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*Research Article*

# Approximation Order for Multivariate Durrmeyer Operators with Jacobi Weights

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Using the equivalence relation between $K$-functional and modulus of smoothness, we establish a strong direct theorem and an inverse theorem of weak type for multivariate Bernstein-Durrmeyer operators with Jacobi weights on a simplex in this paper. We also obtain a characterization for multivariate Bernstein-Durrmeyer operators with Jacobi weights on a simplex. The obtained results not only generalize the corresponding ones for Bernstein-Durrmeyer operators, but also give approximation order of Bernstein-Durrmeyer operators.

## 1. Introduction

Let $S = S_d$ $(d = 1, 2, \ldots)$ be a simplex in $R^d$ defined by

$$S = \left\{ x = (x_1, x_2, \ldots, x_d) : x_i \geq 0,\ i = 1, 2, \ldots, d,\ |x| = \sum_{i=0}^{d} x_i \leq 1 \right\}. \tag{1.1}$$

For $p \geq 1$, we denote by $L^p(S)$ the space of $p$-order Lebesgue integrable functions on $S$ with

$$\|\omega f\|_p = \begin{cases} \left( \int_S |\omega(x) f(x)|^p dx \right)^{1/p} < \infty & 1 \leq p < +\infty, \\ \max_{x \in S} |\omega(x) f(x)| & p = +\infty, \end{cases} \tag{1.2}$$

where $L^{\infty}(S) = C(S)$ denote the space of continuous functions on $S$. For $f \in L(S)$, the multivariate *Bernstein-Durrmeyer* Operators with $d$ variables on $S$ are given by

$$M_{n,d}(f;x) = \sum_{|k|\le n} P_{n\cdot k}(x)\frac{(n+d)!}{n!}\int_S P_{n,k}(u)f(u)du, \tag{1.3}$$

where $P_{n,k}(x) = (n!/(k!(n-|k|)!))x^k(1-|x|)^{n-|k|}$ $(x \in S)$ and $x = (x_1, x_2, \ldots, x_d) \in R^d$, $k = (k_1, k_2, \ldots, k_d) \in N_0^d$, with the convention

$$|x| = \sum_{i=1}^{d} x_i, \quad x^k = x_1^{k_1}x_2^{k_2}\cdots x_d^{k_d}, \quad |k| = \sum_{i=1}^{d} k_i, \quad k! = k_1!k_2!\cdots k_d!. \tag{1.4}$$

For multivariate Jacobi weights $\omega(x) = x^{\alpha}(1-|x|)^{\beta}$, $(x \in S, \alpha = (\alpha_1, \ldots, \alpha_d) \in R^d, 0 < \alpha_i, \beta < 1, i = 1, 2, \ldots, d, x^{\alpha} = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_d^{\alpha_d})$. We give some further notations, for $x \in S$, and we write $\varphi_i(x) = \varphi_{ii}(x) = \sqrt{x_i(1-|x|)}$ $(1 \le i \le d)$, $\varphi_{ij}(x) = \sqrt{x_i x_j}$, $(1 \le i < j \le d)$ and

$$\begin{gathered} D_i = D_{ii} = \frac{\partial}{\partial x_i}, \quad 1 \le i \le d, \qquad D_{ij} = D_i - D_j, \quad 1 \le i < j \le d, \\ D_{ij}^r = D_{ij}\left(D_{ij}^{r-1}\right), \quad 1 \le i \le j \le d,\ r \in N, \qquad D^k = D_1^{k_1}D_2^{k_2}\cdots D_d^{k_d}, \quad k \in N_0^d. \end{gathered} \tag{1.5}$$

For $f \in L^p(S)$, the weighted *Sobolev* space is given by

$$\begin{aligned} W_{\phi}^{r,p}(S) = \Big\{ & f \in L^p(S) : \omega f \in L^p(S),\ D^k f \in L_{\mathrm{loc}}\left(\overset{0}{S}\right), \\ & \omega\varphi_{ij}^r D_{ij}^r f \in L^p(S),\ |k| \le r,\ 1 \le i \le j \le d,\ r \in N \Big\}, \\ W_{\phi}^{r,\infty}(S) = \Big\{ & f \in C(S) : \omega f \in C(S),\ f \in C^r\left(\overset{0}{S}\right),\ \omega\varphi_{ij}^r D_{ij}^r f \in C(S),\ 1 \le i \le j \le d,\ r \in N \Big\}, \end{aligned} \tag{1.6}$$

where $\overset{0}{S}$ is the interior of $S$. To characterize the approximation capability of multivariate Bernstein-Durrmeyer operators, we introduce the weighted $K$-functional

$$K_{\varphi}^r(f,t^r)_{\omega} = \inf_{g \in W_{\phi}^{r,p}}\left\{ \|\omega(f-g)\|_p + t^r \sum_{1\le i\le j\le d}\left\|\omega\varphi_{ij}^r D_{ij}^r g\right\|_p \right\} \tag{1.7}$$

and a measure of smoothness of $f$

$$\omega_{\varphi}^r(f,t)_{\omega} = \sup_{0<h\le t}\sum_{1\le i\le j\le d}\left\|\omega\Delta_{h\varphi_{ij}e_{ij}}^r f\right\|_p. \tag{1.8}$$

Since 1967, Durrmeyer introduced Bernstein-Durrmeyer operators, and there are many papers which studied theirproperties [1–7]. In 1991, Zhang studied the characterization of convergence for $M_{n,1}(f;x)$ with Jacobi weights. In 1992, Zhou [5] considered multivariate Bernstein-Durrmeyer operators $M_{n,d}(f;x)$ and obtained a characterization of convergence. In 2002, Xuan et al. studied the equivalent characterization of convergence for $M_{n,d}(f;x)$ with Jacobi weights and obtained the following result.

**Theorem 1.1.** *For $\omega f \in L^p(S)$, $0 < r < 1$, the following results are equivalent:*

(i) $\|\omega(M_{n,d}f - f)\|_p = O(n^{-r})$;

(ii) $K_\varphi^2(f,t)_\omega = O(t^r)$.

In this paper, using the Ditzian-Totik modulus of smoothness, we will give the upper bound and lower bound of approximation function by $M_{n,d}(f;x)$ on simplex. The main results are as follows.

**Theorem 1.2.** *If $\omega f \in L^p(S)$, then*

$$\|\omega(M_{n,d}f - f)\|_p \le C\left\{\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\omega + \frac{\|\omega f\|_p}{n}\right\}. \tag{1.9}$$

*And there exists a positive number $\delta$ $(0 < \delta < 1)$ such that the following inequality is satisfied:*

$$\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right)_\omega \le \frac{C}{n}\sum_{k=1}^{n}\left(\frac{n}{k}\right)^\delta \|\omega(M_{n,d}f - f)\|_\omega. \tag{1.10}$$

Throughout the paper, the letter $C$, appearing in various formulas, denotes a positive constant independent of $n$, $x$, and $f$. Its value may be different at different occurrences, even within the same formula.

From Theorem 1.2, we can easily obtain the following corollary.

**Corollary 1.3.** *If $\omega f \in L^p(S)$, $0 < r < 1$, we has the following equivalent results:*

(i) $\|\omega(M_{n,d}f - f)\|_p = O(n^{-r})$;

(ii) $K_\varphi^2(f,t)_\omega = O(t^r)$;

(iii) $\omega_\varphi^2(f,t)_\omega = O(t^{2r})$.

## 2. Some Lemmas

To prove Theorem 1.2, we will show some lemmas in this section. For the simplex $S$, the transformation $T\colon S \to S^{[10]}$ defined by

$$T(x_1, x_2, \ldots, x_d) = (u_1, u_2, \ldots, u_d), \quad u_l = \begin{cases} x_j & l = j, \\ 1 - |x| & l \ne j \end{cases} \tag{2.1}$$

satisfies $T^2 = I$, and $I$ is the identity operator. So we have

$$\frac{\partial}{\partial u_l} = \frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_j} \quad (l \neq j), \quad \frac{\partial}{\partial u_j} = -\frac{\partial}{\partial x_j},$$
$$M_{n,d}(f;x) = M_{n,d}(f_T;Tx); \qquad M_{n,d}(f;Tx) = M_{n,d}(f_T;x), \tag{2.2}$$

where $f_T(u) = f(Tx)$.

**Lemma 2.1.** *If $\omega f \in L^p(S)$, then*

$$\|\omega M_{n,d} f\|_p \leq \|\omega f\|_p,$$
$$\|\omega(M_{n,d} f - f)\|_p \leq \frac{C}{n}\left(\|\omega f\|_p + \sum_{1\leq i\leq j\leq d}\left\|\omega\varphi_{ij}^2 D_{ij}^2 f\right\|_p\right), \quad f \in W_\phi^{r,p}(S). \tag{2.3}$$

*Proof.* Letting $S' = \{\overline{x} : (x_1, \overline{x}) \in S_d\}$, $\overline{x} = (x_2, x_3, \ldots, x_d)$, $\overline{k} = (k_2, k_3 \ldots, k_d)$, $k = (k_1, \overline{k})$, $P_{n,k_1}(x_1) = (n!/k_1!(n-k_1)!)x_1^{k_1}(1-x_1)^{n-k_1}$, then

$$\begin{aligned}
M_{n,d}(f;x) &= \sum_{k_1=0}^{n} P_{n,k_1}(x_1) \sum_{|\overline{k}|\leq n-k_1} P_{n-k_1,\overline{k}}\left(\frac{\overline{x}}{1-x_1}\right)\frac{(n+d)!}{n!} \\
&\quad \times \int_0^1 P_{n,k_1}(u_1)\int_{S'} P_{n-k_1,\overline{k}}\left(\frac{\overline{u}}{1-u_1}\right) f(u)\,d\overline{u}\,du_1 \\
&= \sum_{k_1=0}^{n} P_{n,k_1}(x_1)\frac{(n+d)!}{n!}\int_0^1 P_{n,k_1}(u_1)(1-u_1)^{d-1} \sum_{|\overline{k}|\leq n-k_1} P_{n-k_1,\overline{k}}\left(\frac{\overline{x}}{1-x_1}\right) \\
&\quad \times \int_{S_{d-1}} P_{n-k_1,\overline{k}}(t) f(u_1, (1-u_1)t)\,dt\,du_1 \\
&= \sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1) M_{n-k_1,d-1}\left(f(u_1,(1-u_1)\cdot); \frac{\overline{x}}{1-x_1}\right)du_1.
\end{aligned} \tag{2.4}$$

Using the transformation $T$, (2.2), (2.4), the method of [7], we can easily get (2.3). □

**Lemma 2.2** (see [8]). *If $f \in L^p(S)$, then*

$$C^{-1}\omega_\varphi^r(f,t)_\omega \leq K_\varphi^r(f,t^r)_\omega \leq C\omega_\varphi^r(f,t)_\omega. \tag{2.5}$$

*Proof.* Lemma 2.2 is proved when $f \in C(S)$ in [8]. Similarly, we can prove $f \in L^p(S)$. □

**Lemma 2.3.** *If $0 < a < 1$, $b > 0$, $x \in (0,1)$, $P_{n,k}(x) = C_n^k x^k (1-x)^{n-k}$ is basis function of the classical Bernstein operators, then*

$$\begin{gathered}\sum_{k=1}^{n-1} P_{n,k}(x)\left(\frac{n}{k}\right)^a \le Cx^{-a},\\ \sum_{k=1}^{n-1} P_{n,k}(x)\left(\frac{n}{n-k}\right)^b \le C(1-x)^{-b}.\end{gathered} \tag{2.6}$$

*Proof.* The first inequality can be inferred by Hölder inequality. In the following we prove the second inequality.

(i) If $0 < b < 1$, using Hölder inequality, we can easily obtain the result.

(ii) If $b \ge 1$, let $b = m + r$, $m \in N$, $0 \le r < 1$, then

$$\begin{aligned}\sum_{k=1}^{n-1} P_{n,k}(x)\left(\frac{n}{n-k}\right)^b &= \sum_{k=1}^{n-1} P_{n,k}(x)\left(\frac{n}{n-k}\right)^m\left(\frac{n}{n-k}\right)^r\\ &\le C(1-x)^{-m}\sum_{k=1}^{n-1} P_{n+m,k}(x)\left(\frac{n+m}{n+m-k}\right)^r\\ &\le C(1-x)^{-m-r} = C(1-x)^{-b}.\end{aligned} \tag{2.7}$$

Lemma 2.3 is completed. □

**Lemma 2.4.** *If $f \in L^p(S)$, $1 \le p \le \infty$, then*

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d} f\right\|_p \le Cn\left\|\omega f\right\|_p \quad 1 \le i \le j \le d. \tag{2.8}$$

*Proof.* In the following we use the induction on the dimension number $d$ to prove the result. The case $d = 1$ was proved by Lemma 4 of [6]. Next, suppose that Lemma 2.4 is valid for $d = r$ $(r \ge 1)$; we prove it is also true for $d = r + 1$. To observe this, we use a decomposition formula (2.4), and we have

$$\begin{aligned}&\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f;x)\\ &\quad= x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d}\\ &\qquad\times(1-|z|)^{\beta}\varphi_{11}^2(z)\int_0^1 P_{n+d-1,k_1}(u_1)D_{11}^2 M_{n-k_1,d-1}\big(f(u_1,(1-u_1)\cdot);z\big)du_1,\end{aligned} \tag{2.9}$$

where $z = (z_1, z_2, \ldots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \ldots, x_d/(1-x_1))$. Thus we have

$$\begin{aligned}
&\int_S \left|\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f;x)\right| ds \\
&\quad \le C\int_0^1 x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)(n-k_1) \\
&\qquad \times \int_{z\in S_{d-1}} \left|\omega(z)f(u_1,(1-u_1)z)\right| dz\, dx_1 du_1 \\
&\quad \le C\frac{n+d}{n+1}n\int_0^1 \sum_{k_1=0}^{n}\left(\frac{k_1+1}{n+1}\right)^{\alpha_1}\left(\frac{n-k_1+1}{n+1}\right)^{|\bar{\alpha}|+\beta} P_{n+d-1,k_1}(u_1) \\
&\qquad \times \int_{z\in S_{d-1}} \left|\omega(z)f(u_1,(1-u_1)z)\right| dz\, du_1 \\
&\quad \le Cn\int_0^1 u_1^{\alpha_1}(1-u_1)^{|\bar{\alpha}|+\beta}\left(\frac{1}{u_1}\right)^{\alpha_1}(1-u_1)^{-|\bar{\alpha}|-\beta}\int_{z\in S_{d-1}} \left|(\omega f)(u_1,(1-u_1)z)\right| dz\, du_1 \\
&\quad = Cn\|\omega f\|_1.
\end{aligned} \tag{2.10}$$

In the above derivation, we have used the formula [6]

$$\int_0^1 x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}P_{n,k_1}(x_1)dx_1 \le C\frac{1}{n+1}\left(\frac{k_1+1}{n+1}\right)^{\alpha_1}\left(\frac{n-k_1+1}{n+1}\right)^{|\bar{\alpha}|+\beta} \tag{2.11}$$

and the inequality

$$\sum_{k_1=0}^{n}\left(\frac{k_1+1}{n+1}\right)^{\alpha_1}\left(\frac{n-k_1+1}{n+1}\right)^{|\bar{\alpha}|+\beta} P_{n+d-1,k_1}(u_1) \le Cu_1^{\alpha_1}(1-u_1)^{|\bar{\alpha}|+\beta}. \tag{2.12}$$

When $p = \infty$, we have

$$\begin{aligned}
&\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f;x) \\
&\quad = x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d} \\
&\qquad \times (1-|z|)^{\beta}\varphi_{11}^2(z)\int_0^1 P_{n+d-1,k_1}(u_1)D_{11}^2 M_{n-k_1,d-1}\left(f(u_1,(1-u_1)\cdot);z\right)du_1,
\end{aligned} \tag{2.13}$$

where $z = (z_1, z_2, \ldots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \ldots, x_d/(1-x_1))$.

From the Cauchy-Swartz inequality, Hölder inequality, and Lemma 2.3, we have

$$
\begin{aligned}
&\left|\omega(x)\varphi_{22}^{2}(x)D_{22}^{2}M_{n,d}(f;x)\right| \\
&\quad\le Cx_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n}P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)(n-k_1) \\
&\qquad\times\max_{z\in S_{d-1}}\left|z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d}(1-|z|)^{\beta}f(u_1,(1-u_1)z)\right|du_1 \\
&\quad\le Cn\|\omega f\|_{\infty}x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n}P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)\left(\frac{1}{u_1}\right)^{\alpha_1}(1-u_1)^{-|\bar{\alpha}|-\beta}du_1 \\
&\quad\le Cn\|\omega f\|_{\infty}x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n}P_{n,k_1}(x_1)(n+d)\left(\int_0^1 P_{n+d-1,k_1}(u_1)u_1^{-2\alpha_1}du_1\right)^{1/2} \\
&\qquad\times\left(\int_0^1 P_{n+d-1,k_1}(u_1)(1-u_1)^{-2|\bar{\alpha}|-2\beta}du_1\right)^{1/2} \\
&\quad\le Cn\|\omega f\|_{\infty}x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n}P_{n,k_1}(x_1)(n+d)\left(\int_0^1 P_{n+d-1,k_1}(u_1)u_1^{-2}du_1\right)^{\alpha_1/2} \\
&\qquad\times\left(\int_0^1 P_{n+d-1,k_1}(u_1)du_1\right)^{(1-\alpha_1)/2}\left(\int_0^1 P_{n+d-1,k_1}(u_1)(1-u_1)^{-2d}du_1\right)^{(|\bar{\alpha}|+\beta)/2d} \\
&\qquad\times\left(\int_0^1 P_{n+d-1,k_1}(u_1)du_1\right)^{1/2-(|\bar{\alpha}|+\beta)/2d} \\
&\quad\le Cn\|\omega f\|_{\infty}x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=2}^{n}P_{n,k_1}(x_1)(n+d)\left(\frac{n+d-1}{k_1(k_1-1)}\right)^{\alpha_1/2} \\
&\qquad\times\left(\frac{(n+d-1)!}{(n-d)!}\frac{(n-d-k_1-1)!}{(n+d-k_1-1)!}\right)^{(|\bar{\alpha}|+\beta)/2d}\left((n+d)^{1-(|\bar{\alpha}|+\beta)/2d-\alpha_1/2}\right)^{-1} \\
&\quad\le Cn\|\omega f\|_{\infty}x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=1}^{n-1}P_{n,k_1}(x_1)\left(\frac{n}{k_1}\right)^{\alpha_1}\left(\frac{n}{n-k_1}\right)^{|\bar{\alpha}|+\beta} \\
&\quad\le Cn\|\omega f\|_{\infty}.
\end{aligned}
\tag{2.14}
$$

By *Riesz* interpolation theorem, we get

$$
\left\|\omega\varphi_{22}^{2}D_{22}^{2}M_{n,d}f\right\|_p\le Cn\|\omega f\|_p. \tag{2.15}
$$

Similarly, the other cases for $i = 1, 3, 4, \ldots, d(=j)$ can be proved. For $i \neq j$, by the transformation $T$, we have

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d} f\right\|_p = \left\|\omega_T \varphi_{jj}^2 D_{jj}^2 M_{n,d} f_T\right\|_p \leq Cn\left\|\omega_T f_T\right\|_p = Cn\left\|\omega f\right\|_p. \tag{2.16}$$

Lemma 2.4 is completed. □

**Lemma 2.5.** *If $f \in W_\phi^{r,p}(S) \subset L^p(S)$, $1 \leq p \leq \infty$, then*

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d} f\right\|_p \leq C\left\|\omega\varphi_{ij}^2 D_{ij}^2 f\right\|_p \quad 1 \leq i \leq j \leq d. \tag{2.17}$$

*Proof.* We use the induction on the dimension number $d$ to prove Lemma 2.5. The case $d = 1$ was proved by Lemma 3 of [6], that is,

$$\left\|\omega\varphi^2 D^2 M_{n,1} f\right\|_p \leq C\left\|\omega\varphi^2 D^2 f\right\|_p. \tag{2.18}$$

Next, suppose that Lemma 2.5 is valid for $d = r$ $(r \geq 1)$, and we prove it is also true for $d = r + 1$. Noticing formula (2.4), we have

$$\begin{aligned}
&\omega(x)\varphi_{22}^2(x) D_{22}^2 M_{n,d}(f;x) \\
&\quad = x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta} \sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d) z_1^{\alpha_2} z_2^{\alpha_3} \cdots z_{d-1}^{\alpha_d} \\
&\qquad \times (1-|z|)^\beta \varphi_{11}^2(z) \int_0^1 P_{n+d-1,k_1}(u_1) D_{11}^2 M_{n-k_1,d-1}\big(f(u_1,(1-u_1)\cdot);z\big)\,du_1,
\end{aligned} \tag{2.19}$$

where $z = (z_1, z_2, \ldots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \ldots, x_d/(1-x_1))$. When $p = 1$, from the inductive assumption of $p = 1$, we have

$$\begin{aligned}
&\int_S \left|\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f;x)\right| ds \\
&\quad \le C\int_0^1 x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1) \\
&\qquad \times \int_{z\in S_{d-1}} \left|\omega(z)\varphi_{11}^2(z)D_{11}^2 f(u_1,(1-u_1)z)\right| dz\, dx_1\, du_1 \\
&\quad \le C\frac{n+d}{n+1}\int_0^1 \sum_{k_1=0}^{n}\left(\frac{k_1+1}{n+1}\right)^{\alpha_1}\left(\frac{n-k_1+1}{n+1}\right)^{|\bar{\alpha}|+\beta} P_{n+d-1,k_1}(u_1) \\
&\qquad \times \int_{z\in S_{d-1}} \left|\omega(z)\varphi_{11}^2(z)D_{11}^2 f(u_1,(1-u_1)z)\right| dz\, du_1 \\
&\quad \le C\int_0^1 u_1^{\alpha_1}(1-u_1)^{|\bar{\alpha}|+\beta}\left(\frac{1}{u_1}\right)^{\alpha_1}(1-u_1)^{-|\bar{\alpha}|-\beta}\int_{z\in S_{d-1}} \left|\left(\omega\varphi_{22}^2 D_{22}^2 f\right)(u_1,(1-u_1)z)\right| dz\, du_1 \\
&\quad \le C\left\|\omega\varphi_{22}^2 D_{22}^2 f\right\|_1.
\end{aligned} \tag{2.20}$$

When $p = \infty$, we have

$$\begin{aligned}
&\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f;x) \\
&\quad = x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d} \\
&\qquad \times (1-|z|)^{\beta}\varphi_{11}^2(z)\int_0^1 P_{n+d-1,k_1}(u_1)D_{11}^2 M_{n-k_1,d-1}\left(f(u_1,(1-u_1)\cdot);z\right)du_1,
\end{aligned} \tag{2.21}$$

where $z = (z_1, z_2, \ldots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \ldots, x_d/(1-x_1))$. From the inductive assumption, the Cauchy-Swartz inequality, Holder inequality, and Lemma 2.4, we get

$$\begin{aligned}
&\left|\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f;x)\right| \\
&\quad \le Cx_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1) \\
&\qquad \times \max_{z\in S_{d-1}}\left|z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d}(1-|z|)^{\beta}\varphi_z^2 D_z^2 f(u_1,(1-u_1)z)\right| du_1
\end{aligned}$$

$$\begin{aligned}
&\leq C x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)x_1^{-\alpha_1}(1-x_1)^{-|\bar{\alpha}|-\beta} \\
&\quad\times \left\|\omega\varphi_{22}^2 D_{22}^2 f\right\|_\infty du_1 \\
&\leq C\left\|\omega\varphi_{22}^2 D_{22}^2 f\right\|_\infty.
\end{aligned} \tag{2.22}$$

By *Riesz* interpolation theorem, we get

$$\left\|\omega\varphi_{22}^2 D_{22}^2 M_{n,d} f\right\|_p \leq C\left\|\omega\varphi_{22}^2 D_{22}^2 f\right\|_p. \tag{2.23}$$

Similarly, the other cases for $i = 1, 3, 4, \ldots, d(=j)$ can be proved. For $i \neq j$, by the transformation $T$, we have

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d} f\right\|_p = \left\|\omega_T\varphi_{jj}^2 D_{jj}^2 M_{n,d} f_T\right\|_p \leq C\left\|\omega_T\varphi_{jj}^2 D_{jj}^2 f_T\right\|_p \leq C\left\|\omega\varphi_{ij}^2 D_{ij}^2 f\right\|_p \tag{2.24}$$

Lemma 2.5 is completed. □

**Lemma 2.6** (see [9]). *Let $\{\sigma_n\}$, $\{\phi_n\}$ be nonnegative sequences ($\sigma_1 = 0$, $n \in N$). For $l > 0$, if the sequences $\{\sigma_n\}$, $\{\phi_n\}$ satisfy*

$$\sigma_n \leq Q\left(\frac{k}{n}\right)^l \sigma_k + \phi_k \quad (Q \geq 1,\ 1 \leq k \leq n,\ n \in N), \tag{2.25}$$

*one has*

$$\sigma_n \leq M n^{-s}\sum_{k=1}^{n} k^{s-1}\phi_k. \tag{2.26}$$

*If $Q = 1$, then $l = s$. If $Q > 1$, then $0 < s < l$.*

## 3. The Proof of Theorems

Now we prove (1.9) of Theorem 1.2. By using Lemma 2.1, for arbitrary $g \in W_{\phi}^{r,p}(S) \subset L^p(S)$, we have

$$\begin{aligned}\|\omega(M_{n,d}f - f)\|_p &\le C\left(\|\omega M_{n,d}(f-g)\|_p + \|\omega M_{n,d}g - \omega g\|_p + \|\omega(f-g)\|_p\right)\\ &\le C\left(\|\omega(f-g)\|_p + \frac{1}{n}\left(\sum_{1\le i\le j\le d}\left\|\omega\varphi_{ij}^2 D_{ij}^2 g\right\|_p + \|\omega g\|_p\right)\right)\\ &\le C\left(\|\omega(f-g)\|_p + \frac{1}{n}\sum_{1\le i\le j\le d}\left\|\omega\varphi_{ij}^2 D_{ij}^2 g\right\|_p + \frac{1}{n}\|\omega f\|_p\right).\end{aligned} \tag{3.1}$$

Hence, from Lemma 2.2, we obtain

$$\begin{aligned}\|\omega(M_{n,d}f - f)\|_p &\le C\left(K_{\varphi}^2\left(f, \frac{1}{n}\right)_{\omega} + \frac{1}{n}\|\omega f\|_p\right)\\ &\le C\left(\omega_{\varphi}^2(f,t)_{\omega} + \frac{1}{n}\|\omega f\|_p\right).\end{aligned} \tag{3.2}$$

Next, we prove (1.10) of Theorem 1.2. Leting $\sigma_n = C(1/n)\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d}(f)\|_p$ $(1 \le i \le j \le d)$, $\phi_n = C\|\omega(M_{n,d}(f) - f)\|_p$, then $\sigma_1 = 0$. By Lemmas 2.4 and 2.5, we have

$$\begin{aligned}\sigma_n &\le C\frac{1}{n}\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d}(f - M_{k,d}f)\right\|_p + C\frac{1}{n}\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d}M_{k,d}f\right\|_p\\ &\le C\|\omega(f - M_{k,d}f)\|_p + C\frac{1}{n}\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{k,d}f\right\|_p\\ &= C\frac{k}{n}\sigma_k + \phi_k \quad (C > 1).\end{aligned} \tag{3.3}$$

Using Lemma 2.6, we get $\sigma_n \le C(1/n)\sum_{k=1}^{n}(n/k)^{\delta}\phi_k$ $(0 < \delta < 1)$. That is,

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d}(f)\right\|_p \le C\sum_{k=1}^{n}\left(\frac{n}{k}\right)^{\delta}\|\omega(M_{k,d}f - f)\|_p. \tag{3.4}$$

When $n \ge 2$, there exists $(m \in N)$ such that $n/2 \le m \le n$ and satisfies the equation

$$\|\omega(M_{m,d}f - f)\|_p = \min_{n/2\le k\le n}\|\omega(M_{k,d}f - f)\|_p. \tag{3.5}$$

Thus,

$$\|\omega(M_{m,d}f - f)\|_p \le \frac{2}{n}\sum_{n/2\le k\le n}\|\omega(M_{k,d}f - f)\|_p. \tag{3.6}$$

Using Lemma 2.2, we have

$$\begin{aligned}\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right)_\omega &\le CK_\varphi^2\left(f,\frac{1}{n}\right)\\ &\le C\left(\left\|\omega(M_{m,d}f-f)\right\|_p+\frac{1}{n}\sum_{1\le i\le j\le d}\left\|\omega\varphi_{ij}^2D_{ij}^2M_{m,d}f\right\|_p\right)\\ &\le C\frac{1}{n}\sum_{k=1}^{n}\left(\frac{n}{k}\right)^{\delta}\left\|\omega(M_{k,d}f-f)\right\|_p.\end{aligned}\tag{3.7}$$

Theorem 1.2 is completed.

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