1. Introduction

A double sequence \( x = [x_{jk}]_{j,k=0}^{\infty} \) is said to be convergent in the Pringsheim sense or \( P \)-convergent if for every \( \varepsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that \( |x_{jk} - \ell| < \varepsilon \) whenever \( j, k > N \), [1]. In this case, we write \( P \lim x = \ell \). By \( c_2 \), we mean the space of all \( P \)-convergent sequences.

A double sequence \( x \) is bounded if

\[
\|x\| = \sup_{j,k \geq 0} |x_{jk}| < \infty. \tag{1.1}
\]

By \( \ell_2^2 \), we denote the space of all bounded double sequences.

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. So, we denote by \( c_2^2 \) the space of double sequences which are bounded and convergent.

A double sequence \( x = [x_{jk}] \) is said to converge regularly if it converges in Pringsheim’s sense and, in addition, the following finite limits exist:

\[
\lim_{k \to \infty} x_{jk} = \ell_j, \quad (j = 1, 2, 3, \ldots),
\]

\[
\lim_{j \to \infty} x_{jk} = t_k, \quad (k = 1, 2, 3, \ldots). \tag{1.2}
\]
Let $A = [a_{jk}^{mn}]_{j,k=0}^{\infty}$ be a four-dimensional infinite matrix of real numbers for all $m, n = 0, 1, \ldots$. The sums

$$y_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk}$$

are called the $A$-transforms of the double sequence $x = [x_{jk}]$. We say that a sequence $x = [x_{jk}]$ is $A$-summable to the limit $\ell$ if the $A$-transform of $x = [x_{jk}]$ exists for all $m, n = 0, 1, \ldots$ and is convergent to $\ell$ in the Pringsheim sense, that is,

$$\lim_{m,n \to \infty} y_{mn} = \ell.$$ 

We say that a matrix $A$ is bounded-regular if every bounded-convergent sequence $x$ is $A$-summable to the same limit and the $A$-transforms are also bounded. The necessary and sufficient conditions for $A$ to be bounded-regular or RH-regular (cf., Robison [2]) are

$$\lim_{m,n \to \infty} a_{jk}^{mn} = 0, \quad (j, k = 0, 1, \ldots),$$

$$\lim_{m,n \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} = 1,$$

$$\lim_{m,n \to \infty} \sum_{j=0}^{\infty} |a_{jk}^{mn}| = 0, \quad (k = 0, 1, \ldots),$$

$$\lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{jk}^{mn}| = 0, \quad (j = 0, 1, \ldots),$$

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{jk}^{mn}| \leq C < \infty \quad (m, n = 0, 1, \ldots).$$

A double sequence $x = [x_{jk}]$ is said to be almost convergent (see [3]) to a number $L$ if

$$\lim_{p,q \to \infty} \sup_{s,t \geq 0} \frac{1}{pq} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{s+j+t+k} = L.$$ 

Let $\sigma$ be a one-to-one mapping from $\mathbb{N}$ into itself. The almost convergence of double sequences has been generalized to the $\sigma$-convergence in [4] as follows:

$$\lim_{p,q \to \infty} \sup_{s,t \geq 0} \frac{1}{pq} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)} = \ell.$$ 

(1.7)
where $\sigma^i(s) = \sigma(\sigma^{i-1}(s))$. In this case, we write $\sigma - \lim x = \ell$. By $V_2^\sigma$, we denote the set of all $\sigma$-convergent and bounded double sequences. One can see that in contrast to the case for single sequences, a convergent double sequence need not be $\sigma$-convergent. But every bounded convergent double sequence is $\sigma$-convergent. So, $c_2^\infty \subset V_2^\sigma \subset c_2^\infty$. In the case $\sigma(i) = i + 1$, $\sigma$-convergence of double sequences reduces to the almost convergence. A matrix $A = [a_{m,n}]$ is said to be $\sigma$-regular if $Ax \in V_2^\sigma$ for $x = [x_{j,k}] \in c_2^\infty$ with $\sigma - \lim Ax = \lim x$, and we denote this by $A \in (c_2^\infty, V_2^\sigma)_{reg}$ (see [5, 6]). Mursaleen and Mohiuddine defined and characterized $\sigma$-conservative and $\sigma$-coercive matrices for double sequences [6].

A double sequence $x = [x_{j,k}]$ of real numbers is said to be Cesàro convergent (or $C_1$-convergent) to a number $L$ if and only if

$$C_1 = \left\{ x \in c_2^\infty : \lim_{p,q \to \infty} T_{pq}(x) = L ; L = C_1 - \lim x \right\},$$

$$T_{pq}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=1}^{p} \sum_{k=1}^{q} x_{j,k}^m,$$

We shall denote by $C_1$ the space of Cesàro convergent ($C_1$-convergent) double sequences.

A matrix $A = (a_{j,k})$ is said to be $C_1$-multiplicative if $Ax \in C_1$ for $x = [x_{j,k}] \in c_2^\infty$ with $C_1 - \lim Ax = \alpha \lim x$, and in this case we write $A \in (c_2^\infty, C_1)_\alpha$. Note that if $\alpha = 1$, then $C_1$-multiplicative matrices are said to be $C_1$-regular matrices.

Recall that the Knopp core (or K-core) of a real number single sequence $x = (x_k)$ is defined by the closed interval $[\ell(x), L(x)]$, where $\ell(x) = \lim \inf x$ and $L(x) = \lim \sup x$. The well-known Knopp core theorem states (cf., Maddox [7] and Knopp [8]) that in order that $L(Ax) \leq L(x)$ for every bounded real sequence $x$, it is necessary and sufficient that $A = (a_{j,k})$ should be regular and $\lim_{n \to \infty} \sum_{k=0}^{\infty} |a_{n,k}| = 1$. Patterson [9] extended this idea for double sequences by defining the Pringsheim core (or P-core) of a real bounded double sequence $x = [x_{j,k}]$ as the closed interval $[P - \lim \inf x, P - \lim \sup x]$. Some inequalities related to the these concepts have been studied in [5, 9, 10]. Let

$$L^*(x) = \lim sup_{p,q \to \infty} \sup_{s,t \geq 0} \frac{1}{pq} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j,k} s + t,$$

$$C_\sigma(x) = \lim sup_{p,q \to \infty} \sup_{s,t \geq 0} \frac{1}{pq} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j,k} s^\sigma(s), t^\sigma(t).$$

Then, MR- (Moricz-Rhoades) and $\sigma$-core of a double sequence have been introduced by the closed intervals $[-L^*(x), L^*(x)]$ and $[-C_\sigma(x), C_\sigma(x)]$, and also the inequalities

$$L(Ax) \leq L^*(x), L^*(Ax) \leq L(x), L^*(Ax) \leq L^*(x), L(Ax) \leq C_\sigma(x), C_\sigma(Ax) \leq L(x)$$

have been studies in [3-5, 11].
Abstract and Applied Analysis

In this paper, we introduce the concept of $C_1$-multiplicative matrices and determine the necessary and sufficient conditions for a matrix $A = (a_{jk}^{mn})$ to belong to the class $(c_2^\infty, C_1)$. Further we investigate the necessary and sufficient conditions for the inequality

$$C_1^*(Ax) \leq \alpha L(x)$$

(1.11)

for all $x \in \ell_2^\infty$.

2. Main Results

Let us write

$$C_1^*(x) = \limsup_{p,q \to \infty} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{jk}.$$  

(2.1)

Then, we will define the $P_C$-core of a realvalued bounded double sequence $x = [x_{jk}]$ by the closed interval $[-C_1^*(-x), C_1^*(x)]$. Since every bounded convergent double sequence is Cesàro convergent, we have $C_1^*(x) \leq P - \limsup x$, and hence it follows that $P_C$-core$(x) \subseteq P$-core$(x)$ for a bounded double sequence $x = [x_{jk}]$.

**Lemma 2.1.** A matrix $A = (a_{jk}^{mn})$ is $C_1$-multiplicative if and only if

$$\lim_{p,q \to \infty} \beta(j,k,p,q) = 0 \quad (j,k = 0, 1, \ldots),$$

(2.2)

$$\lim_{p,q \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta(j,k,p,q) = \alpha,$$  

(2.3)

$$\lim_{p,q \to \infty} \sum_{j=0}^{\infty} |\beta(j,k,p,q)| = 0 \quad (k = 0, 1, \ldots),$$

(2.4)

$$\lim_{p,q \to \infty} \sum_{k=0}^{\infty} |\beta(j,k,p,q)| = 0 \quad (j = 0, 1, \ldots),$$

(2.5)

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{jk}^{mn}| \leq C < \infty, \quad (m,n = 0, 1, \ldots),$$

(2.6)

where the $\lim$ means $P - \lim$ and

$$\beta(j,k,p,q) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}^{mn}.$$  

(2.7)

**Proof.** Sufficiency. Suppose that the conditions (2.2)-(2.6) hold and $x = [x_{jk}] \in c_2^\infty$ with $P - \lim_{j,k} x_{jk} = L$, say. So that for every $\varepsilon > 0$ there exists $N > 0$ such that $|x_{jk}| < |\ell| + \varepsilon$ whenever $j, k > N$. 

Then, we can write
\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta(j, k, p, q) x_{jk} = \sum_{j=0}^{N} \sum_{k=0}^{N} \beta(j, k, p, q) x_{jk} + \sum_{j=N+1}^{\infty} \sum_{k=0}^{\infty} \beta(j, k, p, q) x_{jk} \\
+ \sum_{j=0}^{N-1} \sum_{k=N}^{\infty} \beta(j, k, p, q) x_{jk} + \sum_{j=N+1}^{\infty} \sum_{k=N+1}^{\infty} \beta(j, k, p, q) x_{jk}
\] (2.8)

Therefore,
\[
\left| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta(j, k, p, q) x_{jk} \right| \leq \|x\| \left( \sum_{j=0}^{N} \sum_{k=0}^{N} |\beta(j, k, p, q)| + \sum_{j=N+1}^{\infty} \sum_{k=0}^{\infty} |\beta(j, k, p, q)| \right) \\
+ \|x\| \sum_{j=0}^{N-1} \sum_{k=N}^{\infty} |\beta(j, k, p, q)| \\
+ (|L| + \epsilon) \sum_{j=N+1}^{\infty} \sum_{k=N+1}^{\infty} \beta(j, k, p, q)
\] (2.9)

Letting \(p, q \to \infty\) and using the conditions (2.2)–(2.6), we get
\[
\lim_{p, q \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta(j, k, p, q) x_{jk} \leq (|L| + \epsilon) \alpha. 
\] (2.10)

Since \(\epsilon\) is arbitrary, \(C_1 - \lim Ax = \alpha L\). Hence \(A \in (c_2^\infty, C_1)_\alpha\), that is, \(A\) is \(C_1\)-multiplicative.

**Necessity 1.** Suppose that \(A\) is \(C_1\)-multiplicative. Then, by the definition, the \(A\)-transform of \(x\) exists and \(Ax \in C_1\) for each \(x \in c_2^\infty\). Therefore, \(Ax\) is also bounded. Then, we can write
\[
\sup_{m,n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{jk}^{mn} x_{jk}| < M < \infty, 
\] (2.11)

for each \(x \in c_2^\infty\). Now, let us define a sequence \(y = [y_{jk}]\) by
\[
y_{jk} = \begin{cases} 
\text{sgn} a_{jk}^{mn}, & 0 \leq j \leq r, \ 0 \leq k \leq r, \\
0, & \text{otherwise},
\end{cases}
\] (2.12)

\(m, n = 0, 1, 2, \ldots\). Then, the necessity of (10) follows by considering the sequence \(y = [y_{jk}]\) in (2.11).
Also, by the assumption, we have
\[
\lim_{p,q \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta(j, k, p, q)x_{jk} = a \lim_{j,k \to \infty} x_{jk}. \quad (2.13)
\]

Now let us define the sequence \(e^{il}\) as follows:
\[
e^{il} = \begin{cases} 
1, & (j, k) = (i, l), \\
0, & \text{otherwise},
\end{cases}
\quad (2.14)
\]

and write \(s^i = \sum_l e^{il} (i \in \mathbb{N})\), \(r^i = \sum_l e^{il} (i \in \mathbb{N})\). Then, the necessity of (2.2), (2.4), and (2.5) follows from \(C_1 - \lim A e^{il}, C_1 - \lim A r^i\) and \(C_1 - \lim A s^k\), respectively.

Note that when \(a = 1\), the above theorem gives the characterization of \(A \in (c_2^\infty, C_1)_{\text{reg}}\).

Now, we are ready to construct our main theorem.

**Theorem 2.2.** For every bounded double sequence \(x\),
\[
C_1^*(Ax) \leq aL(x), \quad (2.15)
\]

or \(\langle P_{c} - \text{core} \{Ax\}\rangle \subseteq a(\langle P - \text{core} \{x\}\rangle)\) if and only if \(A\) is \(C_1\)-multiplicative and
\[
\limsup_{p,q \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\beta(j, k, p, q)| = a. \quad (2.16)
\]

**Proof. Necessity.** Let (2.15) hold and for all \(x \in \ell^2_\infty\). So, since \(c_2^\infty \subset \ell^2_\infty\), then, we get
\[
a(-L(-x)) \leq -C_1^*(-Ax) \leq C_1^*(Ax) \leq aL(x). \quad (2.17)
\]

That is,
\[
a \liminf x \leq -C_1^*(-Ax) \leq C_1^*(Ax) \leq a \limsup x, \quad (2.18)
\]

where
\[
-C_1^*(-Ax) = \liminf_{p,q \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta(j, k, p, q)x_{jk}. \quad (2.19)
\]

By choosing \(x = [x_{jk}] \in c_2^\infty\), we get from (2.17) that
\[
-C_1^*(-Ax) = C_1^*(Ax) = C_1 - \lim Ax = a \lim x. \quad (2.20)
\]

This means that \(A\) is \(C_1\)-multiplicative.
By Lemma 3.1 of Patterson [9], there exists a \( y \in \ell^2_{\infty} \) with \( \|y\| \leq 1 \) such that

\[
C_1^*(Ay) = \limsup_{p,q \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta(j,k,p,q).
\] (2.21)

If we choose \( y = \nu = [v_{jk}] \), it follows

\[
v_{jk} = \begin{cases} 
1 & \text{if } j = k, \\
0, & \text{elsewhere.}
\end{cases}
\] (2.22)

Since \( \|v_{jk}\| \leq 1 \), we have from (2.15) that

\[
\alpha = C_1^*(Av) = \limsup_{p,q \to \infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\beta(j,k,p,q)| \leq \alpha L(v_{jk}) \leq \alpha \|v\| \leq \alpha.
\] (2.23)

This gives the necessity of (2.16).

\[\square\]

**Sufficiency 1.** Suppose that \( A \) is \( C_1 \)-regular and (2.16) holds. Let \( x = [x_{jk}] \) be an arbitrary bounded sequence. Then, there exist \( M, N > 0 \) such that \( x_{jk} \leq K \) for all \( j, k \geq 0 \). Now, we can write the following inequality:

\[
\left| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta(j,k,p,q) x_{jk} \right| = \left| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{|\beta(j,k,p,q)| + \beta(j,k,p,q)}{2} - \frac{|\beta(j,k,p,q)| - \beta(j,k,p,q)}{2} \right) x_{jk} \right|
\]

\[
\leq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\beta(j,k,p,q)| |x_{jk}|
\]

\[
+ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |(\beta(j,k,p,q) - \beta(j,k,p,q)) x_{jk}|
\]

\[
\leq \|x\| \sum_{j=0}^{M} \sum_{k=0}^{N} |\beta(j,k,p,q)|
\]

\[
+ \|x\| \sum_{j=M+1}^{\infty} \sum_{k=0}^{N} |\beta(j,k,p,q)|
\]

\[
+ \|x\| \sum_{j=0}^{M} \sum_{k=N+1}^{\infty} |\beta(j,k,p,q)|
\]

\[
+ \|x\| \sum_{j=M+1}^{\infty} \sum_{k=N+1}^{\infty} |\beta(j,k,p,q)|
\]
Using the condition of $C_1$-multiplicative and condition (2.16), we get

$$C_1^*(Ax) \leq aL(x). \quad (2.25)$$

This completes the proof of the theorem.

**Theorem 2.3.** For $x, y \in \ell_2^\infty$, if $C_1 - \lim |x - y| = 0$, then $C_1 - \text{core} |x| = C_1 - \text{core} |y|.$

**Proof.** Since $C_2 - \lim |x - y| = 0$, we have $C_1 - \lim (x - y) = 0$ and $C_1 - \lim (- (x - y)) = 0$. Using definition of $C_1 - \text{core}$, we take $C_1^*(x - y) = C_1^*(- (x - y)) = 0$. Since $C_1^*$ is sublinear,

$$0 = -C_1^*(- (x - y)) \leq -C_1^*(- x) - C_1^*(y). \quad (2.26)$$

Therefore, $C_1^*(y) \leq -C_1^*(- x)$. Since $-C_1^*(- x) \leq C_1^*(x)$, this implies that $C_1^*(y) \leq C_1^*(x)$. By an argument similar as above, we can show that $C_1^*(x) \leq C_1^*(y)$. This completes the proof. \(\square\)

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**References**


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