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Research Article

Perturbation Results and Monotone Iterative Technique for Fractional Evolution Equations

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We mainly study the fractional evolution equation in an ordered Banach space X ${}^{C}D_0^{\alpha} + u(t) + Au(t) = f(t, u(t), Gu(t))$, $1 < \alpha < 2$, $u(0) = x \in X$, $u'(0) = \theta$. Using the monotone iterative technique based on lower and upper solutions, the existence and uniqueness results are obtained. The necessary perturbation results for accomplishing this approach are also developed.

1. Introduction

In this paper, we use the perturbation theory and the monotone iterative technique based on lower and upper solutions to investigate the existence and uniqueness of mild solutions for the fractional evolution equation in an ordered Banach space X:

$$\begin{gathered} {}^{C}D_{0+}^{\alpha} u(t) + Au(t) = f(t, u(t), Gu(t)), \quad t \in I = [0, T], \\ u(0) = x \in X, \qquad u'(0) = \theta, \end{gathered} \tag{1.1}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative, $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a linear closed densely defined operator, $f : I \times X \times X \to X$ is continuous, θ is the zero element of X, and

$$Gu(t) = \int_0^t K(t, s) u(s) ds \tag{1.2}$$

is a Volterra integral operator with integral kernel $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \{(t, s) \mid 0 \le s \le t \le T\}$.

In particular, when $f(t,u(t),Gu(t)) = f(t,u(t))$, we study the existence and uniqueness of mild solutions for the fractional evolution equation in an ordered Banach space X:

$$\begin{gathered} {}^{C}D_{0+}^{\alpha}u(t) + Au(t) = f(t,u(t)), \quad t \in I, \\ u(0) = x \in X, \qquad u'(0) = \theta, \end{gathered} \tag{1.3}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative, $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a linear closed densely defined operator, $f : I \times X \to X$ is continuous, and θ is the zero element of X.

The fractional calculus (i.e., calculus of integrals and derivatives of any arbitrary real or complex order) goes back to Newton and Leibnitz in the seventieth century. It has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields such as physics, chemistry, aerodynamics, viscoelasticity, porous media, electrodynamics of complex medium, and electrochemistry, control, electromagnetic. For instance, fractional calculus concepts have been used in the modeling of transmission lines [1], neurons [2], viscoelastic materials [3], and electrical capacitors [4]. Other examples from fractional order dynamics can be found in [5, 6] and the references therein.

One of the branches of fractional calculus is the theory of fractional evolution equations, that is evolution equations where the integer derivative with respect to time is replaced by a derivative of any order. Also, in recent years, fractional evolution equations have attracted increasing attention; see [7–21].

The monotone iterative technique based on lower and upper solutions is an effective and a flexible mechanism that offers theoretical as well as constructive existence results in a closed set. It yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. Since under suitable conditions each member of the sequences happens to be the unique solution of a certain nonlinear problem, the advantage and importance of the technique is remarkable. For differential equations of integer order, many papers used the monotone iterative technique based on lower and upper solutions; see [22–24] and the references therein. Recently, there have been some papers which deal with the existence of the solutions of initial value problems or boundary value problems for fractional ordinary differential equations by using this method; see [25–31]. They mainly involve Riemann-Liouville fractional derivatives.

However, to the best of the authors' knowledge, no results yet exist for the fractional evolution equations by using the monotone iterative technique based on lower and upper solutions. Our results can be considered as a contribution to this emerging field.

In comparison with fractional ordinary differential equations, we have great difficulty in using the monotone iterative technique for the fractional evolution equations. Firstly, how to introduce a suitable concept of a mild solution for fractional evolution equations based on the corresponding solution operator? A pioneering work has been reported by El-Borai [10, 11]. Later on, some authors introduced the definitions of mild solutions for fractional evolution equations. Wang and Zhou [17], Wang et al. [16, 18], and Zhou and Jiao [20, 21] also introduced a suitable definition of the mild solutions based on the well-known theory of Laplace transform and probability density functions. Moreover, Hernández et al. [12] used an approach to treat abstract equations with fractional derivatives based on the well-developed theory of resolvent operators for integral equations. Shu et al. [15] give the definition of a mild solution by investigating the classical solutions of the corresponding

system. Secondly, do the solution operators for fractional evolution equations have the perturbation properties analogous to those for the C_0-semigroup? For evolution equations of integer order, perturbation properties play a significant role in monotone iterative technique; see [24].

Our paper copes with the above difficulties, and the new features of this paper mainly include the following aspects. We firstly introduce a new concept of a mild solution based on the well-known theory of Laplace transform, and the form is very easy. Secondly, we discuss the perturbation properties for the corresponding solution operators. Thirdly, by the monotone iterative technique based on lower and upper solutions, we obtain results on the existence and uniqueness of mild solutions for problem (1.1) and (1.3).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition 2.1 (see [5]). The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of function $f \in L_1(\mathbb{R}^+)$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 (see [5]). The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, is defined as

$${}^{C}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $n-1$. If f is an abstract function with values in X, then the integrals and derivatives which appear in (2.1) and (2.2) are taken in Bochner sense.

Proposition 2.3. *For $\alpha, \beta > 0$ and f as a suitable function (e.g., [5]) one has the following:*

(i) $I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\alpha+\beta} f(t)$;

(ii) $I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\beta} I_{0+}^{\alpha} f(t)$;

(iii) $I_{0+}^{\alpha}(f(t) + g(t)) = I_{0+}^{\alpha} f(t) + I_{0+}^{\alpha} g(t)$;

(iv) ${}^{C}D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t)$;

(v) ${}^{C}D_{0+}^{\alpha}\, {}^{C}D_{0+}^{\beta} f(t) \neq {}^{C}D_{0+}^{\alpha+\beta} f(t)$;

(vi) ${}^{C}D_{0+}^{\alpha}\, {}^{C}D_{0+}^{\beta} f(t) \neq {}^{C}D_{0+}^{\beta}\, {}^{C}D_{0+}^{\alpha} f(t)$.

We observe from the above that the Caputo fractional differential operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives

on integer order. For basic facts about fractional integrals and fractional derivatives one can refer to the books [5, 32–34].

Let X be an ordered Banach space with norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{y \in X \mid y \geq \theta\}$ (θ is the zero element of X) is normal with normal constant N. Let $C(I, X)$ be the Banach space of all continuous X-value functions on interval I with norm $\|u\|_c = \max_{t\in I}\|(t)\|$. For $u, v \in C(I, X)$, $u \leq v \Leftrightarrow u(t) \leq v(t)$ for all $t \in I$. For $v, w \in C(I, X)$, denote the ordered interval $[v, w] = \{u \in C(I, X) \mid v \leq u \leq w\}$, and $[v(t), w(t)] = \{y \in X \mid v(t) \leq y \leq w(t)\}$, $t \in I$. By $\mathcal{B}(X)$ we denote the space of all bounded linear operators from X to X.

Definition 2.4. If ${}^{C}D^{\alpha}_{0+}v_0, Av_0, v_0' \in C(I, X)$, and v_0 satisfies

$$\begin{gathered} {}^{C}D^{\alpha}_{0+}v_0(t) + Av_0(t) \leq f(t, v_0(t), Gv_0(t)), \quad t \in I, \\ v_0 \leq x \in X, \quad v_0'(0) \leq \theta, \end{gathered} \tag{2.3}$$

then $\tilde{v}_0$ is called a lower solution of problem (1.1); if all inequalities of (2.3) are inverse, we call it an upper solution of problem (1.1).

Similarly, we give the definitions of lower and upper solutions of problem (1.3).

Definition 2.5. If ${}^{C}D^{\alpha}_{0+}\tilde{v}_0, A\tilde{v}_0, \tilde{v}_0' \in C(I, X)$, and $\tilde{v}_0$ satisfy

$$\begin{gathered} {}^{C}D^{\alpha}_{0+}\tilde{v}_0(t) + A\tilde{v}_0(t) \leq f(t, \tilde{v}_0(t)), \quad t \in I, \\ \tilde{v}_0(0) \leq x \in X, \quad \tilde{v}_0'(0) \leq \theta, \end{gathered} \tag{2.4}$$

then $\tilde{v}_0$ is called a lower solution of problem (1.3); if all inequalities of (2.4) are inverse, we call it an upper solution of problem (1.3).

Consider the following problem:

$$\begin{gathered} {}^{C}D^{\alpha}_{0+}u(t) + Au(t) = \theta, \quad t \in I, \\ u(0) = x, \qquad u'(0) = \theta. \end{gathered} \tag{2.5}$$

Definition 2.6 (see [9]). A family $\{S_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called a solution operator for (2.5) if the following conditions are satisfied:

(1) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$;

(2) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$;

(3) $S_\alpha(t)$ is a solution of

$$u(t) = x - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Au(s)ds, \tag{2.6}$$

for all $x \in D(A)$, $t \geq 0$.

In this case, $-A$ is called the generator of the solution operator $S_\alpha(t)$ and $S_\alpha(t)$ is called the solution operator generated by $-A$.

Definition 2.7 (see [9]). The solution operator $S_\alpha(t)$ is called exponentially bounded if there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|S_\alpha(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{2.7}$$

An operator $-A$ is said to belong to $\mathcal{C}(X; M, \omega)$, or $\mathcal{C}^\alpha(M, \omega)$ for short, if problem (2.5) has a solution operator $S_\alpha(t)$ satisfying (2.7). Denote $\mathcal{C}^\alpha(\omega) = \cup\{\mathcal{C}^\alpha(M, \omega) \mid M \geq 1\}$, $\mathcal{C}^\alpha = \cup\{\mathcal{C}^\alpha(\omega) \mid \omega \geq 0\}$. In these notations $\mathcal{C}^1$ and $\mathcal{C}^2$ are the sets of all infinitesimal generators of C_0-emigroups and cosine operator families (COF), respectively. Next, we give a characterization of $\mathcal{C}^\alpha(M, \omega)$.

Lemma 2.8 (see [9]). *Let $1 < \alpha < 2$, $-A \in C^\alpha(M, \omega)$ and let $S_\alpha(t)$ be the corresponding solution operator. Then for $\lambda > \omega$, one has $\lambda^\alpha \in \rho(-A)$ and*

$$\lambda^{\alpha-1} R(\lambda^\alpha, -A)x = \int_0^{+\infty} e^{-\lambda t} S_\alpha(t) x dt, \quad x \in X. \tag{2.8}$$

Lemma 2.9 (see [9]). *Let $1 < \alpha < 2$ and $-A \in C^\alpha$. Then the corresponding solution operator is given by*

$$S_\alpha(t)x = \lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{n+1} b_{k,n+1}^\alpha \left(I + \left(\frac{t}{n}\right)^\alpha A\right)^{-k} x = \lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{n+1} b_{k,n+1}^\alpha \left[\left(\frac{n}{t}\right)^\alpha R\left(\left(\frac{n}{t}\right)^\alpha, -A\right)\right]^k x, \tag{2.9}$$

where $b_{k,n}^\alpha$ are given by the recurrence relations:

$b_{1,1}^\alpha = 1$,
$b_{k,n}^\alpha = (n - 1 - ka) b_{k,n-1}^\alpha + \alpha(k-1) b_{k-1,n-1}^\alpha$, $1 \leq k \leq n$, $n = 2, 3, \ldots$,
$b_{k,n}^\alpha = 0$, $k > n$, $n = 1, 2, \ldots$.

The convergence is uniform on bounded subsets of $[0, +\infty)$ for any fixed $x \in X$.

Lemma 2.10 (see [9]). *Let $1 < \alpha < 2$. Then $-A \in C^\alpha(M, \omega)$ if and only if $(\omega^\alpha, \infty) \subset \rho(-A)$ and*

$$\left\| \frac{\partial^n}{\partial \lambda^n} \left(\lambda^{\alpha-1} R(\lambda^\alpha, -A)\right) \right\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}} \quad \lambda > \omega, \ n = 0, 1, \ldots. \tag{2.10}$$

Lemma 2.11. *Assume $h \in C(I, X)$. For the linear Cauchy problem*

$$\begin{gathered} {}^C D_{0+}^\alpha u(t) + Au(t) = h(t), \quad t \in I, \\ u(0) = x \in X, \qquad u'(0) = \theta, \end{gathered} \tag{2.11}$$

$u(t)$ has the form

$$u(t) = S_\alpha(t)x + \int_0^t T_\alpha(t-s) h(s) ds, \tag{2.12}$$

where $S_\alpha(t)$ is the solution operator generated by $-A$, and $T_\alpha(t) = I_{0+}^{\alpha-1} S_\alpha(t)$.

Proof. For $\lambda > \omega$, applying the Laplace transform to (2.11), we have that

$$\lambda^{\alpha}\mathcal{L}u(\lambda) - \lambda^{\alpha-1}u(0) - \lambda^{\alpha-2}u'(0) + A\mathcal{L}u(\lambda) = \lambda^{\alpha}\mathcal{L}u(\lambda) - \lambda^{\alpha-1}x + A\mathcal{L}u(\lambda) = \mathcal{L}h(\lambda). \tag{2.13}$$

By Lemma 2.8, $\lambda^{\alpha} \in \rho(-A)$, from the above equation, we obtain

$$\mathcal{L}u(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha}I + A)^{-1}x + \lambda^{1-\alpha}\lambda^{\alpha-1}(\lambda^{\alpha}I + A)^{-1}\mathcal{L}h(\lambda). \tag{2.14}$$

Since $\mathcal{L}[t^{\alpha-2}/\Gamma(\alpha-1)](\lambda) = \lambda^{1-\alpha}$, by Lemma 2.8 and the inverse Laplace transform, we have that

$$u(t) = S_{\alpha}(t)x + \int_0^t T_{\alpha}(t-s)h(s)ds, \tag{2.15}$$

where

$$T_{\alpha}(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}S_{\alpha}(s)ds = I_{0+}^{\alpha-1}S_{\alpha}(t). \tag{2.16}$$

□

Remark 2.12. If $A = a$ (a is a constant), we know that $u(t) = E_{\alpha}(-at^{\alpha})x + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})h(s)ds$ is the solution of (2.11) by [5, Example 4.10], where $E_{\alpha}(-at^{\alpha})$ and $E_{\alpha,\alpha}(-a(t-s)^{\alpha})$ are the Mittag-Leffler functions. We also find that $E_{\alpha}(-at^{\alpha})x$ is the solution of the problem (2.5), and $t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha}) = I_{0+}^{\alpha-1}E_{\alpha}(-at^{\alpha})$; see [5].

Definition 2.13. A function $u : I \to X$ is called a mild solution of (2.11) if $u \in C(I, X)$ and satisfies the following equation:

$$u(t) = S_{\alpha}(t)x + \int_0^t T_{\alpha}(t-s)h(s)ds, \tag{2.17}$$

where $S_{\alpha}(t)$ is the solution operator generated by $-A$, and $T_{\alpha}(t) = I_{0+}^{\alpha-1}S_{\alpha}(t)$.

Remark 2.14. It is easy to verify that a classical solution of (2.11) is a mild solution of the same system.

Lemma 2.15. *If $1 < \alpha < 2$, $-A \in C^{\alpha}(M, \omega)$, $S_{\alpha}(t)$ is the solution operator generated by $-A$, and $T_{\alpha}(t) = I_{0+}^{\alpha-1}S_{\alpha}(t)$, then one has that*

$$\|T_{\alpha}(t)\| \le \frac{M}{\Gamma(\alpha)}e^{\omega t}t^{\alpha-1}, \quad t \ge 0. \tag{2.18}$$

Proof. By (2.7), for $s \geq 0$, we have that $\|S_\alpha(s)\| \leq Me^{\omega s}$. Thus,

$$\begin{aligned}
\|T_\alpha(t)\| &= \left\| \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} S_\alpha(s) ds \right\| \\
&\leq M \left\| \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} e^{\omega s} ds \right\| \\
&= M e^{\omega t} \left\| \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds \right\| \\
&= M e^{\omega t} \frac{t^{\alpha-1}}{\Gamma(\alpha)}.
\end{aligned} \tag{2.19}$$

□

Now, we discuss the perturbation properties of the solution operators.

Definition 2.16. An operator $S(t) : X \to X (t \geq 0)$ is called a positive operator in X if $u \in P$ and $t \geq 0$ such that $S(t)\ u \geq \theta$.

From Definition 2.1, we can easily obtain the following result.

Lemma 2.17. *Assume that $S_\alpha(t)$ is the solution operator generated by $-A$ and $T_\alpha(t) = I_{0+}^{\alpha-1} S_\alpha(t)$. Then $S_\alpha(t)(t \geq 0)$ is a positive operator if and only if $T_\alpha(t)(t \geq 0)$ is a positive operator.*

By Lemmas 2.8 and 2.9 and the closedness of the positive cone, we can obtain the following result.

Lemma 2.18. *Assume that $-A \in C^\alpha(M, \omega)$ and $S_\alpha(t)$ is the solution operator generated by $-A$. The following results are true.*

(i) *If $S_\alpha(t)(t \geq 0)$ is a positive solution operator, then for any $\lambda > \omega$ and $u \in P$, we have $R(\lambda^\alpha, -A)u \geq \theta$.*

(ii) *If there is a $\lambda_0 > \omega$, for any $\lambda > \lambda_0$ and $u \in P$ such that $R(\lambda^\alpha, -A)u \geq \theta$, then $S_\alpha(t)(t \geq 0)$ is a positive solution operator.*

Lemma 2.19. *Assume $C > 0$, $1 < \alpha < 2$, $-A \in C^\alpha(M, \omega)$; then the following results hold.*

(i) *$-(A + CI) \in C^\alpha(ME_\alpha(MCT^\alpha), \omega)$, where $E_\alpha(MCT^\alpha)$ is the Mittag-Leffler function.*

(ii) *If $\lambda > \omega$ and $u \in P$ such that $R(\lambda^\alpha, -A)u \geq \theta$, then for $\lambda > \omega + CM\omega^{1-\alpha}$ and $u \in P$, one has $R(\lambda^\alpha, -(A + CI))u \geq \theta$.*

Proof. (i) If $\widetilde{S}_\alpha(t)$ is the solution operator generated by $-(A+CI)$, in view of [9, Theorem 2.26], we have that

$$\left\| \widetilde{S}_\alpha(t) \right\| \leq ME_\alpha(MCT^\alpha) e^{\omega t}, \quad t \geq 0. \tag{2.20}$$

That is, $-(A + CI) \in \mathcal{C}^\alpha(ME_\alpha(MCT^\alpha), \omega)$.

(ii) If $\lambda > \omega$, by (i) and Lemma 2.8, we have $\lambda^\alpha \in \rho(-A)$, $\lambda^\alpha \in \rho(-(A+CI))$. Then by Lemma 2.10,

$$\left\| \lambda^{\alpha-1} R(\lambda^\alpha, -A) \right\| \leq \frac{M}{\lambda - \omega}. \tag{2.21}$$

When $\lambda > \omega + CM\omega^{1-\alpha}$,

$$\|CR(\lambda^\alpha, -A)\| \leq \frac{CM\lambda^{1-\alpha}}{\lambda - \omega} \leq \frac{CM\omega^{1-\alpha}}{\lambda - \omega} < 1. \tag{2.22}$$

Therefore, for such λ the operator $I + CR(\lambda^\alpha, -A)$ is invertible and

$$\begin{aligned} R(\lambda^\alpha, -(A+CI)) &= R(\lambda^\alpha, -A)(I + CR(\lambda^\alpha, -A))^{-1} \\ &= R(\lambda^\alpha, -A) \sum_{n=0}^{\infty} (-1)^n (CR(\lambda^\alpha, -A))^n. \end{aligned} \tag{2.23}$$

For any $u \in P$, in view of $R(\lambda^\alpha, -A)u \geq \theta$, $C > 0$ and (2.22), then

$$R(\lambda^\alpha, -(A+CI))u \geq \theta. \tag{2.24}$$

□

Remark 2.20. If $\alpha \in (0,1)$, Lemma 2.19 (i) is not true; see [9, Example 2.24]. However, a classical perturbation result for $\mathcal{C}^1$ or $\mathcal{C}^2$ (see [35, 36]) is as follows: if A is the generator of C_0-semigroup (or COF) and $B \in \mathcal{B}(X)$, then $A + B$ is again a generator of a C_0-semigroup (or COF).

By Definition 2.13, we can obtain the following result.

Lemma 2.21. *The linear Cauchy problem*

$$\begin{gathered} {}^{C}D_{0+}^{\alpha} u(t) + Au(t) + Cu(t) = h(t), \quad t \in I, \\ u(0) = x \in X, \qquad u'(0) = \theta, \end{gathered} \tag{2.25}$$

where $C > 0$, $h \in C(I, X)$, has the unique mild solution given by

$$u(t) = \widetilde{S}_\alpha(t)x + \int_0^t \widetilde{T}_\alpha(t-s)h(s)ds, \tag{2.26}$$

where $\widetilde{S}_\alpha(t)$ is the solution operator generated by $-(A+CI)$, and $\widetilde{T}_\alpha(t) = I_{0+}^{\alpha-1}\widetilde{S}_\alpha(t)$.

By Lemmas 2.17, 2.18, and 2.19, the following result holds.

Lemma 2.22. *Assume that $S_\alpha(t)$ and $\widetilde{S}_\alpha(t)$ are the solution operators generated by $-A$ and $-(A+CI)$, respectively, and $\widetilde{T}_\alpha(t) = I_{0+}^{\alpha-1}\widetilde{S}_\alpha(t)$. Then $S_\alpha(t)$ is a positive operator $\Rightarrow$ $\widetilde{S}_\alpha(t)$ and $\widetilde{T}_\alpha(t)$ are positive operators.*

Now, we recall some properties of the measure of noncompactness that will be used later. Let $\mu(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [37]. For any $B \subset C(I,X)$ and $t \in I$, set $B(t) = \{u(t) \mid u \in B\}$. If B is bounded in $C(I,X)$, then $B(t)$ is bounded in X, and $\mu(B(t)) \leq \mu(B)$.

Lemma 2.23 (see [38]). *Let $B = \{u_n\} \subset C(I,X)(n = 1,2,\ldots)$ be a bounded and countable set. Then $\mu(B(t))$ is Lebesgue integral on I, and*

$$\mu\left(\left\{\int_I u_n(t)dt \mid n = 1,2,\ldots\right\}\right) \leq 2\int_I \mu(B(t))dt. \tag{2.27}$$

3. Main Results

Theorem 3.1. *Let X be an ordered Banach space, whose positive* cone *P is normal with normal constant N, $f : I \times X \times X \to X$ is continuous, and $A : D(A) \subset X \to X$ is a linear closed densely defined operator. Assume that $-A \in \mathcal{C}^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.1) *has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \leq w_0$, and the following conditions are satisfied.*

(H_1) *There exists a constant $C \geq 0$ such that*

$$f(t,x_2,y_2) - f(t,x_1,y_1) \geq -C(x_2 - x_1), \tag{3.1}$$

for any $t \in I$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$, and $Gv_0(t) \leq y_1 \leq y_2 \leq Gw_0(t)$.

(H_2) *There exists a constant $L \geq 0$ such that*

$$\mu(\{f(t,x_n,y_n)\}) \leq L(\mu(\{x_n\}) + \mu(\{y_n\})), \tag{3.2}$$

for any $t \in I$, and increasing or decreasing monotonic sequences $\{x_n\} \subset [v_0(t), w_0(t)]$ and $\{y_n\} \subset [Gv_0(t), Gw_0(t)]$.

Then the Cauchy problem (1.1) *has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. Since $-A \in \mathrm{C}^{\alpha}(M,\omega)$, by Lemmas 2.15 and 2.19, we have that

$$\begin{gathered}\left\|\widetilde{S}_{\alpha}(t)\right\| \leq ME_{\alpha}(MCT^{\alpha})e^{\omega t}, \quad t \geq 0, \\ \left\|\widetilde{T}_{\alpha}(t)\right\| \leq \frac{M}{\Gamma(\alpha)}E_{\alpha}(MCT^{\alpha})e^{\omega t}t^{\alpha-1}, \quad t \geq 0.\end{gathered} \tag{3.3}$$

Set

$$\widetilde{M}_T = \max_{t\in I}\left\|\widetilde{T}_\alpha(t)\right\|. \tag{3.4}$$

Since $S_\alpha(t)$ is the positive solution operator generated by $-A$, by Lemma 2.22, $\widetilde{S}_\alpha(t)$ and $\widetilde{T}_\alpha(t)$ are positive operators.

Let $D = [v_0, w_0]$; we define a mapping $Q : D \to C(I, X)$ by

$$Qu(t) = \widetilde{S}_\alpha(t)x + \int_0^t \widetilde{T}_\alpha(t-s)\big(f(s,u(s),Gu(s)) + Cu(s)\big)ds, \quad t \in I. \tag{3.5}$$

By Lemma 2.21, $u \in D$ is a mild solution of problem (1.1) if and only if

$$u = Qu. \tag{3.6}$$

By (H_1), for $u_1, u_2 \in D$ and $u_1 \le u_2$, we have that

$$Qu_1 \le Qu_2. \tag{3.7}$$

That is, Q is an increasing monotonic operator. Now, we show that $v_0 \le Qv_0$, $Qw_0 \le w_0$.

Let $\sigma(t) \triangleq {}^{C}D^{\alpha}_{0+}v_0(t)+Av_0(t)+Cv_0(t)$; by Definition 2.1, Lemma 2.21, and the positivity of operators $\widetilde{S}_\alpha(t)$ and $\widetilde{T}_\alpha(t)$, we have that

$$\begin{aligned} v_0(t) &= \widetilde{S}_\alpha(t)v_0(0) + \int_0^t \widetilde{T}_\alpha(t-s)\sigma(s)ds \\ &\le \widetilde{S}_\alpha(t)x + \int_0^t \widetilde{T}_\alpha(t-s)\big(f(s,v_0(s),Gv_0(s)) + Cv_0(s)\big)ds \\ &= Qv_0(t), \quad t \in I, \end{aligned} \tag{3.8}$$

namely, $v_0 \le Qv_0$. Similarly, we can show that $Qw_0 \le w_0$. For $u \in D$, in view of (3.7), then $v_0 \le Qv_0 \le Qu \le Qw_0 \le w_0$. Thus, $Q : D \to D$. We can now define the sequences

$$v_n = Qv_{n-1}, \quad w_n = Qw_{n-1}, \quad n = 1, 2, \ldots, \tag{3.9}$$

and it follows from (3.7) that

$$v_0 \le v_1 \le \cdots v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{3.10}$$

For convenience, by (1.2), we can denote

$$K_0 = \max_{(t,s)\in\Delta} K(t,s). \tag{3.11}$$

Let $B = \{v_n\}(n = 1,2,\ldots)$ and $B_0 = \{v_{n-1}\}(n = 1,2,\ldots)$. It follows from $B_0 = B \cup \{v_0\}$ that $\mu(B(t)) = \mu(B_0(t))$ for $t \in I$. Let

$$\varphi(t) = \mu(B(t)) = \mu(B_0(t)), \quad t \in I. \tag{3.12}$$

In view of (3.10), since the positive cone P is normal, then B_0 and B are bounded in $C(I,X)$. By Lemma 2.23 and (3.12), $\varphi(t)$ is Lebesgue integrable on I. For $t \in I$, by (3.11) and Lemma 2.23, we have that

$$\mu(GB_0(t)) = \mu\left(\left\{\int_0^t K(t,s)v_{n-1}(s) \mid n = 1,2,\ldots\right\}\right) \le 2K_0 \int_0^t \varphi(s)ds, \tag{3.13}$$

and therefore,

$$\int_0^t \mu(GB_0(s))ds \le 2TK_0 \int_0^t \varphi(s)ds. \tag{3.14}$$

For $t \in I$, from Lemma 2.23, (H_2), (3.4), (3.5), (3.9), (3.12), (3.14), and the positivity of operator $\widetilde{T}_\alpha(t)$, we have that

$$\begin{aligned}
\varphi(t) &= \mu(B(t)) = \mu(QB_0(t)) \\
&= \mu\left(\left\{\int_0^t \widetilde{T}_\alpha(t-s)\big(f(s,v_{n-1}(s),Gv_{n-1}(s)) + Cv_{n-1}(s)\big)ds \mid n = 1,2,\ldots\right\}\right) \\
&\le 2\int_0^t \mu\left(\left\{\widetilde{T}_\alpha(t-s)\big(f(s,v_{n-1}(s),Gv_{n-1}(s)) + Cv_{n-1}(s)\big) \mid n = 1,2,\ldots\right\}\right)ds \\
&\le 2\widetilde{M}_T \int_0^t L\big(\mu(B_0(s)) + \mu(GB_0(s))\big) + C\mu(B_0(s))ds \\
&= 2\widetilde{M}_T(L + 2LTK_0 + C)\int_0^t \varphi(s)ds.
\end{aligned} \tag{3.15}$$

By (3.15) and the Gronwall inequality, we obtain that $\varphi(t) \equiv 0$ on I. This means that $v_n(t)(n = 1,2,\ldots)$ is precompact in X for every $t \in I$. So, $v_n(t)$ has a convergent subsequence in X. In view of (3.7), we can easily prove that $v_n(t)$ itself is convergent in X. That is, there exist $\underline{u}(t) \in X$ such that $v_n(t) \to \underline{u}(t)$ as $n \to \infty$ for every $t \in I$. By (3.5) and (3.9), for any $t \in I$, we have that

$$v_n(t) = \widetilde{S}_\alpha(t)x + \int_0^t \widetilde{T}_\alpha(t-s)\big(f(s,v_{n-1}(s),Gv_{n-1}(s)) + Cv_{n-1}(s)\big)ds. \tag{3.16}$$

Let $n \to \infty$; then by Lebesgue-dominated convergence theorem, for any $t \in I$, we have that

$$\underline{u}(t) = \widetilde{S}_\alpha(t)x + \int_0^t \widetilde{T}_\alpha(t-s)\big(f(s,\underline{u}(s),G\underline{u}(s)) + C\underline{u}(s)\big)ds, \tag{3.17}$$

and $\underline{u} \in C(I,X)$. Then $\underline{u} = Q\underline{u}$. Similarly, we can prove that there exists $\overline{u} \in C(I,X)$ such that $\overline{u} = Q\overline{u}$. By (3.7), if $u \in D$, and u is a fixed point of Q, then $v_1 = Qv_0 \le Qu = u \le Qw_0 = w_1$. By induction, $v_n \le u \le w_n$. By (3.10) and taking the limit as $n \to \infty$, we conclude that $v_0 \le \underline{u} \le u \le \overline{u} \le w_0$. That means that $\underline{u}, \overline{u}$ are the minimal and maximal fixed points of Q on $[v_0, w_0]$, respectively. By (3.6), they are the minimal and maximal mild solutions of the Cauchy problem (1.1) on $[v_0, w_0]$, respectively. □

Remark 3.2. Even if $A = 0$, our results are also new.

Corollary 3.3. *Let X be an ordered Banach space, whose positive* cone *P is regular, $f : I \times X \times X \to X$ is continuous, and $A : D(A) \subset X \to X$ is a linear closed densely defined operator. Assume that $-A \in C^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.1) *has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \le w_0$, and (H_1) holds. Then the Cauchy problem* (1.1) *has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. Since (H_1) is satisfied, then (3.10) holds. In regular positive cone P, any monotonic and ordered-bounded sequence is convergent. Then there exist $\underline{u} \in C(I,E)$, $\overline{u} \in C(I,E)$ and $\lim_{n\to\infty} v_n = \underline{u}$, $\lim_{n\to\infty} w_n = \overline{u}$. Then by the proof of Theorem 3.1, the proof is then complete. □

Corollary 3.4. *Let X be an ordered and weakly sequentially complete Banach space, whose positive* cone *P is normal with normal constant N, $f : I \times X \times X \to X$ is continuous, and $A : D(A) \subset X \to X$ is a linear closed densely defined operator. Assume that $-A \in C^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.1) *has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \le w_0$, and (H_1) holds. Then the Cauchy problem* (1.1) *has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. Since X is an ordered and weakly sequentially complete Banach space, then the assumption (H_2) holds. In fact, by [39, Theorem 2.2], any monotonic and ordered-bounded sequence is precompact. Let x_n and y_n be two increasing or decreasing sequences. By (H_1), $\{f(t,x_n,y_n) + Cx_n\}$ is monotonic and ordered-bounded sequence. Then, by the properties of the measure of noncompactness, we have

$$\mu(\{f(t,x_n,y_n)\}) \le \mu(f(t,x_n,y_n) + Cx_n) + C\mu(\{x_n\}) = 0. \tag{3.18}$$

So, (H_2) holds. By Theorem 3.1, the proof is then complete. □

Theorem 3.5. *Let X be an ordered Banach space, whose positive* cone *P is normal with normal constant N, $f : I \times X \times X \to X$ is continuous, $A : D(A) \subset X \to X$ is a linear closed densely defined operator. Assume $-A \in \mathcal{C}^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.1) *has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \le w_0$, (H_1) holds, and the following condition is satisfied:*

(H_3) *There are constants $S_1, S_2 \ge 0$ such that*

$$f(t,x_2,y_2) - f(t,x_1,y_1) \le S_1(x_2 - x_1) + S_2(y_2 - y_1), \tag{3.19}$$

for any $t \in I$, $v_0(t) \le x_1 \le x_2 \le w_0(t)$, *and* $Gv_0(t) \le y_1 \le y_2 \le Gw_0(t)$.

Then the Cauchy problem (1.1) *has the unique mild solution between* v_0 *and* w_0, *which can be obtained by a monotone iterative procedure starting from* v_0 *or* w_0.

Proof. We can find that (H_1) and (H_3) imply (H_2). In fact, for $t \in I$, let $\{x_n\} \subset [v_0(t), w_0(t)]$ and $\{y_n\} \subset [Gv_0(t), Gw_0(t)]$ be two increasing or decreasing monotonic sequence. For $m, n = 1, 2, \ldots$ with $m > n$, by (H_1) and (H_3), we have that

$$\theta \le f(t, x_m, y_m) - f(t, x_n, y_n) + C(x_m - x_n) \le (S_1 + C)(x_m - x_n) + S_2(y_m - y_n). \tag{3.20}$$

By (3.20) and the normality of positive cone P, we have

$$\|f(t, x_m, y_m) - f(t, x_n, y_n)\| \le (NS_1 + NC + C)\|x_m - x_n\| + NS_2\|y_m - y_n\|. \tag{3.21}$$

From (3.21) and the definition of the measure of noncompactness, we have that

$$\mu(\{f(t, x_n, y_n)\}) \le (NS_1 + NC + C)\mu(\{x_n\}) + NS_2\mu(\{y_n\}) \le L(\mu(\{x_n\}) + \mu(\{y_n\})), \tag{3.22}$$

where $L = NS_1 + NC + C + NS_2$. Hence, (H_2) holds.

Therefore, by Theorem 3.1, the Cauchy problem (1.1) has the minimal solution $\underline{u}$ and the maximal solution $\overline{u}$ on $D = [v_0, w_0]$. In view of the proof of Theorem 3.1, we show that $\underline{u} = \overline{u}$. For $t \in I$, by (3.4), (3.5), (3.6), (3.11), (H_3), and the positivity of operator $\widetilde{T}_\alpha(t)$, we have that

$$\begin{aligned}
\theta &\le \overline{u}(t) - \underline{u}(t) = Q\overline{u}(t) - Q\underline{u}(t) \\
&= \int_0^t \widetilde{T}_\alpha(t-s)\left[f(s, \overline{u}(s), G\overline{u}(s)) - f(s, \underline{u}(s), G\underline{u}(s)) + C(\overline{u}(s) - \underline{u}(s))\right] ds \\
&\le \int_0^t \widetilde{T}_\alpha(t-s)\left[(S_1 + C)(\overline{u}(s) - \underline{u}(s)) + S_2(G\overline{u}(s) - G\underline{u}(s))\right] ds \\
&\le \widetilde{M}_T(S_1 + C + S_2K_0T)\int_0^t \overline{u}(s) - \underline{u}(s) ds.
\end{aligned} \tag{3.23}$$

By (3.23) and the normality of the positive cone P, for $t \in I$, we obtain that

$$\|\overline{u}(s) - \underline{u}(s)\| \le N\widetilde{M}_T(S_1 + C + S_2K_0T)\int_0^t \|\overline{u}(s) - \underline{u}(s)\| ds. \tag{3.24}$$

By the Gronwall inequality, then $\underline{u}(t) \equiv \overline{u}(t)$ on I. Hence $\underline{u} = \overline{u}$ is the the unique mild solution of the Cauchy problem (1.1) on $[v_0, w_0]$. By the proof of Theorem 3.1, we know that it can be obtained by a monotone iterative procedure starting from v_0 or w_0. □

By Corollaries 3.3 and 3.4, and Theorem 3.5, we have the following results.

Corollary 3.6. *Let $f : I \times X \times X \to X$ be continuous, and let $A : D(A) \subset X \to X$ be a linear closed densely defined operator. Assume that $-A \in \mathcal{C}^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.1) *has a lower solution $v_0 \in C(I,X)$ and an upper solution $w_0 \in C(I,X)$ with $v_0 \le w_0$, (H_1) and (H_3) hold, and one of the following conditions is satisfied:*

(i) *X is an ordered Banach space, whose positive* cone *P is regular;*

(ii) *X is an ordered and weakly sequentially complete Banach space, whose positive* cone *P is normal with normal constant N.*

Then the Cauchy problem (1.1) *has the unique mild solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 or w_0.*

Next, we consider the existence and uniqueness results of the Cauchy problem (1.3). Substituting $f(t,,u(t))$ for $f(t,u(t),Gu(t))$ in Theorem 3.1, Corollaries 3.3 and 3.4, and Theorem 3.5, we can obtain the following results.

Corollary 3.7. *Let X be an ordered Banach space, whose positive* cone *P is normal with normal constant N, $f : I \times X \to X$ is continuous, $A : D(A) \subset X \to X$ is a linear closed densely defined operator. Assume $-A \in \mathcal{C}^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.3) *has a lower solution $\tilde{v}_0 \in C(I,X)$ and an upper solution $\tilde{w}_0 \in C(I,X)$ with $\tilde{v}_0 \le \tilde{w}_0$, and the following conditions are satisfied.*

$(\widetilde{H}_1)$ *There exists a constant $\tilde{C} \ge 0$ such that*

$$f(t,x_2) - f(t,x_1) \ge -\tilde{C}(x_2 - x_1), \tag{3.25}$$

for any $t \in I$, and $\tilde{v}_0(t) \le x_1 \le x_2 \le \tilde{w}_0(t)$;

$(\widetilde{H}_2)$ *There exists a constant $\tilde{L} \ge 0$ such that*

$$\mu(\{f(t,x_n)\}) \le \tilde{L}\mu(\{x_n\}), \tag{3.26}$$

for any $t \in I$, and increasing or decreasing monotonic sequence $\{x_n\} \subset [\tilde{v}_0(t), \tilde{w}_0(t)]$.

Then the Cauchy problem (1.3) *has the minimal and maximal mild solutions between $\tilde{v}_0$ and $\tilde{w}_0$, which can be obtained by a monotone iterative procedure starting from $\tilde{v}_0$ and $\tilde{w}_0$, respectively.*

Corollary 3.8. *Let X be an ordered Banach space, whose positive* cone *P is regular, $f : I \times X \to X$ is continuous, and $A : D(A) \subset X \to X$ is a linear closed densely defined operator. Assume that $-A \in \mathcal{C}^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.3) *has a lower solution $\tilde{v}_0 \in C(I,X)$ and an upper solution $\tilde{w}_0 \in C(I,X)$ with $\tilde{v}_0 \le \tilde{w}_0$, and $(\widetilde{H}_1)$ holds. Then the Cauchy problem* (1.3) *has the minimal and maximal mild solutions between $\tilde{v}_0$ and $\tilde{w}_0$, which can be obtained by a monotone iterative procedure starting from $\tilde{v}_0$ and $\tilde{w}_0$, respectively.*

Corollary 3.9. *Let X be an ordered and weakly sequentially complete Banach space, whose positive* cone *P is normal with normal constant N, $f : I \times X \to X$ is continuous, and $A : D(A) \subset X \to X$ is a linear closed densely defined operator. Assume that $-A \in \mathcal{C}^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.3) *has a lower solution $\tilde{v}_0 \in C(I,X)$ and*

an upper solution $\widetilde{w}_0 \in C(I,X)$ with $\widetilde{v}_0 \le \widetilde{w}_0$, and $(\widetilde{H}_1)$ holds. Then the Cauchy problem (1.3) *has the minimal and maximal mild solutions between $\widetilde{v}_0$ and $\widetilde{w}_0$, which can be obtained by a monotone iterative procedure starting from $\widetilde{v}_0$ and $\widetilde{w}_0$, respectively.*

Corollary 3.10. *Let X be an ordered Banach space, whose positive* cone *P is normal with normal constant N, $f : I \times X \to X$ is continuous, and $A : D(A) \subset X \to X$ is a linear closed densely defined operator. Assume that $-A \in \mathcal{C}^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.3) *has a lower solution $\widetilde{v}_0 \in C(I,X)$ and an upper solution $\widetilde{w}_0 \in C(I,X)$ with $\widetilde{v}_0 \le \widetilde{w}_0$, $(\widetilde{H}_1)$ holds, and the following condition is satisfied.*

$(\widetilde{H}_3)$ *There exists a constant $\widetilde{S}_1 \ge 0$ such that*

$$f(t,x_2) - f(t,x_1) \le \widetilde{S}_1(x_2 - x_1), \tag{3.27}$$

for any $t \in I$, and $\widetilde{v}_0(t) \le x_1 \le x_2 \le \widetilde{w}_0(t)$.
Then the Cauchy problem (1.3) *has the unique mild solution between $\widetilde{v}_0$ and $\widetilde{w}_0$, which can be obtained by a monotone iterative procedure starting from $\widetilde{v}_0$ or $\widetilde{w}_0$.*

By Corollaries 3.8, 3.9, and 3.10, we have the following results.

Corollary 3.11. *Let $f : I \times X \to X$ be continuous, and let $A : D(A) \subset X \to X$ be a linear closed densely defined operator. Assume that $-A \in \mathcal{C}^{\alpha}(M,\omega)$, $S_{\alpha}(t)$ is the positive solution operator generated by $-A$, the Cauchy problem* (1.3) *has a lower solution $\widetilde{v}_0 \in C(I,X)$ and an upper solution $\widetilde{w}_0 \in C(I,X)$ with $\widetilde{v}_0 \le \widetilde{w}_0$, $(\widetilde{H}_1)$ and $(\widetilde{H}_3)$ hold, and one of the following conditions is satisfied:*

(i) *X is an ordered Banach space, whose positive* cone *P is regular;*

(ii) *X is an ordered and weakly sequentially complete Banach space, whose positive* cone *P is normal with normal constant N.*

Then the Cauchy problem (1.3) *has the unique mild solution between $\widetilde{v}_0$ and $\widetilde{w}_0$, which can be obtained by a monotone iterative procedure starting from $\widetilde{v}_0$ or $\widetilde{w}_0$.*

4. Examples

Example 4.1. In order to illustrate our main results, we consider the Cauchy problem in $X = \mathbb{R}^n$ (n-dimensional Euclidean space and $\|y\| = (\sum_{i=1}^{n} y_i^2)^{1/2}$):

$$\begin{gathered} {}^{C}D_{0+}^{\alpha} u(t) + Au(t) = f(t,u(t),Gu(t)), \quad t \in I = [0,T], \\ u(0) = x \in X, \quad u' = \theta, \end{gathered} \tag{4.1}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative, $1 < \alpha < 2$, $A = (a_{ij})_{n\times n}(a_{ij} \le 0)$ is a real matrix, $f : I \times X \times X \to X$ is continuous, $\theta = (0,0,\ldots,0)$ is the zero element of X, and

$$Gu(t) = \int_0^t K(t,s)u(s)ds \tag{4.2}$$

is a Volterra integral operator with integral kernel $K \in C(\Delta,\mathbb{R}^+)$, $\Delta = \{(t,s) \mid 0 \le s \le t \le T\}$.

For $y = (y_1, y_2, \ldots, y_n)$, and $z = (z_1, z_2, \ldots, z_n)$, we define the partial order $y \leq z \Leftrightarrow y_i \leq z_i (i = 1, 2, \ldots, n)$. Set $P = \{y \in X \mid y \geq \theta\}$; then P is a normal cone in X and normal constant $N = 1$. It is easy to verify that $-A$ generates a uniformly continuous positive cosine operator family $\mathcal{S}_2(t)$:

$$\mathcal{S}_2(t) = \sum_{n=0}^{\infty} \frac{t^{2n}(-A)^n}{2n!}, \qquad t \geq 0. \tag{4.3}$$

By [9, Theorem 3.1], there exist $M \geq 1$ and $\omega \geq 0$ such that $-A \in C^{\alpha}(M, \omega^{2/\alpha})$, and the corresponding solution operator is

$$\mathcal{S}_{\alpha}(t) = \int_0^{\infty} \varphi_{t,\alpha/2}(s) \mathcal{S}_2(s) ds, \quad t > 0, \tag{4.4}$$

where $\varphi_{t,\alpha/2}(s) = t^{-\alpha/2}\Phi_{\alpha/2}(st^{-\alpha/2})$, $\Phi_{\alpha/2}(\tau)$ is a probability density function, $\Phi_{\alpha/2}(\tau) \geq 0$, $\tau > 0$, and $\int_0^{\infty} \Phi_{\alpha/2}(\tau) d\tau = 1$. Thus, $\mathcal{S}_{\alpha}(t)$ is the positive solution operator generated by $-A$. In order to solve the problem (4.1), we give the following assumptions.

(O_1) $x \geq \theta$, $f(t, \theta, \theta) \geq \theta$ for $t \in I$.

(O_2) There exist $x \leq \overline{x} \in X$ such that $A\overline{x} \geq f(t, \overline{x}, G\overline{x})$ for $t \in I$.

(O_3) The partial derivative $f'_u(t, u, v)$ is continuous on any bounded domain and $f'_v(t, u, v)$ has upper bound.

Theorem 4.2. *If (O_1), (O_2), and (O_3) are satisfied, then the problem* (4.1) *has the unique mild solution $u(t)$, and $\theta \leq u \leq \overline{x}$.*

Proof. From (O_1) and (O_2), we obtain that θ is a lower solution of (4.1), and $\overline{x}$ is an upper solution of (4.1). Form (O_3), it is easy to verify that (H_1) and (H_3) are satisfied. Therefore, by Theorem 3.5, the problem (4.1) has the unique solution $u(t)$, and $\theta \leq u \leq \overline{x}$. □

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