Research Article

Approximation Order for Multivariate Durrmeyer Operators with Jacobi Weights

Jianjun Wang,1 Chan-Yun Yang,2 and Shukai Duan3

1 School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
2 Department of Mechanical Engineering, Technology and Science Institute of Northern Taiwan, No. 2 Xue-Yuan Road, Beitou, Taipei 112, Taiwan
3 School of Electronics and Information Engineering, Southwest University, Chongqing 400715, China

Correspondence should be addressed to Jianjun Wang, wjj@swu.edu.cn

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Using the equivalence relation between $K$-functional and modulus of smoothness, we establish a strong direct theorem and an inverse theorem of weak type for multivariate Bernstein-Durrmeyer operators with Jacobi weights on a simplex in this paper. We also obtain a characterization for multivariate Bernstein-Durrmeyer operators with Jacobi weights on a simplex. The obtained results not only generalize the corresponding ones for Bernstein-Durrmeyer operators, but also give approximation order of Bernstein-Durrmeyer operators.

1. Introduction

Let $S = S_d$ ($d = 1, 2, \ldots$) be a simplex in $R^d$ defined by

$$S = \left\{ x = (x_1, x_2, \ldots, x_d) : x_i \geq 0, \ i = 1, 2, \ldots, d, \ |x| = \sum_{i=0}^{d} x_i \leq 1 \right\}.$$  \hfill (1.1)

For $p \geq 1$, we denote by $L^p(S)$ the space of $p$-order Lebesgue integrable functions on $S$ with

$$\|\omega f\|_p = \begin{cases} \left(\int_S |\omega(x)f(x)|^p dx\right)^{1/p} < \infty \quad 1 \leq p < +\infty, \\ \max_{x \in S} |\omega(x)f(x)| \quad p = +\infty, \end{cases} \hfill (1.2)$$
where \( L^\infty(S) = C(S) \) denote the space of continuous functions on \( S \). For \( f \in L(S) \), the multivariate Bernstein-Durrmeyer Operators with \( d \) variables on \( S \) are given by

\[
M_{n,d}(f; x) = \sum_{|k|\leq n} P_n(x) \frac{(n + d)!}{n!} \int_S P_{n,k}(u) f(u) du,
\]

(1.3)

where \( P_{n,k}(x) = (n!/(k!(n-|k|)!)) x^k (1-|x|)^{n-|k|} \) (\( x \in S \)) and \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \), \( k = (k_1, k_2, \ldots, k_d) \in N_0^d \), with the convention

\[
|x| = \sum_{i=1}^d x_i, \quad x^k = x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, \quad |k| = \sum_{i=1}^d k_i, \quad k! = k_1! k_2! \cdots k_d!.
\]

(1.4)

For multivariate Jacobi weights \( \omega(x) = x^a (1-|x|)^\beta, (x \in S, a = (a_1, \ldots, a_d) \in \mathbb{R}^d, 0 < a_i, \beta < 1, i = 1, 2, \ldots, d, x^a = x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}) \). We give some further notations, for \( x \in S \), and we write \( \phi_i(x) = \phi_{ii}(x) = \sqrt{x_i(1-|x|)} \) (\( 1 \leq i \leq d \)), \( \phi_{ij}(x) = \sqrt{x_i x_j} \) (\( 1 \leq i < j \leq d \)) and

\[
D_i = D_{ii}, \quad 1 \leq i \leq d, \quad D_{ij} = D_i - D_j, \quad 1 \leq i < j \leq d,
\]

\[
D^r_{ij} = D_{ij} \left( D^{r-1}_{ij} \right), \quad 1 \leq i < j \leq d, \quad r \in \mathbb{N}, \quad D^k = D_{i_1}^{k_1} D_{i_2}^{k_2} \cdots D_{i_d}^{k_d}, \quad k \in N_0^d.
\]

(1.5)

For \( f \in L^p(S) \), the weighted Sobolev space is given by

\[
W^r_{\phi}(S) = \left\{ f \in L^p(S) : \omega f \in L^p(S), D^k f \in L^p_{\text{loc}} \left( S \right), \right. \\
\left. \omega \phi^r_{ij} D^r_{ij} f \in L^p(S), |k| \leq r, 1 \leq i < j \leq d, r \in \mathbb{N} \right\},
\]

\[
W^{r,\infty}_{\phi}(S) = \left\{ f \in C(S) : \omega f \in C(S), f \in C^r \left( S \right), \omega \phi^r_{ij} D^r_{ij} f \in C(S), 1 \leq i < j \leq d, r \in \mathbb{N} \right\},
\]

(1.6)

where \( S \) is the interior of \( S \). To characterize the approximation capability of multivariate Bernstein-Durrmeyer operators, we introduce the weighted \( K \)-functional

\[
K^r_{\phi}(f, t') = \inf \left\{ \omega(f - g)_{L^p} + t \sum_{1 \leq i < j \leq d} \omega \phi^r_{ij} D^r_{ij} g_{L^p} : g \right\}
\]

(1.7)

and a measure of smoothness of \( f \)

\[
\omega^r_{\phi}(f, t) = \sup \sum_{0 \leq h \leq 1} \left\| \omega \Delta^r_{h \phi_i, e_i} f \right\|_{L^p}.
\]

(1.8)
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Since 1967, Durrmeyer introduced Bernstein-Durrmeyer operators, and there are many papers which studied their properties [1–7]. In 1991, Zhang studied the characterization of convergence for \( M_{n,1}(f; x) \) with Jacobi weights. In 1992, Zhou [5] considered multivariate Bernstein-Durrmeyer operators \( M_{n,d}(f; x) \) and obtained a characterization of convergence. In 2002, Xuan et al. studied the equivalent characterization of convergence for \( M_{n,d}(f; x) \) with Jacobi weights and obtained the following result.

**Theorem 1.1.** For \( \omega f \in L^p(S) \), \( 0 < r < 1 \), the following results are equivalent:

1. \( \| \omega(M_{n,d}f - f) \|_p = O(n^{-r}) \);
2. \( K^2_\omega(f, t)_{\omega} = O(t^r) \).

In this paper, using the Ditzian-Totik modulus of smoothness, we will give the upper bound and lower bound of approximation function by \( M_{n,d}(f; x) \) on simplex. The main results are as follows.

**Theorem 1.2.** If \( \omega f \in L^p(S) \), then

\[
\| \omega(M_{n,d}f - f) \|_p \leq C \left\{ \omega^2_\frac{2}{p} \left( f, \frac{1}{\sqrt{n}} \right)_\omega + \frac{\| \omega f \|_p}{n} \right\}. \tag{1.9}
\]

And there exists a positive number \( \delta \) \( (0 < \delta < 1) \) such that the following inequality is satisfied:

\[
\omega^2_\frac{2}{p} \left( f, \frac{1}{\sqrt{n}} \right)_\omega \leq \frac{C}{n} \sum_{k=1}^{n} \left( \frac{n}{k} \right)^\delta \| \omega(M_{n,d}f - f) \|_\omega. \tag{1.10}
\]

Throughout the paper, the letter \( C \), appearing in various formulas, denotes a positive constant independent of \( n, x, \) and \( f \). Its value may be different at different occurrences, even within the same formula.

From Theorem 1.2, we can easily obtain the following corollary.

**Corollary 1.3.** If \( \omega f \in L^p(S) \), \( 0 < r < 1 \), we has the following equivalent results:

1. \( \| \omega(M_{n,d}f - f) \|_p = O(n^{-r}) \);
2. \( K^2_\omega(f, t)_{\omega} = O(t^r) \);
3. \( \omega^2_\frac{2}{p} (f, t)_{\omega} = O(t^{2r}). \)

2. **Some Lemmas**

To prove Theorem 1.2, we will show some lemmas in this section. For the simplex \( S \), the transformation \( T: S \to S^{[10]} \) defined by

\[
T(x_1, x_2, \ldots, x_d) = (u_1, u_2, \ldots, u_d), \quad u_l = \begin{cases} x_l & l = j, \\ 1 - |x| & l \neq j \end{cases} \tag{2.1}
\]
satisfies $T^2 = I$, and $I$ is the identity operator. So we have

$$
\frac{\partial}{\partial u_l} - \frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_j} (l \neq j), \quad \frac{\partial}{\partial u_j} = -\frac{\partial}{\partial x_j},
$$

(2.2)

$$
M_{n,d}(f;x) = M_{n,d}(fT;x), \quad M_{n,d}(fT;x) = M_{n,d}(f;x),
$$

where $f_T(u) = f(Tu)$.

**Lemma 2.1.** If $\omega f \in L^p(S)$, then

$$
\|\omega M_{n,d}f\|_p \leq \|\omega f\|_p
$$

(2.3)

$$
\|\omega(M_{n,d}f - f)\|_p \leq \frac{C}{n}\left(\|\omega f\|_p + \sum_{1 \leq i \leq d} \|\omega \nabla_i^2 D_{ij}^2 f\|_p\right), \quad f \in W^{r,p}_\phi(S).
$$

**Proof.** Letting $S' = \{\overline{x} : (x_1, \overline{x}) \in S_d\}$, $\overline{x} = (x_2, x_3, \ldots, x_d)$, $\overline{k} = (k_2, k_3, \ldots, k_d)$, $k = (k_1, \overline{k})$, $P_{n,k_1}(x_1) = (n!/k_1!(n-k_1)!)x_1^{k_1}(1-x_1)^{n-k_1}$, then

$$
M_{n,d}(f;x) = \sum_{k_1=0}^n P_{n,k_1}(x_1) \sum_{|\overline{k}| \leq n-k_1} P_{n-k_1,\overline{k}}(\overline{x}) \frac{(n+d)!}{n!} \\
\times \int_0^1 P_{n,k_1}(u_1) \int_{S'} P_{n-k_1,\overline{k}}(\overline{u}) f(u)d\overline{u} du_1
$$

$$
= \sum_{k_1=0}^n P_{n,k_1}(x_1) \frac{(n+d)!}{n!} \int_0^1 P_{n,k_1}(u_1)(1-u_1)^{d-1} \sum_{|\overline{k}| \leq n-k_1} P_{n-k_1,\overline{k}}(\overline{x}) \\
\times \int_{S_{d-1}} P_{n-k_1,\overline{k}}(t) f(u_1, (1-u_1)t)dt du_1
$$

$$
= \sum_{k_1=0}^n P_{n,k_1}(x_1)(n+d) \int_0^1 P_{n+d-1,k_1}(u_1)M_{n-k_1,d-1}(f(u_1, (1-u_1)^t); \overline{x}) du_1.
$$

(2.4)

Using the transformation $T$, (2.2), (2.4), the method of [7], we can easily get (2.3). 

**Lemma 2.2** (see [8]). If $f \in L^p(S)$, then

$$
C^{-1}\omega_{\phi}^r(f,t)_{\omega} \leq K_{\phi}^r(f,t')_{\omega} \leq C\omega_{\phi}^r(f,t)_{\omega}.
$$

(2.5)

**Proof.** Lemma 2.2 is proved when $f \in C(S)$ in [8]. Similarly, we can prove $f \in L^p(S)$. 

$\square$
Lemma 2.3. If $0 < a < 1$, $b > 0$, $x \in (0, 1)$, $P_{n,k}(x) = C^k_n x^k (1-x)^{n-k}$ is basis function of the classical Bernstein operators, then

$$
\sum_{k=1}^{n-1} P_{n,k}(x) \left( \frac{n}{n-k} \right)^a \leq C x^{-a},
$$

(2.6)

$$
\sum_{k=1}^{n-1} P_{n,k}(x) \left( \frac{n}{n-k} \right)^b \leq C (1-x)^{-b}.
$$

Proof. The first inequality can be inferred by Hölder inequality. In the following we prove the second inequality.

(i) If $0 < b < 1$, using Hölder inequality, we can easily obtain the result.

(ii) If $b \geq 1$, let $b = m + r$, $m \in \mathbb{N}$, $0 \leq r < 1$, then

$$
\sum_{k=1}^{n-1} P_{n,k}(x) \left( \frac{n}{n-k} \right)^b = \sum_{k=1}^{n-1} P_{n,k}(x) \left( \frac{n}{n-k} \right)^m \left( \frac{n}{n-k} \right)^r
\leq C (1-x)^{-m} \sum_{k=1}^{n-1} P_{n+m,k}(x) \left( \frac{n+m}{n+m-k} \right)^r
\leq C (1-x)^{-m-r} = C (1-x)^{-b}.
$$

Lemma 2.3 is completed.

Lemma 2.4. If $f \in L^p(S)$, $1 \leq p \leq \infty$, then

$$
\left\| \omega q_{ij} D_{ij}^2 M_{n,d} f \right\|_p \leq C n \left\| \omega f \right\|_p \quad 1 \leq i \leq j \leq d.
$$

(2.8)

Proof. In the following we use the induction on the dimension number $d$ to prove the result. The case $d = 1$ was proved by Lemma 4 of [6]. Next, suppose that Lemma 2.4 is valid for $d = r$ ($r \geq 1$); we prove it is also true for $d = r + 1$. To observe this, we use a decomposition formula (2.4), and we have

$$
\omega(x) q_{i_2}^2(x) D_{i_2}^2 M_{n,d}(f; x)
$$

$$
= x_1^{a_1} (1-x_1)^{a_1+\beta} \sum_{k_1=0}^{n+1} P_{n,k_1}(x_1)(n+d) z_1^{a_1} z_2^{a_2} \cdots z_{d-1}^{a_{d-1}}
$$

(2.9)

$$
\times (1-|z|)^{a_1} \int_0^1 P_{n+d-1,k_1}(u_1) D_{i_1}^2 M_{n-k_1,d-1}(f(u_1, (1-u_1)\cdot); z) du_1,
$$
where \( z = (z_1, z_2, \ldots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \ldots, x_d/(1-x_1)) \). Thus we have

\[
\int_S \left| \omega(x) \phi^2_{22}(x) D^2_{22} M_{n,d}(f; x) \right| ds \\
\leq C \int_0^1 x_1^{a_1}(1-x_1)^{\bar{\alpha}+\beta} \sum_{k_i=0}^n P_{n,k_i}(x_1)(n+d) \int_0^1 P_{n+d-1,k_i}(u_1)(n-k_1) \\
\times \int_{z \in S_{d-1}} |\omega(z) f(u_1, (1-u_1)z)| dz \, dx_1 \, du_1 \\
= C n \|\omega f\|_1.
\]

In the above derivation, we have used the formula [6]

\[
\int_0^1 x_1^{a_1}(1-x_1)^{\bar{\alpha}+\beta} P_{n,k_i}(x_1) dx_1 \leq C \frac{1}{n+1} \left( \frac{k_1+1}{n+1} \right)^{a_1} \left( \frac{n-k_1+1}{n+1} \right)^{\bar{\alpha}+\beta}
\]

and the inequality

\[
\sum_{k_i=0}^n \left( \frac{k_1+1}{n+1} \right)^{a_1} \left( \frac{n-k_1+1}{n+1} \right)^{\bar{\alpha}+\beta} P_{n+d-1,k_i}(u_1) \leq C n^{a_1}(1-u_1)^{\bar{\alpha}+\beta}.
\]

When \( p = \infty \), we have

\[
\omega(x) \phi^2_{22}(x) D^2_{22} M_{n,d}(f; x) \\
= x_1^{a_1}(1-x_1)^{\bar{\alpha}+\beta} \sum_{k_i=0}^n P_{n,k_i}(x_1)(n+d) z_1^{a_1} z_2^{a_1} \cdots z_d^{a_d} \\
\times (1-|z|)^{\delta} \phi^2_{11}(z) \int_0^1 P_{n+d-1,k_i}(u_1) D^2_{11} M_{n-k_i,d-1}(f(u_1, (1-u_1)z); z) \, du_1,
\]

where \( z = (z_1, z_2, \ldots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \ldots, x_d/(1-x_1)) \).
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From the Cauchy-Swartz inequality, Hölder inequality, and Lemma 2.3, we have

\[
\left| \omega(x) \phi_2^2(x) D_{22}^2 M_{n,d}(f; x) \right|
\]

\[
\leq C x_1^{a_1} (1 - x_1)^{|a|+\beta} \sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d) \int_{0}^{1} P_{n+d-1,k_1}(u_1)(n-k_1) \]

\[
\times \max_{z \in S_{n,k_1}} \left| z_0^{a_0} z_1^{a_1} \cdots z_{d-1}^{a_{d-1}} (1 - |z|)^{\beta} f(u_1, (1 - u_1)z) \right| du_1
\]

\[
\leq C n \| \omega f \|_\infty x_1^{a_1} (1 - x_1)^{|a|+\beta} \sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d) \left( \int_{0}^{1} P_{n+d-1,k_1}(u_1) \left( \frac{1}{u_1} \right)^{a_1} (1 - u_1)^{-|a|+\beta} du_1 \right)^{1/2}
\]

\[
\times \left( \int_{0}^{1} P_{n+d-1,k_1}(u_1)(1 - u_1)^{-2|a|+2\beta} du_1 \right)^{-1/2}
\]

\[
\leq C n \| \omega f \|_\infty x_1^{a_1} (1 - x_1)^{|a|+\beta} \sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d) \left( \int_{0}^{1} P_{n+d-1,k_1}(u_1) u_1^{-2\alpha_1} du_1 \right)^{\alpha_1/2}
\]

\[
\times \left( \int_{0}^{1} P_{n+d-1,k_1}(u_1) u_1^{1-(|a|+\beta)/2d} du_1 \right)^{(|a|+\beta)/2d}
\]

\[
\times \left( \int_{0}^{1} P_{n+d-1,k_1}(u_1) du_1 \right)^{1/2 - (|a|+\beta)/2d}
\]

\[
\leq C n \| \omega f \|_\infty x_1^{a_1} (1 - x_1)^{|a|+\beta} \sum_{k_1=2}^{n} P_{n,k_1}(x_1)(n+d) \left( \frac{n + d - 1}{k_1(k_1 - 1)} \right)^{\alpha_1/2}
\]

\[
\times \left( \frac{(n + d - 1)! (n - d - k_1 - 1)!}{(n - d)! (n + d - k_1 - 1)!} \right)^{(|a|+\beta)/2d} \left( \frac{(n + d)! - (|a|+\beta)/2d - \alpha_1/2}{(n + d)^{1/2} - (|a|+\beta)/2d - \alpha_1/2} \right)^{-1}
\]

\[
\leq C n \| \omega f \|_\infty x_1^{a_1} (1 - x_1)^{|a|+\beta} \sum_{k_1=1}^{n-1} P_{n,k_1}(x_1) \left( \frac{n}{k_1} \right)^{a_1} \left( \frac{n}{n - k_1} \right)^{|a|+\beta}
\]

\[
\leq C n \| \omega f \|_\infty.
\]

(2.14)

By Riesz interpolation theorem, we get

\[
\left\| \omega \phi_2^2 D_{22}^2 M_{n,d} f \right\|_p \leq C n \| \omega f \|_p.
\]

(2.15)
Similarly, the other cases for \( i = 1, 3, 4, \ldots, d = j \) can be proved. For \( i \neq j \), by the transformation \( T \), we have

\[
\left\| \omega \varphi_{ij}^2 D_{ij}^2 M_{n,d} f \right\|_p = \left\| \omega \varphi_{ij}^2 D_{ij}^2 M_{n,d} f \right\|_p \leq C \left\| \omega f r \right\|_p = C n \left\| \omega f \right\|_p.
\]

(2.16)

Lemma 2.4 is completed. \( \square \)

**Lemma 2.5.** If \( f \in W_p^w(S) \subset L_p(S) \), \( 1 \leq p \leq \infty \), then

\[
\left\| \omega \varphi_{ij}^2 D_{ij}^2 M_{n,d} f \right\|_p \leq C \left\| \omega \varphi_{ij}^2 D_{ij}^2 f \right\|_p \quad 1 \leq i \leq j \leq d.
\]

(2.17)

**Proof.** We use the induction on the dimension number \( d \) to prove Lemma 2.5. The case \( d = 1 \) was proved by Lemma 3 of [6], that is,

\[
\left\| \omega D^2 M_{n,1} f \right\|_p \leq C \left\| \omega D^2 f \right\|_p.
\]

(2.18)

Next, suppose that Lemma 2.5 is valid for \( d = r \) \((r \geq 1)\), and we prove it is also true for \( d = r + 1 \). Noticing formula (2.4), we have

\[
\omega(x) \varphi^2_{22}(x) D_{22}^2 M_{n,d}(f; x)
\]

\[
= x_{11}^{d1}(1 - x_1)^{d1+1} \sum_{k=0}^{n} P_{n,k_1}(x_1) (n + d) z_1^{a_1} z_2^{a_2} \cdots z_{d-1}^{a_{d-1}}
\]

(2.19)

\[
\times (1 - |z|)^{d} \varphi_{11}^2(z) \int_0^1 P_{n+d-1,k_1}(u_1) D_{11}^2 M_{n-k_1,d-1}(f(u_1, (1 - u_1)z); z) du_1,
\]
where \( z = (z_1, z_2, \dots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \ldots, x_d/(1-x_1)) \). When \( p = 1 \), from the inductive assumption of \( p = 1 \), we have

\[
\int_S |\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f; x)| ds
\]

\[
\leq C \int_0^1 x_1^{a_1}(1-x_1)^{[a_1+\beta]} \sum_{k_1=0}^n P_{n,k_1}(x_1)(n+d) \int_0^1 P_{n+d-1,k_1}(u_1)
\times \int_{z \in S_d-1} |\omega(z)\varphi_{11}^2(z)D_{11}^2 f(u_1, (1-u_1)z)| dz dx_1 u_1
\]

\[
\leq C \left( \frac{n+d}{n+1} \right) \sum_{k_1=0}^n P_{n+d-1,k_1}(u_1)
\times \int_{z \in S_d-1} |\omega(z)\varphi_{11}^2(z)D_{11}^2 f(u_1, (1-u_1)z)| dz du_1
\]

\[
\leq C \int_0^1 x_1^{a_1}(1-u_1)^{[a_1+\beta]} \left( \frac{1}{u_1} \right)^{a_1} (1-u_1)^{-[a_1+\beta]} \int_{z \in S_d-1} \left| (\omega D_{22}^2 f)(u_1, (1-u_1)z) \right| dz du_1
\]

\[
\leq C \left\| \omega D_{22}^2 f \right\|_1.
\]

(2.20)

When \( p = \infty \), we have

\[
\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f; x)
\]

\[
= x_1^{a_1}(1-x_1)^{[a_1+\beta]} \sum_{k_1=0}^n P_{n,k_1}(x_1)(n+d)z_1^{a_1}z_2^{a_1} \cdots z_{d-1}^{a_1}
\times (1-|z|)^{\delta} \varphi_{11}^2(z) \int_0^1 P_{n+d-1,k_1}(u_1)D_{11}^2 M_{n-k_1,d-1}(f(u_1, (1-u_1)z); z) du_1,
\]

(2.21)

where \( z = (z_1, z_2, \ldots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \ldots, x_d/(1-x_1)) \). From the inductive assumption, the Cauchy-Swartz inequality, Holder inequality, and Lemma 2.4, we get

\[
\left| \omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f; x) \right|
\]

\[
\leq C x_1^{a_1}(1-x_1)^{[a_1+\beta]} \sum_{k_1=0}^n P_{n,k_1}(x_1)(n+d) \int_0^1 P_{n+d-1,k_1}(u_1)
\times \max_{z \in S_d-1} z_1^{a_1}z_2^{a_1} \cdots z_{d-1}^{a_1}(1-|z|)^{\delta} \varphi_{22}^2 D_{22}^2 f(u_1, (1-u_1)z) du_1
\]
\[ \leq C x_1^{a_1} (1 - x_1)^{|\alpha| + \beta} \sum_{k_1=0}^{n} P_{n,k_1} (x_1)(n + d) \int_0^1 P_{n+d-1,k_1}(u_1) x_1^{-a_1}(1 - x_1)^{-|\alpha| - \beta} \]
\[ \times \left\| \omega_{22}^2 D_{22}^2 f \right\|_{\infty} \, du_1 \]
\[ \leq C \left\| \omega_{22}^2 D_{22}^2 f \right\|_{\infty} . \]  
(2.22)

By Riesz interpolation theorem, we get

\[ \left\| \omega_{22}^2 D_{22}^2 M_{n,d} f \right\|_p \leq C \left\| \omega_{22}^2 D_{22}^2 f \right\|_p , \]  
(2.23)

Similarly, the other cases for \( i = 1, 3, 4, \ldots, d(=j) \) can be proved. For \( i \neq j \), by the transformation \( T \), we have

\[ \left\| \omega_{ij}^2 D_{ij}^2 M_{n,d} f \right\|_p = \left\| \omega_{ij}^2 D_{ij}^2 f \right\|_p . \]  
(2.24)

Lemma 2.5 is completed. \( \square \)

Lemma 2.6 (see [9]). Let \( \{ \sigma_n \}, \{ \phi_n \} \) be nonnegative sequences \( (\sigma_0 = 0, n \in N) \). For \( l > 0 \), if the sequences \( \{ \sigma_n \}, \{ \phi_n \} \) satisfy

\[ \sigma_n \leq Q \left( \frac{k}{n} \right)^l \sigma_k + \phi_k \quad (Q \geq 1, 1 \leq k \leq n, n \in N) , \]  
(2.25)

one has

\[ \sigma_n \leq Mn^{-s} \sum_{k=1}^{n} k^{s-1} \phi_k . \]  
(2.26)

If \( Q = 1 \), then \( l = s \). If \( Q > 1 \), then \( 0 < s < l \).
3. The Proof of Theorems

Now we prove (1.9) of Theorem 1.2. By using Lemma 2.1, for arbitrary $g \in W_{\phi}^{r,p}(S) \subset L^p(S)$, we have

$$
\|\omega(M_{n,d}f - f)\|_p \leq C \left( \|\omega M_{n,d}(f - g)\|_p + \|\omega M_{n,d}g - \omega g\|_p + \|\omega(f - g)\|_p \right)
$$

$$
\leq C \left( \|\omega(f - g)\|_p + \frac{1}{n} \sum_{1 \leq i \leq j \leq d} \|\omega \phi_{ij}^2 D_{ij}^2 g\|_p + \|\omega g\|_p \right)
$$

$$
\leq C \left( \|\omega(f - g)\|_p + \frac{1}{n} \sum_{1 \leq i \leq j \leq d} \|\omega \phi_{ij}^2 D_{ij}^2 \|_p + \frac{1}{n} \|\omega f\|_p \right). \quad (3.1)
$$

Hence, from Lemma 2.2, we obtain

$$
\|\omega(M_{n,d}f - f)\|_p \leq C \left( K_2^2 \left( f, \frac{1}{n} \right)_\omega + \frac{1}{n} \|\omega f\|_p \right)
$$

$$
\leq C \left( \omega_2^2(f, t)_{\omega} + \frac{1}{n} \|\omega f\|_p \right). \quad (3.2)
$$

Next, we prove (1.10) of Theorem 1.2. Letting $\sigma_n = C(1/n)\|\omega \phi_{ij}^2 D_{ij}^2 M_{n,d}(f)\|_p (1 \leq i \leq j \leq d)$, $\phi_n = C\|\omega(M_{n,d}(f) - f)\|_p$, then $\sigma_1 = 0$. By Lemmas 2.4 and 2.5, we have

$$
\sigma_n \leq C \frac{1}{n} \|\omega \phi_{ij}^2 D_{ij}^2 M_{n,d}(f - M_{k,d}f)\|_p + C \frac{1}{n} \|\omega \phi_{ij}^2 D_{ij}^2 M_{n,d}M_{k,d}f\|_p
$$

$$
\leq C \|\omega(f - M_{k,d}f)\|_p + C \frac{1}{n} \|\omega \phi_{ij}^2 D_{ij}^2 M_{k,d}f\|_p
$$

$$
= C \frac{1}{n} \sigma_k + \phi_k \quad (C > 1). \quad (3.3)
$$

Using Lemma 2.6, we get $\sigma_n \leq C(1/n) \sum_{k=1}^{n} (n/k)^\delta \phi_k (0 < \delta < 1)$. That is,

$$
\|\omega \phi_{ij}^2 D_{ij}^2 M_{n,d}(f)\|_p \leq C \sum_{k=1}^{n} \left( \frac{n}{k} \right)^\delta \|\omega(M_{k,d}f - f)\|_p. \quad (3.4)
$$

When $n \geq 2$, there exists $(m \in N)$ such that $n/2 \leq m \leq n$ and satisfies the equation

$$
\|\omega(M_{m,d}f - f)\|_p = \min_{n/2 \leq k \leq n} \|\omega(M_{k,d}f - f)\|_p. \quad (3.5)
$$

Thus,

$$
\|\omega(M_{m,d}f - f)\|_p \leq \frac{2}{n} \sum_{n/2 \leq k \leq n} \|\omega(M_{k,d}f - f)\|_p. \quad (3.6)
$$
Using Lemma 2.2, we have

\[
\omega^2_{\psi} \left( f, \frac{1}{\sqrt{n}} \right) \leq C K^2_{\psi} \left( f, \frac{1}{n} \right)
\]

\[
\leq C \left( \| \omega(M_{m,d}f - f) \|_p + \frac{1}{n} \sum_{1 \leq i \leq j \leq d} \| \omega_{\psi_{ij}}^2 D_{ij}^2 M_{m,d}f \|_p \right)
\]

(3.7)

\[
\leq C \frac{1}{n} \sum_{k=1}^{n} \left( \frac{n}{k} \right)^{\delta} \| \omega(M_{k,d}f - f) \|_p,
\]

Theorem 1.2 is completed.

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**References**


