Research Article

Asymptotic Formula for Oscillatory Solutions of Some Singular Nonlinear Differential Equation

Irena Rachůnková and Lukáš Rachůnek

*Department of Mathematics, Faculty of Science, Palacký University, 17. Listopadu 12,
 771 46 Olomouc, Czech Republic*

Correspondence should be addressed to Irena Rachůnková, irena.rachunkova@upol.cz

Received 28 October 2010; Revised 31 March 2011; Accepted 2 May 2011

Academic Editor: Yuri V. Rogovchenko

Copyright © 2011 I. Rachůnková and L. Rachůnek. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Singular differential equation $(p(t)u')' = p(t)f(u)$ is investigated. Here f is Lipschitz continuous on $\mathbb{R}$ and has at least two zeros 0 and $L > 0$. The function p is continuous on $[0, \infty)$ and has a positive continuous derivative on $(0, \infty)$ and $p(0) = 0$. An asymptotic formula for oscillatory solutions is derived.

1. Introduction

In this paper, we investigate the equation

$$(p(t)u')' = p(t)f(u), \quad t \in (0, \infty), \quad (1.1)$$

where f satisfies

$$f \in Lip_{loc}(\mathbb{R}), \quad f(0) = f(L) = 0, \quad f(x) < 0, \quad x \in (0, L), \quad (1.2)$$

$$\exists \bar{B} \in (-\infty, 0): f(x) > 0, \quad x \in [\bar{B}, 0), \quad (1.3)$$

$$F(\bar{B}) = F(L), \quad \text{where } F(x) = -\int_0^x f(z)dz, \quad x \in \mathbb{R}, \quad (1.4)$$

and p fulfils

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p(0) = 0, \quad (1.5)$$

$$p'(t) > 0, \quad t \in (0, \infty), \quad \lim_{t \rightarrow \infty} \frac{p'(t)}{p(t)} = 0. \quad (1.6)$$

Equation (1.1) is a generalization of the equation

$$u'' + \frac{k-1}{t}u' = f(u), \quad t \in (0, \infty), \quad (1.7)$$

which arises for $k > 1$ and special forms of f in many areas, for example: in the study of phase transitions of Van der Waals fluids [1–3], in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [4, 5], in the homogeneous nucleation theory [6], in the relativistic cosmology for the description of particles which can be treated as domains in the universe [7], in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [8]. Numerical simulations of solutions of (1.1), where f is a polynomial with three zeros have been presented in [9–11]. Close problems about the existence of positive solutions can be found in [12–14].

Due to $p(0) = 0$, (1.1) has a singularity at $t = 0$.

Definition 1.1. A function $u \in C^1[0, \infty) \cap C^2(0, \infty)$ which satisfies (1.1) for all $t \in (0, \infty)$ is called a *solution* of (1.1).

Definition 1.2. Let u be a solution of (1.1) and let L be of (1.2). Denote $u_{\sup} = \sup\{u(t) : t \in [0, \infty)\}$. If $u_{\sup} < L$ ($u_{\sup} = L$ or $u_{\sup} > L$), then u is called a *damped* solution (a *bounding homoclinic* solution or an *escape* solution).

These three types of solutions have been investigated in [15–19]. In particular, the existence of damped oscillatory solutions which converge to 0 has been proved in [19].

The main result of this paper is contained in Section 3 in Theorem 3.1, where we provide an asymptotic formula for damped oscillatory solutions of (1.1).

2. Existence of Oscillatory Solutions

Here, we will study solutions of (1.1) satisfying the initial conditions

$$u(0) = B, \quad u'(0) = 0, \quad (2.1)$$

with a parameter $B \leq L$. Reason is that we focus our attention on damped solutions of (1.1) and that each solution u of (1.1) must fulfil $u'(0) = 0$ (see [19]).

First, we bring two theorems about the existence of damped and oscillatory solutions.

Theorem 2.1 (see [19]). *Assume that (1.2)–(1.6) hold. Then for each $B \in [\bar{B}, L)$ problem (1.1), (2.1) has a unique solution. This solution is damped.*

Theorem 2.2. *Assume that (1.2)–(1.6) hold. Further, let there exists $k_0 \in (0, \infty)$ such that*

$$p \in C^2(0, \infty), \quad \limsup_{t \rightarrow \infty} \left| \frac{p''(t)}{p'(t)} \right| < \infty, \quad \liminf_{t \rightarrow \infty} \frac{p(t)}{t^{k_0}} \in (0, \infty], \quad (2.2)$$

$$\lim_{x \rightarrow 0+} \frac{f(x)}{x} < 0, \quad \lim_{x \rightarrow 0-} \frac{f(x)}{x} < 0. \quad (2.3)$$

Then for each $B \in [\bar{B}, L)$ problem (1.1), (2.1) has a unique solution u . If $B \neq 0$, then the solution u is damped and oscillatory with decreasing amplitudes and

$$\lim_{t \rightarrow \infty} u(t) = 0. \quad (2.4)$$

Proof. The assertion follows from Theorems 2.3, 2.10 and 3.1 in [19]. $\square$

Example 2.3. The functions

- (i) $p(t) = t^k, p(t) = t^k \ln(t^\ell + 1), k, \ell \in (0, \infty),$
- (ii) $p(t) = t + \alpha \sin t, \alpha \in (-1, 1),$
- (iii) $p(t) = t^k / (1 + t^\ell), k, \ell \in (0, \infty), \ell < k$

satisfy (1.5), (1.6), and (2.2).

The functions

- (i) $p(t) = \ln(t + 1), p(t) = \arctan t, p(t) = t^k / (1 + t^k), k \in (0, \infty)$

satisfy (1.5), (1.6), but not (2.2) (the third condition).

The function

- (i) $p(t) = t^k + \alpha \sin t^k, \alpha \in (-1, 1), k \in (1, \infty),$

satisfy (1.5), (1.6) but not (2.2) (the second and third conditions).

Example 2.4. Let $k \in (0, \infty)$.

- (i) The function

$$f(x) = \begin{cases} -kx, & \text{for } x \leq 0, \\ x(x-1), & \text{for } x > 0, \end{cases} \quad (2.5)$$

satisfies (1.2) with $L = 1$, (1.3), (1.4) with $\bar{B} = -(3k)^{-1/2}$ and (2.3).

- (ii) The function

$$f(x) = \begin{cases} kx^2, & \text{for } x \leq 0, \\ x(x-1), & \text{for } x > 0, \end{cases} \quad (2.6)$$

satisfies (1.2) with $L = 1$, (1.3), (1.4) with $\bar{B} = -(2k)^{-1/3}$ but not (2.3) (the second condition).

In the next section, the generalized Matell's theorem which can be found as Theorem 6.5 in the monograph by Kiguradze will be useful. For our purpose, we provide its following special case.

Consider an interval $J \subset \mathbb{R}$. We write $AC(J)$ for the set of functions absolutely continuous on J and $AC_{\text{loc}}(J)$ for the set of functions belonging to $AC(I)$ for each compact

interval $I \subset J$. Choose $t_0 > 0$ and a function matrix $A(t) = (a_{ij}(t))_{i,j \leq 2}$ which is defined on (t_0, ∞) . Denote by $\lambda(t)$ and $\mu(t)$ eigenvalues of $A(t)$, $t \in (t_0, \infty)$. Further, suppose

$$\lambda = \lim_{t \rightarrow \infty} \lambda(t), \quad \mu = \lim_{t \rightarrow \infty} \mu(t) \quad (2.7)$$

be different eigenvalues of the matrix $A = \lim_{t \rightarrow \infty} A(t)$, and let $\mathbf{l}$ and $\mathbf{m}$ be eigenvectors of A corresponding to λ and μ , respectively.

Theorem 2.5 (see [20]). *Assume that*

$$a_{i,j} \in AC_{\text{loc}}(t_0, \infty), \quad \left| \int_{t_0}^{\infty} a'_{i,j}(t) dt \right| < \infty, \quad i, j = 1, 2, \quad (2.8)$$

and that there exists $c_0 > 0$ such that

$$\int_s^t \operatorname{Re}(\lambda(\tau) - \mu(\tau)) d\tau \leq c_0, \quad t_0 \leq s < t, \quad (2.9)$$

or

$$\int_{t_0}^{\infty} \operatorname{Re}(\lambda(\tau) - \mu(\tau)) d\tau = \infty, \quad \int_s^t \operatorname{Re}(\lambda(\tau) - \mu(\tau)) d\tau \geq -c_0, \quad t_0 \leq s < t. \quad (2.10)$$

Then the differential system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) \quad (2.11)$$

has a fundamental system of solutions $\mathbf{x}(t)$, $\mathbf{y}(t)$ such that

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) e^{-\int_{t_0}^t \lambda(\tau) d\tau} = \mathbf{l}, \quad \lim_{t \rightarrow \infty} \mathbf{y}(t) e^{-\int_{t_0}^t \mu(\tau) d\tau} = \mathbf{m}. \quad (2.12)$$

3. Asymptotic Formula

In order to derive an asymptotic formula for a damped oscillatory solution u of problem (1.1), (2.1), we need a little stronger assumption than (2.3). In particular, the function $f(x)/x$ should have a negative derivative at $x = 0$.

Theorem 3.1. *Assume that (1.2)–(1.6), and (2.2) hold. Assume, moreover, that there exist $\eta > 0$ and $c > 0$ such that*

$$\frac{f(x)}{x} \in AC[-\eta, \eta], \quad \lim_{x \rightarrow 0} \frac{f(x)}{x} = -c. \quad (3.1)$$

Then for each $B \in [\bar{B}, L)$ problem (1.1), (2.1) has a unique solution u . If $B \neq 0$, then the solution u is damped and oscillatory with decreasing amplitudes such that

$$\limsup_{t \rightarrow \infty} \sqrt{p(t)} |u(t)| < \infty. \quad (3.2)$$

Proof. We have the following steps:

Step 1 (construction of an auxiliary linear differential system). Choose $B \in [\bar{B}, L)$, $B \neq 0$. By Theorem 2.2, problem (1.1), (2.1) has a unique oscillatory solution u with decreasing amplitudes and satisfying (2.4). Having this solution u , define a linear differential equation

$$v'' + \frac{p'(t)}{p(t)} v' = \frac{f(u(t))}{u(t)} v, \quad (3.3)$$

and the corresponding linear differential system

$$x_1' = x_2, \quad x_2' = \frac{f(u(t))}{u(t)} x_1 - \frac{p'(t)}{p(t)} x_2. \quad (3.4)$$

Denote

$$A(t) = (a_{i,j}(t))_{i,j \leq 2} = \begin{pmatrix} 0 & 1 \\ \frac{f(u(t))}{u(t)} & -\frac{p'(t)}{p(t)} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix}. \quad (3.5)$$

By (1.6), (2.4), and (3.1),

$$A = \lim_{t \rightarrow \infty} A(t). \quad (3.6)$$

Eigenvalues of A are numbers $\lambda = i\sqrt{c}$ and $\mu = -i\sqrt{c}$, and eigenvectors of A are $\mathbf{l} = (1, i\sqrt{c})$ and $\mathbf{m} = (1, -i\sqrt{c})$, respectively. Denote

$$D(t) = \left(\frac{p'(t)}{2p(t)} \right)^2 + \frac{f(u(t))}{u(t)}, \quad t \in (0, \infty). \quad (3.7)$$

Then eigenvalues of $A(t)$ have the form

$$\lambda(t) = -\frac{p'(t)}{2p(t)} + \sqrt{D(t)}, \quad \mu(t) = -\frac{p'(t)}{2p(t)} - \sqrt{D(t)}, \quad t \in (0, \infty). \quad (3.8)$$

We see that

$$\lim_{t \rightarrow \infty} \lambda(t) = \lambda, \quad \lim_{t \rightarrow \infty} \mu(t) = \mu. \quad (3.9)$$

Step 2 (verification of the assumptions of Theorem 2.5). Due to (1.6), (2.4), and (3.1), we can find $t_0 > 0$ such that

$$u(t_0) \neq 0, \quad |u(t)| \leq \eta, \quad D(t) < 0, \quad t \in (t_0, \infty). \quad (3.10)$$

Therefore, by (3.1),

$$a_{21}(t) = \frac{f(u(t))}{u(t)} \in AC_{\text{loc}}(t_0, \infty), \quad (3.11)$$

and so

$$\left| \int_{t_0}^{\infty} \left(\frac{f(u(t))}{u(t)} \right)' dt \right| = \left| \lim_{t \rightarrow \infty} \frac{f(u(t))}{u(t)} - \frac{f(u(t_0))}{u(t_0)} \right| = \left| -c - \frac{f(u(t_0))}{u(t_0)} \right| < \infty. \quad (3.12)$$

Further, by (2.2), $a_{22}(t) = -p'(t)/p(t) \in C^1(t_0, \infty)$. Hence, due to (1.6),

$$\left| \int_{t_0}^{\infty} \left(\frac{p'(t)}{p(t)} \right) dt \right| = \left| \lim_{t \rightarrow \infty} \frac{p'(t)}{p(t)} - \frac{p'(t_0)}{p(t_0)} \right| = \frac{p'(t_0)}{p(t_0)} < \infty. \quad (3.13)$$

Since $a_{11}(t) \equiv 0$ and $a_{12}(t) \equiv 1$, we see that (2.8) is satisfied. Using (3.8) we get $\text{Re}(\lambda(t) - \mu(t)) \equiv 0$. This yields

$$\int_s^t \text{Re}(\lambda(\tau) - \mu(\tau)) d\tau = 0 < c_0, \quad t_0 \leq s < t, \quad (3.14)$$

for any positive constant c_0 . Consequently (2.9) is valid.

Step 3 (application of Theorem 2.5). By Theorem 2.5 there exists a fundamental system $\mathbf{x}(t) = (x_1(t), x_2(t))$, $\mathbf{y}(t) = (y_1(t), y_2(t))$ of solutions of (3.4) such that (2.12) is valid. Hence

$$\lim_{t \rightarrow \infty} x_1(t) e^{-\int_{t_0}^t \lambda(\tau) d\tau} = 1, \quad \lim_{t \rightarrow \infty} y_1(t) e^{-\int_{t_0}^t \mu(\tau) d\tau} = 1. \quad (3.15)$$

Using (3.8) and (3.10), we get

$$\begin{aligned} \exp\left(-\int_{t_0}^t \lambda(\tau) d\tau\right) &= \exp\left(\int_{t_0}^t \left(\frac{p'(\tau)}{2p(\tau)} - \sqrt{D(\tau)}\right) d\tau\right) \\ &= \exp\left(\frac{1}{2} \ln \frac{p(t)}{p(t_0)}\right) \exp\left(-i \int_{t_0}^t \sqrt{|D(\tau)|} d\tau\right), \end{aligned} \quad (3.16)$$

and, hence,

$$\left| e^{-\int_{t_0}^t \lambda(\tau) d\tau} \right| = \sqrt{\frac{p(t)}{p(t_0)}}, \quad t \in (t_0, \infty). \quad (3.17)$$

Similarly

$$\left| e^{-\int_{t_0}^t \mu(\tau) d\tau} \right| = \sqrt{\frac{p(t)}{p(t_0)}}, \quad t \in (t_0, \infty). \quad (3.18)$$

Therefore, (3.15) implies

$$\begin{aligned} 1 &= \lim_{t \rightarrow \infty} \left| x_1(t) e^{-\int_{t_0}^t \lambda(\tau) d\tau} \right| = \lim_{t \rightarrow \infty} |x_1(t)| \sqrt{\frac{p(t)}{p(t_0)}}, \\ 1 &= \lim_{t \rightarrow \infty} \left| y_1(t) e^{-\int_{t_0}^t \mu(\tau) d\tau} \right| = \lim_{t \rightarrow \infty} |y_1(t)| \sqrt{\frac{p(t)}{p(t_0)}}. \end{aligned} \quad (3.19)$$

Step 4 (asymptotic formula). In Step 1, we have assumed that u is a solution of (1.1), which means that

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f(u(t)), \quad \text{for } t \in (0, \infty). \quad (3.20)$$

Consequently

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = \frac{f(u(t))}{u(t)} u(t), \quad \text{for } t \in (0, \infty), \quad (3.21)$$

and, hence, u is also a solution of (3.3). This yields that there are $c_1, c_2 \in \mathbb{R}$ such that $u(t) = c_1 x_1(t) + c_2 y_1(t)$, $t \in (0, \infty)$. Therefore,

$$\limsup_{t \rightarrow \infty} \sqrt{p(t)} |u(t)| \leq (|c_1| + |c_2|) \sqrt{p(t_0)} < \infty. \quad (3.22)$$

□

Remark 3.2. Due to (2.2) and (3.2), we have for a solution u of Theorem 3.1

$$u(t) = O\left(t^{-k_0/2}\right), \quad \text{for } t \rightarrow \infty. \quad (3.23)$$

Figure 1

Example 3.3. Let $k \in (1, \infty)$.

- (i) The functions $f(x) = x(x - 1)$ and $f(x) = x(x - 1)(x + 2)$ satisfy all assumptions of Theorem 3.1.
- (ii) The functions $f(x) = x^{2k-1}(x - 1)$ and $f(x) = x^{2k-1}(x - 1)(x + 2)$ satisfy (1.2)–(1.4) but not (3.1) (the second condition).

Example 3.4. Consider the initial problem

$$\left(t^2 u'\right)' = t^2 u(u - 5)(u + 10), \quad u(0) = -3, \quad u'(0) = 0. \quad (3.24)$$

Here $L_0 = -10$, $L = 5$ and we can check that $\bar{B} < -3$. Further, all assumptions of Theorems 2.2 and 3.1 are fulfilled. Therefore, by Theorem 2.2, there exists a unique solution u of problem (3.24) which is damped and oscillatory and converges to 0. By Theorem 3.1, we have

$$\limsup_{t \rightarrow \infty} t|u(t)| < \infty, \quad \text{that is, } u(t) = O\left(\frac{1}{t}\right), \quad \text{for } t \rightarrow \infty. \quad (3.25)$$

The behaviour of the solution $u(t)$ and of the function $tu(t)$ is presented on Figure 1.

Remark 3.5. Our further research of this topic will be focused on a deeper investigation of all types of solutions defined in Definition 1.2. For example, we have proved in [15, 19] that damped solutions of (1.1) can be either oscillatory or they have a finite number of zeros or no zero and converge to 0. A more precise characterization of behaviour of nonoscillatory solutions are including their asymptotic formulas in as open problem. The same can be said about homoclinic solutions. In [17] we have found some conditions which guarantee their existence, and we have shown that if u is a homoclinic solution of (1.1), then $\lim_{t \rightarrow \infty} u(t) = L$.

In order to discover other existence conditions for homoclinic solutions, we would like to estimate their convergence by proper asymptotic formulas.

Acknowledgments

The authors thank the referees for comments and suggestions. This paper was supported by the Council of Czech Government MSM 6198959214.

References

- [1] V. Bongiorno, L. E. Scriven, and H. T. Davis, "Molecular theory of fluid interfaces," *Journal of Colloid and Interface Science*, vol. 57, pp. 462–475, 1967.
- [2] H. Gouin and G. Rotoli, "An analytical approximation of density profile and surface tension of microscopic bubbles for Van der Waals fluids," *Mechanics Research Communications*, vol. 24, pp. 255–260, 1997.
- [3] J. D. Van Der Waals and R. Kohnstamm, *Lehrbuch der Thermodynamik*, vol. 1, Leipzig, Germany, 1908.
- [4] P. C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, vol. 28 of *Lecture Notes in Biomathematics*, Springer, Berlin, Germany, 1979.
- [5] R. A. Fischer, "The wave of advance of advantageous genes," *Journal of Eugenics*, vol. 7, pp. 355–369, 1937.
- [6] F. F. Abraham, *Homogeneous Nucleation Theory*, Academies Press, New York, NY, USA, 1974.
- [7] A. P. Linde, *Particle Physics and Inflationary Cosmology*, Harwood Academic, Chur, Switzerland, 1990.
- [8] G. H. Derrick, "Comments on nonlinear wave equations as models for elementary particles," *Journal of Mathematical Physics*, vol. 5, pp. 1252–1254, 1964.
- [9] F. Dell'Isola, H. Gouin, and G. Rotoli, "Nucleation of spherical shell-like interfaces by second gradient theory: numerical simulations," *European Journal of Mechanics*, vol. 15, no. 4, pp. 545–568, 1996.
- [10] G. Kitzhofer, O. Koch, P. Lima, and E. Weinmüller, "Efficient numerical solution of the density profile equation in hydrodynamics," *Journal of Scientific Computing*, vol. 32, no. 3, pp. 411–424, 2007.
- [11] P. M. Lima, N. V. Chemetov, N. B. Konyukhova, and A. I. Sukov, "Analytical-numerical investigation of bubble-type solutions of nonlinear singular problems," *Journal of Computational and Applied Mathematics*, vol. 189, no. 1-2, pp. 260–273, 2006.
- [12] H. Berestycki, P. L. Lions, and L. A. Peletier, "An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$," *Indiana University Mathematics Journal*, vol. 30, no. 1, pp. 141–157, 1981.
- [13] D. Bonheure, J. M. Gomes, and L. Sanchez, "Positive solutions of a second-order singular ordinary differential equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 8, pp. 1383–1399, 2005.
- [14] M. Conti, L. Merizzi, and S. Terracini, "Radial solutions of superlinear equations in $\mathbb{R}^N$, part I: a global variational approach," *Archive for Rational Mechanics and Analysis*, vol. 153, no. 4, pp. 291–316, 2000.
- [15] I. Rachůnková and J. Tomeček, "Bubble-type solutions of nonlinear singular problems," *Mathematical and Computer Modelling*, vol. 51, no. 5-6, pp. 658–669, 2010.
- [16] I. Rachůnková and J. Tomeček, "Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 2114–2118, 2010.
- [17] I. Rachůnková and J. Tomeček, "Homoclinic solutions of singular nonautonomous second-order differential equations," *Boundary Value Problems*, vol. 2009, Article ID 959636, 21 pages, 2009.
- [18] I. Rachůnková, J. Tomeček, and J. Stryja, "Oscillatory solutions of singular equations arising in hydrodynamics," *Advances in Difference Equations*, vol. 2010, Article ID 872160, 13 pages, 2010.
- [19] I. Rachůnková, L. Rachůnek, and J. Tomeček, "Existence of oscillatory solutions of singular nonlinear differential equations," *Abstract and Applied Analysis*, vol. 2011, Article ID 408525, 20 pages, 2011.
- [20] I. Kiguradze, *Some Singular Boundary Value Problems for Ordinary Differential Equations*, ITU, Tbilisi, Georgia, 1975.