_1. \end{aligned} \quad (2.4)$$

From Green's formula and the boundary condition (1.5), we have

$$\begin{aligned} \int_{\Omega} \Delta u(x, t) dx &= \int_{\partial\Omega} \frac{\partial u(x, t)}{\partial \gamma} ds = - \int_{\partial\Omega} g(x, t) u(x, t) ds \leq 0, \\ \int_{\Omega} \Delta u(x, t - \rho_k(t)) dx &= \int_{\partial\Omega} \frac{\partial u(x, t - \rho_k(t))}{\partial \gamma} ds \\ &= - \int_{\partial\Omega} g(x, t - \rho_k(t)) u(x, t - \rho_k(t)) ds \leq 0, \quad t \geq t_1, \quad k \in I_s, \end{aligned} \quad (2.5)$$

where ds denotes the surface element on $\partial\Omega$. Moreover, from (H_2) , (H_4) and Jensen's inequality, we have

$$\begin{aligned} \int_{\Omega} q(x, t) f(u(x, t)) dx &\geq q(t) \int_{\Omega} f(u(x, t)) dx \geq |\Omega| q(t) f\left(\frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx\right), \\ \int_{\Omega} q_j(x, t) f_j(u(x, t - \sigma_j)) dx &\geq q_j(t) \int_{\Omega} f_j(u(x, t - \sigma_j)) dx \\ &\geq |\Omega| q_j(t) f_j\left(\frac{1}{|\Omega|} \int_{\Omega} u(x, t - \sigma_j) dx\right), \end{aligned} \quad (2.6)$$

where $|\Omega| = \int_{\Omega} dx$.
Set

$$U(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx, \quad t \geq t_1. \quad (2.7)$$

In view of (2.5)–(2.7), (2.4) yields that

$$(r(t)U'(t))' + p(t)U'(t) + q(t)f(U(t)) + \sum_{j=1}^m q_j(t)f_j(U(t - \sigma_j)) \leq 0, \quad t \geq t_1. \quad (2.8)$$

Note that (H_4) , (2.8) yields that

$$(r(t)U'(t))' + p(t)U'(t) + q(t)f(U(t)) \leq 0, \quad t \geq t_1. \quad (2.9)$$

Put

$$w(t) = v(t)r(t) \left[\frac{U'(t)}{f(U(t))} + y(t) \right], \quad t \geq t_1, \quad (2.10)$$

where $v(t)$ is given by (2.2), then

$$\begin{aligned}
 w'(t) &= \left[-2\mu y(t) + \frac{p(t)}{r(t)} \right] w(t) + v(t) \left[\frac{(r(t)U'(t))'}{f(U(t))} - \frac{r(t)f'(U(t))(U'(t))^2}{f^2(U(t))} + (r(t)y(t))' \right] \\
 &\leq \left[-2\mu y(t) + \frac{p(t)}{r(t)} \right] w(t) + v(t) \left[\frac{(r(t)U'(t))'}{f(U(t))} - \frac{\mu r(t)(U'(t))^2}{f^2(U(t))} + (r(t)y(t))' \right] \\
 &\leq \left[-2\mu y(t) + \frac{p(t)}{r(t)} \right] w(t) - v(t) \left[\frac{p(t)U'(t)}{f(U(t))} + q(t) + \mu r(t) \left(\frac{U'(t)}{f(U(t))} \right)^2 - (r(t)y(t))' \right] \\
 &= \left[-2\mu y(t) + \frac{p(t)}{r(t)} \right] w(t) - v(t) \left[p(t) \left(\frac{w(t)}{v(t)r(t)} - y(t) \right) + q(t) \right. \\
 &\quad \left. + \mu r(t) \left[\frac{w(t)}{v(t)r(t)} - y(t) \right]^2 - (r(t)y(t))' \right] \\
 &= \left[-2\mu y(t) + \frac{p(t)}{r(t)} \right] w(t) \\
 &\quad - v(t) \left[\frac{p(t)}{v(t)r(t)} w(t) - p(t)y(t) + q(t) \right. \\
 &\quad \left. + \mu r(t) \frac{w^2(t)}{v^2(t)r^2(t)} - 2\mu \frac{w(t)y(t)}{v(t)} + \mu r(t)y^2(t) - (r(t)y(t))' \right] \\
 &= -2\mu y(t)w(t) + \frac{p(t)}{r(t)} w(t) - \frac{p(t)}{r(t)} w(t) - v(t) \left[-p(t)y(t) + q(t) + \mu r(t)y^2(t) - (r(t)y(t))' \right] \\
 &\quad + 2\mu y(t)w(t) - \mu \frac{w^2(t)}{v(t)r(t)} \\
 &= -v(t) \left[-p(t)y(t) + q(t) + \mu r(t)y^2(t) - (r(t)y(t))' \right] - \mu \frac{w^2(t)}{v(t)r(t)} \\
 &= -\varphi(t) - \mu \frac{w^2(t)}{v(t)r(t)},
 \end{aligned} \tag{2.11}$$

that is,

$$\varphi(t) \leq -w'(t) - \mu \frac{w^2(t)}{v(t)r(t)}, \tag{2.12}$$

where $\varphi(t)$ is defined by (2.3). Multiplying (2.12) by $H(t, s)$ and integrating from T to t , we have, for some $\beta \geq 1$ and for all $t \geq T \geq t_1$,

$$\begin{aligned}
\int_T^t H(t, s) \varphi(s) ds &\leq - \int_T^t H(t, s) w'(s) ds - \int_T^t H(t, s) \frac{\mu}{v(s)r(s)} w^2(s) ds \\
&= H(t, T) w(T) - \int_T^t \left(h(t, s) \sqrt{H(t, s) w(s)} + H(t, s) \frac{\mu w^2(s)}{v(s)r(s)} \right) ds \\
&= H(t, T) w(T) - \int_T^t \left[\sqrt{\frac{\mu H(t, s)}{\beta v(s)r(s)}} w(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu}} h(t, s) \right]^2 ds \\
&\quad + \frac{\beta}{4\mu} \int_T^t v(s)r(s) h^2(t, s) ds - \int_T^t \frac{(\beta-1)\mu}{\beta v(s)r(s)} H(t, s) w^2(s) ds.
\end{aligned} \tag{2.13}$$

Writing the latter inequality in the form

$$\begin{aligned}
&\int_T^t \left[H(t, s) \varphi(s) - \frac{\beta}{4\mu} v(s)r(s) h^2(t, s) \right] ds \\
&\leq H(t, T) w(T) - \int_T^t \left[\sqrt{\frac{\mu H(t, s)}{\beta v(s)r(s)}} w(s) + \sqrt{\frac{\beta v(s)r(s)}{4\mu}} h(t, s) \right]^2 ds \\
&\quad - \int_T^t \frac{(\beta-1)\mu}{\beta v(s)r(s)} H(t, s) w^2(s) ds.
\end{aligned} \tag{2.14}$$

Using the properties of $H(t, s)$, we have

$$\int_{t_1}^t \left[H(t, s) \varphi(s) - \frac{\beta}{4\mu} v(s)r(s) h^2(t, s) \right] ds \leq H(t, t_1) |w(t_1)| \leq H(t, t_0) |w(t_1)|, \quad t \geq t_1, \tag{2.15}$$

and for all $t \geq t_1 \geq t_0$,

$$\int_{t_0}^t \left[H(t, s) \varphi(s) - \frac{\beta}{4\mu} v(s)r(s) h^2(t, s) \right] ds \leq H(t, t_0) \left[\int_{t_0}^{t_1} |\varphi(s)| ds + |w(t_1)| \right]. \tag{2.16}$$

By (2.16),

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left(H(t, s) \varphi(s) - \frac{\beta}{4\mu} v(s)r(s) h^2(t, s) \right) ds \leq \int_{t_0}^{t_1} |\varphi(s)| ds + |w(t_1)| < \infty, \tag{2.17}$$

which contradicts (2.1). This proves Theorem 2.1. $\square$

Consider a Kamenev-type function $H(t, s)$ defined by $H(t, s) = (t - s)^{n-1}$, $(t, s) \in D$, where $n > 2$ is an integer. Obviously, H belongs to the class ω , and $h(t, s) = (n - 1)(t - s)^{(n-3)/2}$, $(t, s) \in D$. Then, we can get the following results.

Corollary 2.2. *Suppose that there exists a function $y(t) \in C^1([t_0, \infty); \mathbb{R})$ such that for some integer $n > 2$ and some $\beta \geq 1$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{n-1}} \int_{t_0}^t (t - s)^{n-3} \left[(t - s)^2 \psi(s) - \frac{\beta(n-1)^2}{4\mu} v(s)r(s) \right] ds = \infty, \quad (2.18)$$

where $v(t)$ and $\psi(t)$ are as defined in Theorem 2.1. Then every solution $u(x, t)$ of the problem (1.1), (1.5) is oscillatory in G .

Theorem 2.3. *Suppose that*

$$0 < \inf_{s \geq t_0} \left(\liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} \right) \leq \infty. \quad (2.19)$$

Assume that there exist functions $f \in C^1([t_0, \infty); \mathbb{R})$ and $\phi \in C([t_0, \infty); \mathbb{R})$ such that, for all $t \geq T \geq t_0$ and for some $\beta > 1$,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left(H(t, s) \psi(s) - \frac{\beta}{4\mu} v(s)r(s)h^2(t, s) \right) ds \geq \phi(T), \quad (2.20)$$

where $v(t), \psi(t)$ are as defined in Theorem 2.1 and suppose further that

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \frac{\phi_+^2(s)}{v(s)r(s)} ds = \infty, \quad (2.21)$$

where $\phi_+(t) = \max(\phi(t), 0)$. Then every solution $u(x, t)$ of the problem (1.1), (1.5) is oscillatory in G .

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x, t)$ of the problem (1.1), (1.5) which has no zero on $\Omega \times [t_1, \infty)$ for some $t_1 > t_0$, without loss of generality, we assume that $u(x, t) > 0, u(x, t - \rho_k(t)) > 0$ and $u(x, t - \sigma_j) > 0$ in $\Omega \times [t_1, \infty)$, $t \geq t_1 \geq t_0$, $k \in I_s$, $j \in I_m$.

As in the proof of Theorem 2.1, (2.14) holds for all $t \geq T \geq t_1$, we have

$$\begin{aligned}
& \frac{1}{H(t, T)} \int_T^t \left[H(t, s) \psi(s) - \frac{\beta}{4\mu} v(s) r(s) h^2(t, s) \right] ds \\
& \leq w(T) - \frac{1}{H(t, T)} \int_T^t \left[\sqrt{\frac{\mu H(t, s)}{\beta v(s) r(s)}} w(s) + \sqrt{\frac{\beta v(s) r(s)}{4\mu}} h(t, s) \right]^2 ds \\
& \quad - \frac{1}{H(t, T)} \int_T^t \frac{(\beta - 1)\mu}{\beta v(s) r(s)} H(t, s) w^2(s) ds \\
& \leq w(T) - \frac{1}{H(t, T)} \int_T^t \frac{(\beta - 1)\mu}{\beta v(s) r(s)} H(t, s) w^2(s) ds.
\end{aligned} \tag{2.22}$$

Therefore, for $t > T \geq T_0$,

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[H(t, s) \psi(s) - \frac{\beta}{4\mu} v(s) r(s) h^2(t, s) \right] ds \\
& \leq w(T) - \liminf_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \frac{(\beta - 1)\mu}{\beta v(s) r(s)} H(t, s) w^2(s) ds.
\end{aligned} \tag{2.23}$$

It follows from (2.20) that

$$w(T) \geq \phi(T) + \liminf_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \frac{(\beta - 1)\mu}{\beta v(s) r(s)} H(t, s) w^2(s) ds \tag{2.24}$$

for all $T \geq t_1$ and for any $\beta > 1$. Then, for all $T \geq t_1$,

$$w(T) \geq \phi(T), \tag{2.25}$$

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t \frac{H(t, s)}{v(s) r(s)} w^2(s) ds \leq \frac{\beta}{\mu(\beta - 1)} (w(t_1) - \phi(t_1)). \tag{2.26}$$

Now, we claim that

$$\int_{t_1}^{\infty} \frac{w^2(s)}{v(s) r(s)} ds < \infty. \tag{2.27}$$

Suppose the contrary, that is,

$$\int_{t_1}^{\infty} \frac{w^2(s)}{v(s) r(s)} ds = \infty. \tag{2.28}$$

By (2.19), there is a positive constant M_1 , satisfying

$$\inf_{s \geq t_0} \left(\liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} \right) > M_1 > 0. \quad (2.29)$$

Let M be any arbitrary positive number, then from (2.28) we get that there exists a $T_1 > t_1$ such that, for all $t \geq T_1$,

$$\int_{t_1}^t \frac{w^2(s)}{v(s)r(s)} ds \geq \frac{M}{M_1}. \quad (2.30)$$

Using integration by parts, for all $t \geq T_1$, we get

$$\begin{aligned} \frac{1}{H(t, t_1)} \int_{t_1}^t H(t, s) \frac{w^2(s)}{v(s)r(s)} ds &= \frac{1}{H(t, t_1)} \int_{t_1}^t \left(-\frac{\partial H(t, s)}{\partial s} \right) \left(\int_{t_1}^s \frac{w^2(\tau)}{v(\tau)r(\tau)} d\tau \right) ds \\ &\geq \frac{1}{H(t, t_1)} \int_{T_1}^t \left(-\frac{\partial H(t, s)}{\partial s} \right) \left(\int_{t_1}^s \frac{w^2(\tau)}{v(\tau)r(\tau)} d\tau \right) ds \\ &\geq \frac{M}{M_1} \frac{1}{H(t, t_1)} \int_{T_1}^t \left(-\frac{\partial H(t, s)}{\partial s} \right) ds \\ &= \frac{M}{M_1} \frac{H(t, T_1)}{H(t, t_1)}. \end{aligned} \quad (2.31)$$

By (2.29), there exists a $T_2 > T_1$ such that, for all $t \geq T_2$,

$$\frac{H(t, T_1)}{H(t, t_1)} \geq M_1. \quad (2.32)$$

It follows from (2.31) that for all $t \geq T_2$,

$$\frac{1}{H(t, t_1)} \int_{t_1}^t H(t, s) \frac{w^2(s)}{v(s)r(s)} ds \geq M. \quad (2.33)$$

Since M is an arbitrary positive constant,

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t H(t, s) \frac{w^2(s)}{v(s)r(s)} ds = \infty, \quad (2.34)$$

which contradicts (2.26). Consequently, (2.27) holds. And from (2.25), we obtain

$$\int_{t_1}^{\infty} \frac{\phi_+^2(s)}{v(s)r(s)} ds \leq \int_{t_1}^{\infty} \frac{w^2(s)}{v(s)r(s)} ds < \infty, \quad (2.35)$$

which contradicts (2.21). This completes the proof of Theorem 2.3. $\square$

Choosing H as in Corollary 2.2, by Theorem 2.3, we can obtain the following corollary.

Corollary 2.4. *Let $v(t)$ and $\psi(t)$ be as in Theorem 2.1, assume further that there exist functions $f \in C^1([t_0, \infty); \mathbb{R})$ and $\phi \in C([t_0, \infty); \mathbb{R})$ such that, for all $T \geq t_0$, for some integer $n > 2$, and for some $\beta > 1$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-3} \left[(t-s)^2 \psi(s) - \frac{\beta(n-1)^2}{4\mu} v(s)r(s) \right] ds \geq \phi(T) \quad (2.36)$$

and (2.21) hold. Then every solution $u(x, t)$ of the problem (1.1), (1.5) is oscillatory in G .

Theorem 2.5. *Suppose that there exists a function $f \in C^1([t_0, \infty); \mathbb{R})$ such that for some $\beta \geq 1$ and for some $H \in \omega$,*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s) \left(\bar{\psi}(s) - \frac{p^2(s)\bar{v}(s)}{2\mu r(s)} \right) - \frac{\mu\beta}{2} \bar{v}(s)r(s)h^2(t, s) \right] ds = \infty, \quad (2.37)$$

where

$$\bar{v}(t) = \exp \left(-2\mu \int^t y(s) ds \right), \quad (2.38)$$

$$\bar{\psi}(t) = \bar{v}(t) \left(q(t) + \mu r(t)y^2(t) - p(t)y(t) - (r(t)y(t))' \right).$$

Then every solution $u(x, t)$ of the problem (1.1), (1.5) is oscillatory in G .

Proof. As in Theorem 2.1, without loss of generality, we assume that a nonoscillatory solution $u(x, t)$ of the problem (1.1), (1.5) satisfies $u(x, t) > 0$, $u(x, t - \rho_k(t)) > 0$ and $u(x, t - \sigma_j) > 0$ in $\Omega \times [t_1, \infty)$, $t_1 \geq t_0$, $k \in I_s$, $j \in I_m$. Define a generalized Riccati transformation

$$\bar{w}(t) = \bar{v}(t)r(t) \left[\frac{U'(t)}{f(U(t))} + y(t) \right], \quad t \geq t_1, \quad (2.39)$$

where $\bar{v}(t)$ is given by (2.38). Then

$$\begin{aligned} \bar{w}'(t) &\leq -2\mu y(t)\bar{w}(t) + \bar{v}(t) \left\{ -q(t) + (r(t)y(t))' - p(t) \left[\frac{\bar{w}(t)}{\bar{v}(t)r(t)} - y(t) \right] - \mu r(t) \left[\frac{\bar{w}(t)}{\bar{v}(t)r(t)} - y(t) \right]^2 \right\} \\ &= -\bar{\psi}(t) - \frac{p(t)}{r(t)} \bar{w}(t) - \mu \frac{1}{r(t)\bar{v}(t)} \bar{w}^2(t), \quad t \geq t_1. \end{aligned} \quad (2.40)$$

Using an elementary inequality

$$-ax^2 + bx \leq -\frac{a}{2}x^2 + \frac{b^2}{2a}, \quad (2.41)$$

for all $a > 0$ and for all $b, x \in R$, we conclude from (2.40) that

$$\bar{\varphi}(t) - \frac{p^2(t)\bar{v}(t)}{2\mu r(t)} \leq -\bar{w}'(t) - \frac{\mu}{2\bar{v}(t)r(t)}\bar{w}^2(t), \quad t \geq t_1. \quad (2.42)$$

Multiplying (2.42) by $H(t, s)$ and integrating from $T < t$, we obtain, for some $\beta \geq 1$ and for all $t \geq T \geq t_1$,

$$\begin{aligned} \int_T^t H(t, s) \left(\bar{\varphi}(s) - \frac{p^2(s)\bar{v}(s)}{2\mu r(s)} \right) ds &\leq H(t, T)\bar{w}(T) - \int_T^t h(t, s) \sqrt{H(t, s)} \bar{w}(s) ds - \int_T^t \frac{\mu \bar{w}^2(s)}{2\bar{v}(s)r(s)} ds \\ &\leq H(t, T)\bar{w}(T) + \frac{\mu\beta}{2} \int_T^t \bar{v}(s)r(s)h^2(t, s) ds \\ &\quad - \mu \int_T^t \frac{(\beta-1)H(t, s)}{2\beta\bar{v}(s)r(s)} \bar{w}^2(s) ds \\ &\quad - \frac{\mu}{2} \int_T^t \left(\sqrt{\frac{H(t, s)}{\beta\bar{v}(s)r(s)}} \bar{w}(s) + \sqrt{\beta\bar{v}(s)r(s)} h(t, s) \right)^2 ds. \end{aligned} \quad (2.43)$$

Therefore, for all $t \geq T \geq t_1$, we have

$$\begin{aligned} &\int_T^t \left[H(t, s)\bar{\varphi}(s) - H(t, s) \frac{p^2(s)\bar{v}(s)}{2\mu r(s)} - \frac{\mu\beta}{2} \bar{v}(s)r(s)h^2(t, s) \right] ds \\ &\leq H(t, T)\bar{w}(T) - \mu \int_T^t \frac{(\beta-1)H(t, s)}{2\beta\bar{v}(s)r(s)} \bar{w}^2(s) ds \\ &\quad - \frac{\mu}{2} \int_T^t \left(\sqrt{\frac{H(t, s)}{\beta\bar{v}(s)r(s)}} \bar{w}(s) + \sqrt{\beta\bar{v}(s)r(s)} h(t, s) \right)^2 ds. \end{aligned} \quad (2.44)$$

Following the same lines as in the proof of Theorem 2.1, we have

$$\begin{aligned} &\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left(H(t, s)\bar{\varphi}(s) - H(t, s) \frac{p^2(s)\bar{v}(s)}{2\mu r(s)} - \frac{\mu\beta}{2} \bar{v}(s)r(s)h^2(t, s) \right) ds \\ &\leq \int_{t_0}^{t_1} |\bar{\varphi}(s)| ds + |\bar{w}(t_1)| < \infty \end{aligned} \quad (2.45)$$

which contradicts the assumption (2.19).

This completes the proof. $\square$

Theorem 2.6. Let (2.19) holds. Assume that there exist functions $f \in C^1([t_0, \infty); \mathbb{R})$ and $\phi \in C([t_0, \infty), \mathbb{R})$ such that, for all $t \geq t_0$, any $T \geq t_0$, and for some $\beta > 1$,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[H(t, s) \left(\bar{\psi}(s) - \frac{p^2(s) \bar{v}(s)}{2\mu r(s)} \right) - \frac{\mu\beta}{2} \bar{v}(s) r(s) h^2(t, s) \right] ds \geq \phi(T) \quad (2.46)$$

and (2.21) holds, where $\psi(t), \bar{v}(t)$ are defined as in Theorem 2.6 and $\phi_+(t) = \max(\phi(t), 0)$, then every solution $u(x, t)$ of the problem (1.1), (1.5) is oscillatory in G .

Theorem 2.7. Let all assumptions of Theorem 2.6 be satisfied except that condition (2.46) be replaced by

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, T)} \int_T^t \left[H(t, s) \left(\bar{\psi}(s) - \frac{p^2(s) \bar{v}(s)}{2\mu r(s)} \right) - \frac{\mu\beta}{2} \bar{v}(s) r(s) h^2(t, s) \right] ds \geq \phi(T). \quad (2.47)$$

Then every solution $u(x, t)$ of the problem (1.1), (1.5) is oscillatory in G .

Remark 2.8. By introducing the parameter β in Theorem 2.3, we derive new oscillation criteria of the problem (1.1), (1.5) which are simpler than that in Theorem A; furthermore, modifications of the proofs through the refinement of the standard integral averaging method allowed us to shorten significantly the proofs of Theorem 2.3. We can also derive a number of oscillation criteria with the appropriate choice of the function H and ρ , here, we omit the details.

3. Examples

Now, we consider these following examples.

Example 3.1. Consider the partial differential equation

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\frac{1}{t} \frac{\partial}{\partial t} \left(u(x, t) + \frac{1}{2} u(x, t - \pi) \right) \right] + 2 \cos t \frac{\partial u(x, t)}{\partial t} \\ &= \Delta u(x, t) + \frac{1}{t^2} \Delta u \left(x, t - \frac{3}{2} \pi \right) \\ & - \left(\frac{2}{t^3} + t \cos^2 t + \sin t \right) f(u(x, t)) - \frac{1}{t^2} f_1(u(x, t - \pi)), \quad (x, t) \in (0, \pi) \times (0, \infty), \end{aligned} \quad (3.1)$$

with the boundary condition

$$u_x(0, t) = u_x(\pi, t) = 0, \quad t > 0, \quad (3.2)$$

where $f(u) = u^3 + u$, $f_1(u) = ue^u + u$.

Here, $N = 1$, $l = 1$, $s = 1$, $m = 1$, $\mu = 1$, $r(t) = 1/t$, $p(t) = 2 \cos t$, $q(x, t) = q(t) = (2/t^3 + t \cos^2 t + \sin t)$, $q_1(x, t) = 1/t^2$, $f(u) = f_1(u) = u$, $a(t) = 1$, $a_1(t) = 1/t^2$, $\rho_1(t) = (3/2)\pi$, $\sigma_1 = \pi$, $\tau_1 = \pi$.

Let

$$y(t) = \frac{1}{t} + t \cos t, \quad (3.3)$$

then

$$v(t) = t^2, \quad \psi(t) = t^{-1}. \quad (3.4)$$

Let $n = 3$, for any $\beta \geq 1$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t^2} \int_1^t \left[(t-s)^2 s^{-1} - \beta s^2 \frac{1}{s} \right] ds = \limsup_{t \rightarrow \infty} \frac{1}{t^2} \int_1^t \left[(t-s)^2 s^{-1} - \beta s \right] ds = \infty. \quad (3.5)$$

Therefore, Corollary 2.2 holds, then every solution $u(x, t)$ of the problem (3.1), (3.2) oscillates in $(0, \pi) \times (0, \infty)$.

Example 3.2. Consider the partial differential equation

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\left(1 + \frac{1}{2t^3} \right) (2 + \sin t) \frac{\partial u(x, t)}{\partial t} \right] + \frac{3}{t} \left(1 + \frac{1}{2t^3} \right) (2 + \sin t) \frac{\partial u(x, t)}{\partial t} \\ &= 3\Delta u(x, t) + (2 - \cos t) \Delta u \left(x, t - \frac{3}{2}\pi \right) - t^{-3} \left((1 - t^3 + 2t^2 - 6t) \sin t + 12t \right) f(u(x, t)) \\ & \quad - (2t + \sin t) f_1(u(x, t - \pi)) - 2f_2 \left(u \left(x, t - \frac{\pi}{2} \right) \right), \quad (x, t) \in (0, \pi) \times (0, \infty) \end{aligned} \quad (3.6)$$

with the boundary condition (3.2), where $f(u) = u^5 + u$, $f_1(u) = u \sin^2 u$, $f_2(u) = u^3 \cos^2 u$.

Here $N = 1$, $s = 1$, $m = 2$, $\mu = 1$, $r(t) = (1 + 1/2t^3)(2 + \sin t)$, $p(t) = (3/t)(1 + 1/2t^3)(2 + \sin t)$, $q(t) = t^{-3}[(1 - t^3 + 2t^2 - 6t) \sin t + 12t]$, $a(t) = 3$, $a_1(t) = 2 - \cos t$, $q_1(x, t) = 2 + \sin t$, $q_2(x, t) = 2$, $\rho_1(t) = (3/2)\pi$, $\sigma_1 = \pi$, $\sigma_2 = \pi/2$.

Let $y(t) = 0$, then $v(t) = t^3$ and $\psi(t) = v(t)q(t) = (1 - t^3 + 2t^2 - 6t) \sin t + 12t$.

Choose $\beta = 2$, $n = 3$, a straightforward computation yields

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^2} \int_T^t \left[(t-s)^2 \left((1 - s^3 + 2s^2 - 6s) \sin s + 12s \right) - (2s^3 + 1)(2 + \sin s) \right] ds \\ &= 16 - T^3 \cos T + T^2(2 \cos T - 6 + 3 \sin T) - 4T \sin T - 3 \cos T = \phi(T). \end{aligned} \quad (3.7)$$

Let $\phi_+(t) = \max(\phi(t), 0)$. It is not difficult to see that

$$\limsup_{t \rightarrow \infty} \int_1^t \frac{\phi_+^2(s)}{(s^3 + (1/2))(2 + \sin s)} ds \geq \limsup_{t \rightarrow \infty} \int_1^t \frac{\phi_+^2(s)}{3(s^3 + (1/2))} ds = \infty. \quad (3.8)$$

By Corollary 2.4, we obtain that every solution of problem (3.6), (3.2) oscillates in $(0, \pi) \times (0, \infty)$.

Note that in this example,

$$\limsup_{t \rightarrow \infty} \frac{1}{t^2} \int_1^t 4 \left(s^3 + \frac{1}{2} + (2 + \sin s) \right) ds = \infty, \quad (3.9)$$

so the condition (C_2) would not have been satisfied with the same choices of $v(t)$.

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References

- [1] D. P. Mishev and D. D. Bařnov, "Oscillation of the solutions of parabolic differential equations of neutral type," *Applied Mathematics and Computation*, vol. 28, no. 2, pp. 97–111, 1988.
- [2] X. L. Fu and W. Zhuang, "Oscillation of certain neutral delay parabolic equations," *Journal of Mathematical Analysis and Applications*, vol. 191, no. 3, pp. 473–489, 1995.
- [3] B. T. Cui, "Oscillation properties for parabolic equations of neutral type," *Journal of Computational and Applied Mathematics*, vol. 33, no. 4, pp. 581–588, 1992.
- [4] B. T. Cui, Y. H. Yu, and S. Z. Lin, "Oscillation of solutions to hyperbolic differential equations with delays," *Acta Mathematicae Applicatae Sinica*, vol. 19, no. 1, pp. 80–88, 1996 (Chinese).
- [5] B. S. Lalli, Y. H. Yu, and B. T. Cui, "Oscillation of hyperbolic equations with functional arguments," *Applied Mathematics and Computation*, vol. 53, no. 2-3, pp. 97–110, 1993.
- [6] W. N. Li, "Oscillation for solutions of partial differential equations with delays," *Demonstratio Mathematica*, vol. 33, no. 2, pp. 319–332, 2000.
- [7] R. P. Agarwal, F. W. Meng, and W. N. Li, "Oscillation of solutions of systems of neutral type partial functional differential equations," *Computers and Mathematics with Applications*, vol. 44, no. 5-6, pp. 777–786, 2002.
- [8] W. N. Li and F. W. Meng, "Forced oscillation for certain systems of hyperbolic differential equations," *Applied Mathematics and Computation*, vol. 141, no. 2-3, pp. 313–320, 2003.
- [9] W. N. Li and F. W. Meng, "Oscillation for systems of neutral partial differential equations with continuous distributed deviating arguments," *Demonstratio Mathematica*, vol. 34, no. 3, pp. 619–633, 2001.
- [10] W. N. Li and B. T. Cui, "Oscillation of solutions of neutral partial functional-differential equations," *Journal of Mathematical Analysis and Applications*, vol. 234, no. 1, pp. 123–146, 1999.
- [11] W. N. Li, "Oscillation properties for systems and hyperbolic differential equations of neutral type," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 2, pp. 369–384, 2000.
- [12] W. N. Li and B. T. Cui, "Oscillation of solutions of neutral partial functional-differential equations," *Journal of Mathematical Analysis and Applications*, vol. 234, no. 1, pp. 123–146, 1999.
- [13] Y. V. Rogovchenko and F. Tuncay, "Oscillation criteria for second-order nonlinear differential equations with damping," *Nonlinear Analysis*, vol. 69, no. 1, pp. 208–221, 2008.