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Research Article

On the Convolution Equation Related to the Diamond Klein-Gordon Operator

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We study the distribution $e^{\alpha x}(\diamond + m^2)^k\delta$ for $m \geq 0$, where $(\diamond + m^2)^k$ is the diamond Klein-Gordon operator iterated k times, δ is the Dirac delta distribution, $x = (x_1, x_2, \ldots, x_n)$ is a variable in $\mathbb{R}^n$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a constant. In particular, we study the application of $e^{\alpha x}(\diamond + m^2)^k\delta$ for solving the solution of some convolution equation. We find that the types of solution of such convolution equation, such as the ordinary function and the singular distribution, depend on the relationship between k and M.

1. Introduction

The n-dimensional ultrahyperbolic operator $\Box^k$ iterated k times is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \tag{1.1}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$, and k is a nonnegative integer. We consider the linear differential equation of the form

$$\Box^k u(x) = f(x), \tag{1.2}$$

where $u(x)$ and $f(x)$ are generalized functions, and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Gelfand and Shilov [1] have first introduced the fundamental solution of (1.2), which was initially complicated. Later, Trione [2] has shown that the generalized function $R_{2k}^{H}(x)$ defined by (2.2) with $\gamma = 2k$ is the unique fundamental solution of (1.2). Tellez [3] has also proved that $R_{2k}^{H}(x)$ exists only when $n = p + q$ with odd p.

Kananthai [4] has first introduced the operator $\Diamond^k$ called the diamond operator iterated k times, which is defined by

$$\Diamond^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \tag{1.3}$$

where $n = p + q$ is the dimension of $\mathbb{R}^n$, for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and nonnegative integers k. The operator $\Diamond^k$ can be expressed in the form

$$\Diamond^k = \triangle^k \Box^k = \Box^k \triangle^k, \tag{1.4}$$

where $\Box^k$ is defined by (1.1), and $\triangle^k$ is the Laplace operator iterated k times defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k. \tag{1.5}$$

Note that in case $k = 1$, the generalized form of (1.5) is called the local fractional Laplace operator; see [5] for more details. On finding the fundamental solution of this product, he uses the convolution of functions which are fundamental solutions of the operators $\Box^k$ and $\triangle^k$. He found that the convolution $(-1)^k R_{2k}^{e}(x) * R_{2k}^{H}(x)$ is the fundamental solution of the operator $\Diamond^k$, that is,

$$\Diamond^k\left((-1)^k R_{2k}^{e}(x) * R_{2k}^{H}(x)\right) = \delta, \tag{1.6}$$

where $R_{2k}^{H}(x)$ and $R_{2k}^{e}(x)$ are defined by (2.2) and (2.7), respectively (with $\gamma = 2k$), and δ is the Dirac-delta distribution. The fundamental solution $(-1)^k R_{2k}^{e}(x) * R_{2k}^{H}(x)$ is called the diamond kernel of Marcel Riesz. A number of effective results on the diamond kernel of Marcel Riesz have been presented by Kananthai [6–12].

In 1997, Kananthai [13] has studied the properties of the distribution $e^{\alpha x}\Box^k\delta$ and the application of the distribution $e^{\alpha x}\Box^k\delta$ for finding the fundamental solution of the ultrahyperbolic equation by using the convolution method. Later in 1998, he has also studied the properties of the distribution $e^{\alpha x}\Diamond^k\delta$ and its application for solving the convolution equation

$$e^{\alpha x}\Diamond^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r \Diamond^r \delta. \tag{1.7}$$

Recently, Nonlaopon gave some generalizations of his paper [6]; see [14] for more details.

In 2000, Kananthai [15] has studied the application of the distribution $e^{\alpha x}\Box^k\delta$ for solving the convolution equation

$$e^{\alpha x}\Box^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r \Box^r \delta, \tag{1.8}$$

which is related to the ultrahyperbolic equation.

In 2009, Sasopa and Nonlaopon [16] have studied the properties of the distribution $e^{\alpha x}\Box_c^k\delta$ and its application to solve the convolution equation

$$e^{\alpha x}\Box_c^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r \Box_c^r \delta. \tag{1.9}$$

Here, $\Box_c^k$ is the operator related to the ultrahyperbolic type operator iterated k times, which is defined by

$$\Box_c^k = \left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k, \tag{1.10}$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$.

In 1988, Trione [17] has studied the fundamental solution of the ultrahyperbolic Klein-Gordon operator iterated k times, which is defined by

$$\left(\Box + m^2\right)^k = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2} + m^2\right)^k. \tag{1.11}$$

The fundamental solution of the operator $(\Box + m^2)^k$ is given by

$$W_{2k}(x,m) = \sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(k+r)}{r!\Gamma(k)}\left(m^2\right)^r(-1)^r R_{2k+2r}^H(x), \tag{1.12}$$

where $R_{2k+2r}^H(x)$ is defined by (2.2) with $\gamma = 2k+2r$. Next, Tellez [18] has studied the convolution product of $W_\alpha(x,m) * W_\beta(x,m)$, where α and β are any complex parameter. In addition, Trione [19] has studied the fundamental $(P \pm i0)^\lambda$-ultrahyperbolic solution of the Klein-Gordon operator iterated k times and the convolution of such fundamental solution.

Liangprom and Nonlaopon [20] have studied the properties of the distribution $e^{\alpha x}(\Box + m^2)^k\delta$ and its application for solving the convolution equation

$$e^{\alpha x}\left(\Box + m^2\right)^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{M} C_r\left(\Box + m^2\right)^r\delta, \tag{1.13}$$

where $(\Box + m^2)^k$ is defined by (1.11).

In 2007, Tariboon and Kananthai [21] have introduced the operator $(\Diamond + m^2)^k$ called diamond Klein-Gordon operator iterated k times, which is defined by

$$\left(\Diamond + m^2\right)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k, \tag{1.14}$$

where $p+q = n$ is the dimension of $\mathbb{R}^n$, for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $m \geq 0$ and nonnegative integers k. Later, Lunnaree and Nonlaopon [22, 23] have studied the fundamental solution of operator $(\Diamond + m^2)^k$, and this fundamental solution is called the diamond Klein-Gordon kernel. They have also studied the Fourier transform of the diamond Klein-Gordon kernel and its convolution.

In this paper, we aim to study the properties of the distribution $e^{\alpha x}(\Diamond + m^2)^k \delta$ and the application of $e^{\alpha x}(\Diamond + m^2)^k \delta$ for solving the convolution equation

$$e^{\alpha x}\left(\Diamond + m^2\right)^k \delta * u(x) = e^{\alpha x} \sum_{r=0}^{M} C_r \left(\Diamond + m^2\right)^r \delta, \tag{1.15}$$

where $(\Diamond + m^2)^k$ is defined by (1.14), $u(x)$ is the generalized function, and C_r is a constant. On finding the type of solution $u(x)$ of (1.15), we use the method of convolution of the generalized functions.

Before we proceed to that point, the following definitions and concepts require clarifications.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Let

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{2.1}$$

be the nondegenerated quadratic form, where $p + q = n$ is the dimension of $\mathbb{R}^n$. Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ be the interior of a forward cone, and let $\overline{\Gamma}_+$ denote its closure. For any complex number γ, we define the function

$$R_\gamma^H(x) = \begin{cases} \dfrac{u^{(\gamma-n)/2}}{K_n(\gamma)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\gamma)$ is given by

$$K_n(\gamma) = \frac{\pi^{(n-1)/2}\Gamma((2+\gamma-n)/2)\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((2+\gamma-p)/2)\Gamma((p-\gamma)/2)}. \tag{2.3}$$

The function $R_\gamma^H(x)$ is called the ultrahyperbolic kernel of Marcel Riesz, which was introduced by Nozaki [24]. It is well known that $R_\gamma^H(x)$ is an ordinary function if $\mathrm{Re}(\gamma) \geq n$ and is a distribution of γ if $\mathrm{Re}(\gamma) < n$. Let supp $R_\gamma^H(x)$ denote the support of $R_\gamma^H(x)$ and suppose that supp $R_\gamma^H(x) \subset \overline{\Gamma}_+$, that is, supp $R_\gamma^H(x)$ is compact.

By putting $p = 1$ in $R_{2k}^H(x)$ and taking into account Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.4}$$

we obtain

$$I_\gamma^H(x) = \frac{v^{(\gamma-n)/2}}{H_n(\gamma)}, \tag{2.5}$$

$v = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, where

$$H_n(\gamma) = \pi^{(n-2)/2}2^{\gamma-1}\Gamma\left(\frac{\gamma+2-n}{2}\right)\Gamma\left(\frac{\gamma}{2}\right). \tag{2.6}$$

The function $I_\gamma^H(x)$ is called the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and $\omega = x_1^2 + x_2^2 + \cdots + x_n^2$. The elliptic kernel of Marcel Riesz is defined by

$$R_\gamma^e(x) = \frac{\omega^{(\gamma-n)/2}}{W_n(\gamma)}, \tag{2.7}$$

where n is the dimension of $\mathbb{R}^n$, $\gamma \in \mathbb{C}$, and

$$W_n(\gamma) = \frac{\pi^{n/2}2^\gamma\Gamma(\gamma/2)}{\Gamma((n-\gamma)/2)}. \tag{2.8}$$

Note that $n = p + q$. By putting $q = 0$ (i.e., $n = p$) in (2.2) and (2.3), we can reduce $u^{(\gamma-n)/2}$ to $\omega_p^{(\gamma-p)/2}$, where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$, and reduce $K_n(\gamma)$ to

$$K_p(\gamma) = \frac{\pi^{(p-1)/2}\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((p-\gamma)/2)}. \tag{2.9}$$

Using the Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.10}$$

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\pi\sec(\pi z), \tag{2.11}$$

we obtain

$$K_p(\gamma)=\frac{1}{2}\sec\left(\frac{\gamma\pi}{2}\right)W_p(\gamma). \tag{2.12}$$

Thus, in case $q=0$, we have

$$R_\gamma^H(x)=\frac{u^{(\gamma-p)/2}}{K_p(\gamma)}=2\cos\left(\frac{\gamma\pi}{2}\right)\frac{u^{(\gamma-p)/2}}{W_p(\gamma)}=2\cos\left(\frac{\gamma\pi}{2}\right)R_\gamma^e(x). \tag{2.13}$$

In addition, if $\gamma=2k$ for some nonnegative integer k, then

$$R_{2k}^H(x)=2(-1)^kR_{2k}^e(x). \tag{2.14}$$

Lemma 2.3. *The convolution $(-1)^kR_{2k}^e(x)*R_{2k}^H(x)$ is the fundamental solution of the diamond operator iterated k times, that is,*

$$\Diamond^k\left((-1)^kR_{2k}^e(x)*R_{2k}^H(x)\right)=\delta. \tag{2.15}$$

For the proof of this Lemma, see [4, 12].

It can be shown that $R_{-2k}^e(x)*R_{-2k}^H(x)=(-1)^k\Diamond^k\delta$, for all nonnegative integers k.

Definition 2.4. Let $x=(x_1,x_2,\ldots,x_n)$ be a point of $\mathbb{R}^n$. The function $T_\gamma(x,m)$ is defined by

$$T_\gamma(x,m)=\sum_{r=0}^{\infty}\binom{-\frac{\gamma}{2}}{r}\left(m^2\right)^r(-1)^{\gamma/2+r}R_{\gamma+2r}^e(x)*R_{\gamma+2r}^H(x), \tag{2.16}$$

where γ is a complex parameter, and m is a nonnegative real number. Here, $R_{\gamma+2r}^H(x)$ and $R_{\gamma+2r}^e(x)$ are defined by (2.2) and (2.7), respectively.

From the definition of $T_\gamma(x,m)$, by putting $\gamma=-2k$, we have

$$T_{-2k}(x,m)=\sum_{r=0}^{\infty}\binom{k}{r}\left(m^2\right)^r(-1)^{-k+r}R_{2(-k+r)}^e(x)*R_{2(-k+r)}^H(x). \tag{2.17}$$

Since the operator $(\Diamond + m^2)^k$ defined by (1.14) is linearly continuous and has 1-1 mapping, this possesses its own inverses. From Lemma 2.3, we obtain

$$T_{-2k}(x,m) = \sum_{r=0}^{\infty} \binom{k}{r} \left(m^2\right)^r \Diamond^{k-r} \delta = \left(\Diamond + m^2\right)^k \delta. \tag{2.18}$$

Substituting $k = 0$ in (2.18) yields that we have $T_0(x,m) = \delta$. On the other hand, putting $\gamma = 2k$ in (2.16) yields

$$\begin{aligned} T_{2k}(x,m) = {} & \binom{-k}{0} \left(m^2\right)^0 (-1)^{k+0} R^e_{2k+0}(x) * R^H_{2k+0}(x) \\ & + \sum_{r=1}^{\infty} \binom{-k}{r} \left(m^2\right)^r (-1)^{k+r} R^e_{2k+2r}(x) * R^H_{2k+2r}(x). \end{aligned} \tag{2.19}$$

The second summand of the right-hand side of (2.19) vanishes when $m = 0$. Hence, we obtain

$$T_{2k}(x, m = 0) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x), \tag{2.20}$$

which is the fundamental solution of the diamond operator.

For the proofs of Lemmas 2.5 and 2.6, see [23].

Lemma 2.5. *Given the equation*

$$\left(\Diamond + m^2\right)^k u(x) = \delta, \tag{2.21}$$

where $(\Diamond + m^2)^k$ is the diamond Klein-Gordon operator iterated k times, defined by

$$\left(\Diamond + m^2\right)^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2 + m^2\right]^k \tag{2.22}$$

with a nonnegative integer k and the Dirac-delta distribution δ, then $u(x) = T_{2k}(x,m)$ is the fundamental solution of the diamond Klein-Gordon operator iterated k times $(\Diamond + m^2)^k$, where $T_{2k}(x,m)$ is defined by (2.16) *with $\gamma = 2k$.*

Lemma 2.6. *Let $T_{2k}(x,m)$ be the diamond Klein-Gordon kernel defined by* (2.16)*, then $T_{2k}(x,m)$ is a tempered distribution and can be expressed by*

$$T_{2k}(x,m) = T_{2k-2v}(x,m) * T_{2v}(x,m), \tag{2.23}$$

where v is a nonnegative integer and $v < k$. Moreover, if one puts $l = k - v$ and $h = v$, then one obtains

$$T_{2l}(x,m) * T_{2h}(x,m) = T_{2l+2h}(x,m) \tag{2.24}$$

for $l + h = k$.

3. Properties of the Distribution $e^{\alpha x}(\Diamond + m^2)^k \delta$

Lemma 3.1. *The following equality holds:*

$$e^{\alpha x}\left(\Diamond + m^2\right)^k \delta = L^k \delta, \tag{3.1}$$

and $e^{\alpha x}(\Diamond + m^2)^k \delta$ is the tempered distribution of order $4k$ with support $\{0\}$, where L is the partial differential operator and is defined by

$$\begin{aligned} L \equiv{}& \left(\Diamond + m^2\right) + \sum_{r=1}^{n} \alpha_r^2 \Box - 2\sum_{r=1}^{n}\sum_{i=1}^{p}\left(\alpha_r \frac{\partial^3}{\partial x_i^2 \partial x_r} + \alpha_i \frac{\partial^3}{\partial x_i \partial x_r^2}\right) \\ &+ 2\sum_{r=1}^{n}\sum_{j=p+1}^{p+q}\left(\alpha_r \frac{\partial^3}{\partial x_j^2 \partial x_r} + \alpha_j \frac{\partial^3}{\partial x_j \partial x_r^2}\right) + 4\sum_{r=1}^{n}\alpha_r\left(\sum_{i=1}^{p}\alpha_i \frac{\partial^2}{\partial x_i \partial x_r} - \sum_{j=p+1}^{p+q}\alpha_j \frac{\partial^2}{\partial x_j \partial x_r}\right) \\ &- 2\sum_{r=1}^{n}\alpha_r^2\left(\sum_{i=1}^{p}\alpha_i \frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q}\alpha_j \frac{\partial}{\partial x_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\triangle \\ &- 2\left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\sum_{r=1}^{n}\alpha_r \frac{\partial}{\partial x_r} + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\sum_{r=1}^{n}\alpha_r^2. \end{aligned} \tag{3.2}$$

As before, $\Box$ is the ultrahyperbolic operator defined by (1.1) *(with $k = 1$), and $\triangle$ is the Laplace operator defined by*

$$\triangle = \frac{\partial^2}{\partial x_1} + \frac{\partial^2}{\partial x_2} + \cdots + \frac{\partial^2}{\partial x_n}. \tag{3.3}$$

Proof. Let $\varphi \in \mathfrak{D}$ be the space of testing functions which are infinitely differentiable with compact supports, and let $\mathfrak{D}'$ be the space of distributions. Now,

$$\left\langle e^{\alpha x}\left(\Diamond + m^2\right)\delta, \varphi(x)\right\rangle = \left\langle \delta, \left(\Diamond + m^2\right)e^{\alpha x}\varphi(x)\right\rangle, \tag{3.4}$$

for $e^{\alpha x}(\Diamond + m^2)\delta \in \mathfrak{D}'$. A direct computation shows that

$$\left(\Diamond + m^2\right)e^{\alpha x}\varphi(x) = e^{\alpha x}T\varphi(x), \tag{3.5}$$

where T is the partial differential operator defined by

$$\begin{aligned}
T \equiv{}& \left(\Diamond + m^2\right) + \sum_{r=1}^{n} \alpha_r^2 \Box + 2\sum_{r=1}^{n}\sum_{i=1}^{p}\left(\alpha_r \frac{\partial^3}{\partial x_i^2 \partial x_r} + \alpha_i \frac{\partial^3}{\partial x_i \partial x_r^2}\right) \\
&- 2\sum_{r=1}^{n}\sum_{j=p+1}^{p+q}\left(\alpha_r \frac{\partial^3}{\partial x_j^2 \partial x_r} + \alpha_j \frac{\partial^3}{\partial x_j \partial x_r^2}\right) + 4\sum_{r=1}^{n}\alpha_r\left(\sum_{i=1}^{p}\alpha_i \frac{\partial^2}{\partial x_i \partial x_r} - \sum_{j=p+1}^{p+q}\alpha_j \frac{\partial^2}{\partial x_j \partial x_r}\right) \\
&+ 2\sum_{r=1}^{n}\alpha_r^2\left(\sum_{i=1}^{p}\alpha_i \frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q}\alpha_j \frac{\partial}{\partial x_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\triangle \\
&+ 2\left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\sum_{r=1}^{n}\alpha_r \frac{\partial}{\partial x_r} + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right)\sum_{r=1}^{n}\alpha_r^2.
\end{aligned} \tag{3.6}$$

Thus,

$$\left\langle \delta, \left(\Diamond + m^2\right) e^{\alpha x}\varphi(x)\right\rangle = \left\langle \delta, e^{\alpha x} T\varphi(x)\right\rangle = T\varphi(0). \tag{3.7}$$

Since $\langle e^{\alpha x}(\Diamond + m^2)^k \delta, \varphi(x)\rangle = \langle (\Diamond + m^2)^k \delta, e^{\alpha x}\varphi(x)\rangle$ for every $\varphi(x) \in \mathfrak{D}$ and $e^{\alpha x}(\Diamond + m^2)^k \delta \in \mathfrak{D}'$, we have

$$\begin{aligned}
\left\langle \left(\Diamond + m^2\right)^k \delta, e^{\alpha x}\varphi(x)\right\rangle &= \left\langle \left(\Diamond + m^2\right)^{k-1} \delta, \left(\Diamond + m^2\right) e^{\alpha x}\varphi(x)\right\rangle \\
&= \left\langle \left(\Diamond + m^2\right)^{k-1} \delta, e^{\alpha x} T\varphi(x)\right\rangle \\
&= \left\langle \left(\Diamond + m^2\right)^{k-2} \delta, \left(\Diamond + m^2\right) e^{\alpha x} T\varphi(x)\right\rangle \\
&= \left\langle \left(\Diamond + m^2\right)^{k-2} \delta, e^{\alpha x} T\left(T\varphi(x)\right)\right\rangle \\
&= \left\langle \left(\Diamond + m^2\right)^{k-2} \delta, e^{\alpha x} T^2\varphi(x)\right\rangle.
\end{aligned} \tag{3.8}$$

Repeating this process $(\Diamond + m^2)$ with $k-2$ times, we finally obtain

$$\left\langle \left(\Diamond + m^2\right)^{k-2} \delta, e^{\alpha x} T^2\varphi(x)\right\rangle = \left\langle \delta, e^{\alpha x} T^k\varphi(x)\right\rangle = T^k\varphi(0), \tag{3.9}$$

where T^k is the operator of (3.6) iterated k times. Now,

$$T^k\varphi(0) = \left\langle \delta, T^k\varphi(x)\right\rangle = \left\langle L\delta, T^{k-1}\varphi(x)\right\rangle, \tag{3.10}$$

by the operator L in (3.2) and the derivative of distribution. Continuing this process, we obtain $T^k\varphi(0) = \langle L^k\delta, \varphi(x)\rangle$ or $\langle e^{\alpha x}(\Diamond + m^2)^k\delta, \varphi(x)\rangle = \langle L^k\delta, \varphi(x)\rangle$. By equality of distributions, we obtain (3.1) as required. Since δ and its partial derivatives have support $\{0\}$ which is compact, hence, by Schwartz [25], $L^k\delta$ are tempered distributions and $L^k\delta$ has order $4k$. It follows that $e^{\alpha x}(\Diamond + m^2)^k\delta$ is a tempered distribution of order $4k$ with point support $\{0\}$ by (3.1). This completes the proof. □

Lemma 3.2 (boundedness property). *Let $\mathfrak{D}$ be the space of testing functions and $\mathfrak{D}'$ the space of distributions. For every $\varphi \in \mathfrak{D}$ and $e^{\alpha x}(\Diamond + m^2)^k\delta \in \mathfrak{D}'$,*

$$\left|\left\langle e^{\alpha x}\left(\Diamond + m^2\right)^k\delta, \varphi(x)\right\rangle\right| \leq M, \tag{3.11}$$

for some constant M.

Proof. Note that we have $\langle e^{\alpha x}(\Diamond + m^2)^k\delta, \varphi(x)\rangle = \langle (\Diamond + m^2)^k\delta, e^{\alpha x}\varphi(x)\rangle$ for every $\varphi(x) \in \mathfrak{D}$ and $e^{\alpha x}(\Diamond + m^2)^k\delta \in \mathfrak{D}'$. Hence,

$$\left\langle \left(\Diamond + m^2\right)^k\delta, e^{\alpha x}\varphi(x)\right\rangle = \left\langle \left(\Diamond + m^2\right)^{k-1}\delta, \left(\Diamond + m^2\right)e^{\alpha x}\varphi(x)\right\rangle = \left\langle \left(\Diamond + m^2\right)^{k-1}\delta, e^{\alpha x}T\varphi(x)\right\rangle, \tag{3.12}$$

where T is defined by (3.6). Continuing this process for $k-1$ times, we will obtain

$$\left\langle e^{\alpha x}\left(\Diamond + m^2\right)^k\delta, \varphi(x)\right\rangle = \left\langle \delta, e^{\alpha x}T^k\varphi(x)\right\rangle = T^k\varphi(0). \tag{3.13}$$

Since $\varphi \in \mathfrak{D}$, so $\varphi(0)$ is bounded, and also $T^k\varphi(0)$ is bounded. It then follows that

$$\left|\left\langle e^{\alpha x}\left(\Diamond + m^2\right)^k\delta, \varphi(x)\right\rangle\right| = T^k\varphi(0) \leq M, \tag{3.14}$$

for some constant M. □

4. The Application of Distribution $e^{\alpha x}(\Diamond + m^2)^k\delta$

Theorem 4.1. *Let L be the partial differential operator defined by* (3.2)*, and consider the equation*

$$Lu(x) = \delta, \tag{4.1}$$

where $u(x)$ is any distribution in $\mathfrak{D}'$, then $u(x) = e^{\alpha x}T_2(x, m)$ is the fundamental solution of the operator L, where $T_2(x, m)$ is defined by (2.16) *with $\gamma = 2$.*

Proof. From (3.1) and (4.1), we can write $e^{\alpha x}(\Diamond + m^2)\delta * u(x) = Lu(x) = \delta$. Convolving both sides by $e^{\alpha x}T_2(x,m)$, we have

$$e^{\alpha x}T_2(x,m) * e^{\alpha x}\left(\Diamond + m^2\right)\delta * u(x) = e^{\alpha x}T_2(x,m) * \delta, \tag{4.2}$$

then

$$e^{\alpha x}\left(T_2(x,m) * \left(\Diamond + m^2\right)\delta\right) * u(x) = e^{\alpha x}T_2(x,m), \tag{4.3}$$

or equivalently,

$$e^{\alpha x}\left(\left(\Diamond + m^2\right)T_2(x,m)\right) * u(x) = e^{\alpha x}T_2(x,m). \tag{4.4}$$

Since $(\Diamond + m^2)T_2(x,m) = \delta$ by Lemma 2.5 with $k = 1$, we obtain

$$(e^{\alpha x}\delta) * u(x) = e^{\alpha x}T_2(x,m). \tag{4.5}$$

Moreover, since $e^{\alpha x}\delta = \delta$, we have $\delta * u(x) = e^{\alpha x}T_2(x,m)$. It then follows that $u(x) = e^{\alpha x}T_2(x,m)$ is the fundamental solution of the operator L. □

Theorem 4.2 (the generalization of Theorem 4.1). *From Lemma 3.1, consider that*

$$e^{\alpha x}\left(\Diamond + m^2\right)^k\delta * u(x) = \delta, \tag{4.6}$$

or

$$L^k u(x) = \delta, \tag{4.7}$$

then $u(x) = e^{\alpha x}T_{2k}(x,m)$ is the fundamental solution of the operator L^k.

Proof. We can prove it by using either (4.6) or (4.7). If we start with (4.6), by convolving both sides by $e^{\alpha x}T_{2k}(x,m)$, we obtain

$$e^{\alpha x}T_{2k}(x,m) * \left(e^{\alpha x}\left(\Diamond + m^2\right)^k\delta * u(x)\right) = e^{\alpha x}T_{2k}(x,m) * \delta, \tag{4.8}$$

or $e^{\alpha x}((\Diamond + m^2)^k T_{2k}(x,m)) * u(x) = e^{\alpha x}T_{2k}(x,m)$. Since $(\Diamond + m^2)^k T_{2k}(x,m) = \delta$ by Lemma 2.5, we have $(e^{\alpha x}\delta) * u(x) = e^{\alpha x}T_{2k}(x,m)$ or $u(x) = e^{\alpha x}T_{2k}(x,m)$ as required.

If we use (4.7), by convolving both sides by $e^{\alpha x}T_2(x,m)$, we obtain

$$e^{\alpha x}T_2(x,m) * L^k u(x) = e^{\alpha x}T_2(x,m) * \delta, \tag{4.9}$$

or $L(e^{\alpha x}T_2(x,m)) * L^{k-1}u(x) = e^{\alpha x}T_2(x,m)$. By Theorem 4.1, we obtain $L^{k-1}u(x) = e^{\alpha x}T_2(x,m)$. Keeping on convolving $e^{\alpha x}T_2(x,m)$ for $k-1$ times, we finally obtain

$$u(x) = e^{\alpha x}(T_2(x,m) * T_2(x,m) * \cdots * T_2(x,m)) = e^{\alpha x}T_{2k}(x,m), \tag{4.10}$$

by Lemma 2.6 and [26, page 196]. □

In particular, if we put $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$ in (4.6), then (4.6) reduces to (2.21), and we obtain $u(x) = T_{2k}(x,m)$ as the fundamental solution of the diamond Klein-Gordon operator iterated k times.

Theorem 4.3. *Given the convolution equation*

$$e^{\alpha x}\left(\Diamond + m^2\right)^k \delta * u(x) = e^{\alpha x}\sum_{r=0}^{M} C_r\left(\Diamond + m^2\right)^r \delta, \tag{4.11}$$

where $(\Diamond + m^2)^k$ is the diamond Klein-Gordon operator iterated k times defined by

$$\left(\Diamond + m^2\right)^k = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2} + m^2\right)^k, \tag{4.12}$$

the variable $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the constant $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, m is a nonnegative real number, δ is the Dirac-delta distribution with $(\Diamond + m^2)^0\delta = \delta$, $(\Diamond + m^2)^1\delta = (\Diamond + m^2)\delta$, and C_r is a constant, then the type of solution $u(x)$ of (4.11) depends on k, M, and α as follows:

(1) *if $M < k$ and $M = 0$, then the solution of (4.11) is*

$$u(x) = C_0 e^{\alpha x}T_{2k}(x,m), \tag{4.13}$$

where $T_{2k}(x,m)$ is defined by (2.16) with $\gamma = 2k$. If $2k \geq n$ and for any α, then $e^{\alpha x}T_{2k}(x,m)$ is the ordinary function,

(2) *if $0 < M < k$, then the solution of (4.11) is*

$$u(x) = e^{\alpha x}\sum_{r=1}^{M} C_r T_{2k-2r}(x,m), \tag{4.14}$$

which is an ordinary function for $2k - 2r \geq n$ with any arbitrary constant α,

(3) *if $M \geq k$ and for any α one supposes that $k \leq M \leq N$, then (4.11) has*

$$u(x) = e^{\alpha x}\sum_{r=k}^{N} C_r\left(\Diamond + m^2\right)^{r-k}\delta \tag{4.15}$$

as a solution which is the singular distribution.

Proof. (1) For $M < k$ and $M = 0$, then (4.11) becomes

$$e^{\alpha x}\left(\Diamond + m^2\right)^k \delta * u(x) = C_0 e^{\alpha x}\delta = C_0\delta, \tag{4.16}$$

and by Theorem 4.2, we obtain

$$u(x) = C_0 e^{\alpha x} T_{2k}(x, m). \tag{4.17}$$

Now, by (2.2) and (2.7), $R^e_{2k}(x)$ and $R^H_{2k}(x)$ are ordinary functions, respectively, for $2k \geq n$. It then follows that $C_0 e^{\alpha x} T_{2k}(x, m)$ is an ordinary function for $2k \geq n$ with any α.

(2) For $0 < M < k$, then we can write (4.11) as

$$e^{\alpha x}\left(\Diamond + m^2\right)^k \delta * u(x) = e^{\alpha x}\left[C_1\left(\Diamond + m^2\right)\delta + C_2\left(\Diamond + m^2\right)^2\delta + \cdots + C_M\left(\Diamond + m^2\right)^M\delta\right]. \tag{4.18}$$

Convolving both sides by $e^{\alpha x}T_{2k}(x, m)$ and applying Lemma 2.5, we obtain

$$u(x) = e^{\alpha x}\left[C_1\left(\Diamond + m^2\right)T_{2k}(x, m) + C_2\left(\Diamond + m^2\right)^2 T_{2k}(x, m) + \cdots + C_M\left(\Diamond + m^2\right)^M T_{2k}(x, m)\right]. \tag{4.19}$$

It is known that $(\Diamond + m^2)^k T_{2k}(x, m) = \delta$, thus $(\Diamond + m^2)^{k-r}(\Diamond + m^2)^r T_{2k}(x, m) = \delta$ for $r < k$. Convolving both sides by $T_{2k-2r}(x, m)$, we obtain

$$T_{2k-2r}(x, m) * \left(\Diamond + m^2\right)^{k-r}\left(\Diamond + m^2\right)^r T_{2k}(x, m) = T_{2k-2r}(x, m), \tag{4.20}$$

or

$$\left(\Diamond + m^2\right)^{k-r} T_{2k-2r}(x, m) * \left(\Diamond + m^2\right)^r T_{2k}(x, m) = T_{2k-2r}(x, m), \tag{4.21}$$

which leads to

$$\left(\Diamond + m^2\right)^r T_{2k}(x, m) = T_{2k-2r}(x, m), \tag{4.22}$$

for $r < k$. It follows that

$$u(x) = e^{\alpha x}[C_1 T_{2k-2}(x, m) + C_2 T_{2k-4}(x, m) + \cdots + C_M T_{2k-2M}(x, m)], \tag{4.23}$$

or

$$u(x) = e^{\alpha x}\sum_{r=1}^{M} C_r T_{2k-2r}(x, m). \tag{4.24}$$

Similarly, by case (1), $e^{\alpha x}T_{2k-2r}(x,m)$ is the ordinary function for $2k-2r \geq n$ with any α. It follows that

$$u(x) = e^{\alpha x}\sum_{r=1}^{M} C_r T_{2k-2r}(x,m) \tag{4.25}$$

is also the ordinary function with any α.

(3) if $M \geq k$ and for any α, we suppose that $k \leq M \leq N$, then (4.11) becomes

$$e^{\alpha x}\left(\Diamond + m^2\right)^k \delta * u(x) = e^{\alpha x}\left[C_k\left(\Diamond + m^2\right)^k \delta + C_{k+1}\left(\Diamond + m^2\right)^{k+1}\delta + \cdots + C_N\left(\Diamond + m^2\right)^N \delta\right]. \tag{4.26}$$

Convolving both sides by $e^{\alpha x}T_{2k}(x,m)$ and applying Lemma 2.5, we have

$$u(x) = e^{\alpha x}\left[C_k\left(\Diamond + m^2\right)^k T_{2k}(x,m) + C_{k+1}\left(\Diamond + m^2\right)^{k+1} T_{2k}(x,m) + \cdots + C_N\left(\Diamond + m^2\right)^N T_{2k}(x,m)\right]. \tag{4.27}$$

Now,

$$\left(\Diamond + m^2\right)^M T_{2k}(x,m) = \left(\Diamond + m^2\right)^{M-k}\left(\Diamond + m^2\right)^k T_{2k}(x,m) = \left(\Diamond + m^2\right)^{M-k}, \tag{4.28}$$

for $k \leq M \leq N$. Thus,

$$\begin{aligned} u(x) &= e^{\alpha x}\left[C_k\delta + C_{k+1}\left(\Diamond + m^2\right)\delta + C_{k+2}\left(\Diamond + m^2\right)^2\delta + \cdots + C_N\left(\Diamond + m^2\right)^{N-k}\delta\right] \\ &= e^{\alpha x}\sum_{r=k}^{N} C_r\left(\Diamond + m^2\right)^{r-k}\delta. \end{aligned} \tag{4.29}$$

Now, by (3.1) and (3.2), we have

$$e^{\alpha x}\left(\Diamond + m^2\right)^{r-k}\delta = \left(\Diamond + m^2\right)^{r-k}\delta + (\text{terms of lower order of partial derivative of } \delta) \tag{4.30}$$

for $k \leq r \leq N$. Since all terms on the right-hand side of this equation are singular distribution, it follows that

$$u(x) = e^{\alpha x}\sum_{r=k}^{N} C_r\left(\Diamond + m^2\right)^{r-k}\delta \tag{4.31}$$

is the singular distribution. This completes the proof. □

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