Research Article

On the Difference Equation

\[ x_{n+1} = \frac{x_n x_{n-k}}{(x_{n-k+1}(a + b x_n x_{n-k}))} \]

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We show that the difference equation \( x_{n+1} = \frac{x_n x_{n-k}}{(x_{n-k+1}(a + b x_n x_{n-k}))} \), \( n \in \mathbb{N}_0 \), where \( k \in \mathbb{N} \), the parameters \( a, b \) and initial values \( x_{-i} = 0, k \) are real numbers, can be solved in closed form considerably extending the results in the literature. By using obtained formulae, we investigate asymptotic behavior of well-defined solutions of the equation.

1. Introduction

Recently, there has been some reestablished interest in difference equations which can be solved, as well as in their applications, see, for example, [1–16]. For some old results in the topic see, for example, the classical book [17].

In recently accepted paper [18] are given formulae for the solutions of the following four difference equations:

\[ x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}(\pm 1 \pm x_n x_{n-3})}, \quad n \in \mathbb{N}_0, \quad (1.1) \]

and some of these formulae are proved by induction.
Here, we show that the formulae obtained in [18] follow from known results in a natural way. Related idea was exploited in paper [7].

Moreover, we will consider here the following more general equation:

\[ x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1} (a + b x_{n-k})}, \quad n \in \mathbb{N}, \quad (1.2) \]

where \( k \in \mathbb{N} \) and the parameters \( a, b \) as well as initial values \( x_{-i}, \) \( i = 0, \ldots, k \) are real numbers, and describe the behaviour of all well-defined solutions of the equation.

For a solution \( (x_n)_{n\geq-k}, \) \( k \in \mathbb{N} \) of the difference equation

\[ x_n = f(x_{n-1}, \ldots, x_{n-k}), \quad n \in \mathbb{N}, \quad (1.3) \]

is said that it is eventually periodic with period \( p, \) if there is an \( n_1 \geq -k \) such that

\[ x_{n+p} = x_n, \quad \text{for } n \geq n_1. \quad (1.4) \]

If \( n_1 = -k, \) then it is said that the solution is periodic with period \( p. \) For some results in this area see, for example, [19–26] and the references therein.

2. Solutions of Equation \((1.2)\)

By using the change of variables

\[ y_n = \frac{1}{x_n x_{n-k}}, \quad n \in \mathbb{N}, \quad (2.1) \]

equation \((1.2)\) is transformed into the following linear first-order difference equation:

\[ y_{n+1} = ay_n + b, \quad n \in \mathbb{N}, \quad (2.2) \]

for which it is known (and easy to see) that

\[ y_n = y_0 a^n + b \frac{1 - a^n}{1 - a} = \frac{b + a^n (y_0 (1 - a) - b)}{1 - a}, \quad n \in \mathbb{N}, \quad (2.3) \]

if \( a \neq 1, \) and

\[ y_n = y_0 + bn, \quad n \in \mathbb{N}, \quad (2.4) \]

if \( a = 1. \)

From \((2.1),\) we have

\[ x_n = \frac{1}{y_n x_{n-k}} = \frac{y_{n-k}}{y_n x_{n-2k}}, \quad (2.5) \]
for \( n \geq k \), from which it follows that

\[
x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{y(2j-1)k+i}{y^{2j}k+i},
\]

(2.6)

for every \( m \in \mathbb{N}_0 \) and \( i \in \{k, k+1, \ldots, 3k-1\} \).

Using (2.3) and (2.4) in (2.6), we get

\[
x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{b x_0 x_k + a^{(2j-1)k+i}(1 - a - b x_0 x_k)}{b x_0 x_k + a^{2j}k+i(1 - a - b x_0 x_k)},
\]

(2.7)

if \( a \neq 1 \), and

\[
x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{x_0 x_k b((2j - 1)k + i) + 1}{x_0 x_k b(2jk + i) + 1},
\]

(2.8)

if \( a = 1 \), for every \( m \in \mathbb{N}_0 \) and \( i \in \{k, k+1, \ldots, 3k-1\} \).

By using formulae (2.7) and (2.8), the behavior of well-defined solutions of equation (1.2) can be obtained. This is done in the following section.

**Remark 2.1.** It is easy to check that the formulae in Theorems 2.1 and 4.1 from [18] are direct consequences of formula (2.8), whereas formulae in Theorem 3.1 and Theorem 5.1 from [18] are direct consequences of formula (2.7).

**Remark 2.2.** Note that from formula (2.8), it follows that in the case \( a = 1, b = 0 \), all well-defined solutions of equation (1.2) are periodic with period 2\( k \). This can be also obtained from (1.2), without knowing explicit formulae for its solutions. Namely, in this case, (1.2) can be written as follows:

\[
x_{n+1} x_{n-k+1} = x_n x_{n-k},
\]

(2.9)

since we assume \( x_n \neq 0 \), for all \( n \geq -k \), from which it follows that the sequence \( x_n x_{n-k} \) is constant, that is, \( x_n x_{n-k} = c, n \in \mathbb{N}_0 \) for some \( c \in \mathbb{R} \setminus \{0\} \). Hence,

\[
x_n = \frac{c}{x_{n-k}} = x_{n-2k}, \quad n \geq k,
\]

(2.10)

as claimed.

**Remark 2.3.** Note also that in the case \( a \notin \{0, 1\}, b = 0 \), from (2.7), we have

\[
x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{a^{(2j-1)k+i}}{a^{2jk+i}} = \frac{x_{i-2k}}{a^{k(m+1)}},
\]

(2.11)
for every \( m \in \mathbb{N}_0 \) and \( i \in \{k, k + 1, \ldots, 3k - 1\} \), from which the behaviour of the solutions in the case easily follows.

### 3. Asymptotic Behavior of Well-Defined Solutions of Equation (1.2)

In this section, we derive some results on asymptotic behavior of well-defined solutions of equation (1.2). We will use well-known asymptotic formulae as follows:

\[
\ln(1 + x) = x - \frac{x^2}{2} + O\left(x^3\right),
\]

\[
(1 + x)^{-1} = 1 - x + O\left(x^2\right),
\]

for \( x \to 0 \), where \( O \) is the Landau “big-oh” symbol.

**Theorem 3.1.** Let \( a = 1 \) and \( b \neq 0 \) in (1.2). Then, every well-defined solution \( (x_n)_{n \geq k} \) of equation (1.2) converges to zero.

**Proof.** By formula (2.8), we have

\[
\lim_{m \to \infty} x_{2km+i} = \lim_{m \to \infty} x_{i-2k} \prod_{j=0}^{m} \frac{x_0x_{-k}b(2j - 1)k + i + 1}{x_0x_{-k}b(2jk + i) + 1}
\]

\[
= x_{i-2k} \lim_{m \to \infty} \prod_{j=0}^{m} \left( 1 - \frac{x_0x_{-k}b(k - 1)}{x_0x_{-k}b(2jk + i)} \right)
\]

\[
= x_{i-2k} \lim_{m \to \infty} C(m_0) \prod_{j=m_0+1}^{m} \left( 1 - \frac{x_0x_{-k}b}{x_0x_{-k}(2jk + i)} \right),
\]

where \( m_0 \) is sufficiently large so that (3.1) can be applied below, and

\[
C(m_0) = \prod_{j=0}^{m_0} \left( 1 - \frac{x_0x_{-k}b}{x_0x_{-k}(2jk + i)} \right).
\]

Since

\[
\sum_{j=m_0+1}^{\infty} \frac{k}{2jk + i} = +\infty,
\]

\[
\sum_{j=m_0+1}^{\infty} \frac{k}{2jk + i} = +\infty,
\]
we conclude, using (3.1), that

\[
\lim_{m \to \infty} \prod_{j=m_0+1}^m \left( 1 - \frac{x_0x_{-k}bk}{1 + x_0x_{-k}b(2jk + i)} \right)
\]

\[
= \lim_{m \to \infty} \prod_{j=m_0+1}^m \exp \left[ \ln \left( 1 - \frac{x_0x_{-k}bk}{1 + x_0x_{-k}b(2jk + i)} \right) \right]
\]

\[
= \lim_{m \to \infty} \prod_{j=m_0+1}^m \exp \left[ -\frac{x_0x_{-k}bk}{1 + x_0x_{-k}b(2jk + i)} - \frac{1}{2} \left( \frac{x_0x_{-k}bk}{1 + x_0x_{-k}b(2jk + i)} \right)^2 \right]
\]

\[+ O \left( \left( \frac{x_0x_{-k}bk}{1 + x_0x_{-k}b(2jk + i)} \right)^3 \right) \]  

\[
= \lim_{m \to \infty} \prod_{j=m_0+1}^m \exp \left[ -\frac{k}{2jk + i} + O \left( \frac{1}{j^2} \right) \right]
\]

\[
= \lim_{m \to \infty} \exp \left[ -\sum_{j=m_0+1}^m \frac{k}{2jk + i} \right] \prod_{j=m_0+1}^m \exp \left[ O \left( \frac{1}{j^2} \right) \right] = 0.
\]

Therefore,

\[
\lim_{m \to \infty} x_{2km+i} = 0,
\]

for every \( i \in \{k, k + 1, \ldots, 3k - 1\} \), and consequently, \( \lim_{n \to \infty} x_n = 0 \), as claimed.

Before we formulate and prove our next result, we will prove an auxiliary result which is incorporated in the lemma that follows.

**Lemma 3.2.** If \( a \neq 1 \), then equation (1.2) has \( 2k \)-periodic solutions.

**Proof.** Let

\[
z_n = x_n x_{n-k}, \quad n \in \mathbb{N}_0.
\]

Then

\[
z_{n+1} = \frac{z_n}{a + b z_n}, \quad n \in \mathbb{N}_0.
\]

Since \( a \neq 1 \), we see that equation (3.8) has equilibrium solution as follows:

\[
z_n = \bar{z} = \frac{1 - a}{b}, \quad n \in \mathbb{N}_0.
\]
For this solution of equation (3.8), we have

\[ x_n = \frac{x}{x_{n-k}} = x_{n-2k}, \quad n \geq k, \]  

(3.10)

from which the lemma follows. \(\square\)

**Remark 3.3.** Note that 2k-periodic solutions in the previous lemma could be prime 2k-periodic. For this it is enough to choose initial conditions \(x_{-i}, i = 0, k\), such that the string

\[
\left( x_{-k}, x_{-k+1}, \ldots, x_{-1}, \frac{x}{x_{-k}}, \frac{x}{x_{-k+1}}, \ldots, \frac{x}{x_{-1}} \right)
\]

(3.11)

is not periodic with a period less than 2k.

**Theorem 3.4.** Let \(|a| < 1\) and \(b \neq 0\) in (1.2). Then, every well-defined solution \((x_n)_{n \geq -k}\) of equation (1.2) converges to \(a\), not necessarily prime, 2k-periodic solution of the equation.

**Proof.** First note that by Lemma 3.2, in this case, there are 2k-periodic solutions of the equation. We know that in this case well-defined solutions of the equation are given by formula (2.7). From this and by using asymptotic formulae (3.1), we obtain that for sufficiently large \(m_1\)

\[
x_{2km_1} = x_{i-2k} \prod_{j=0}^{m} \frac{bx_0x_{-k} + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})}{bx_0x_{-k} + a^{2j+k+i}(1 - a - bx_0x_{-k})}
\]

\[
= x_{i-2k} C(m_1) \prod_{j=m_1+1}^{m} \frac{1 + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})/bx_0x_{-k}}{1 + a^{2j+k+i}(1 - a - bx_0x_{-k})/bx_0x_{-k}}
\]

\[
= x_{i-2k} C(m_1) \prod_{j=m_1+1}^{m} \left( 1 + a^{(2j-1)k+i} \left( 1 - a \right) \frac{x_0}{x_{-k}} + O \left( a^{jk} \right) \right)
\]

\[
= x_{i-2k} C(m_1) \exp \left( 1 - a^k \frac{x_0}{x_{-k}} \sum_{j=m_1+1}^{m} \left( a^{(2j-1)k+i} + O \left( a^{jk} \right) \right) \right)
\]

(3.12)

where

\[
C(m_1) = \prod_{j=0}^{m_1} \frac{bx_0x_{-k} + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})}{bx_0x_{-k} + a^{2j+k+i}(1 - a - bx_0x_{-k})}.
\]

(3.13)

From (3.12) and since \(|a| < 1\), it easily follows that the sequences \((x_{2km_1})_{m \in \mathbb{N}_0}\) are convergent for each \(i \in \{k, k+1, \ldots, 3k-1\}\), from which the theorem follows. \(\square\)

**Theorem 3.5.** Let \(|a| > 1\) and \(b \neq 0\) in (1.2). Then, every well-defined solution \((x_n)_{n \geq -k}\) of equation (1.2) converges to zero.
Proof. In this case, well-defined solutions of equation (1.2) are also given by formula (2.7). Further note that for each \( i \in \{k, k + 1, \ldots, 3k - 1\} \) holds

\[
\lim_{j \to \infty} \frac{bx_0x_{-k} + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1 - a - bx_0x_{-k})} = \frac{1}{a^k}.
\]

(3.14)

Now note that \( 1/|a|^k < 1 \), due to the assumption \( |a| > 1 \). Using this fact and (3.14), it follows that for sufficiently large \( j \), say \( j \geq m_2 \) we have

\[
\left| \frac{bx_0x_{-k} + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1 - a - bx_0x_{-k})} \right| \leq \frac{1}{2} \left( 1 + \frac{1}{|a|^k} \right).
\]

(3.15)

From this, we have

\[
|x_{2km+1}| = |x_{i-2k}|C(m_2) \prod_{j=m_2+1}^{m} \left| \frac{bx_0x_{-k} + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1 - a - bx_0x_{-k})} \right|
\]

\[
\leq |x_{i-2k}|C(m_2) \prod_{j=m_2+1}^{m} \left( \frac{1}{2} \left( 1 + \frac{1}{|a|^k} \right) \right)
\]

\[
= |x_{i-2k}|C(m_2) \left( \frac{1}{2} \left( 1 + \frac{1}{|a|^k} \right) \right)^{m-m_2} \to 0,
\]

as \( m \to \infty \), where

\[
C(m_2) = \prod_{j=0}^{m_2} \left| \frac{bx_0x_{-k} + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1 - a - bx_0x_{-k})} \right|
\]

(3.16)

from which the theorem follows. \( \square \)

**Theorem 3.6.** Let \( a = -1, b \neq 0 \), and \( k \) be even in (1.2). Then, every well-defined solution \( (x_n)_{n \geq -k} \) of equation (1.2) is eventually periodic with, not necessarily prime, period \( 4k \).

Proof. From (2.2), in this case, we have

\[
y_n = -y_{n-1} + b = y_{n-2}, \quad n \geq 2,
\]

(3.18)

which means that the sequence \( y_n \) is two-periodic, and consequently the sequence \( x_nx_{n-k} \) is two-periodic. Hence

\[
x_{2n}x_{2n-k} = x_0x_{-k}, \quad x_{2n+1}x_{2n-k+1} = x_1x_{-k+1},
\]

(3.19)
from which it follows that
\[ x_{2n+i} = \frac{x_0 x_{-k+i}}{x_{2n-k+i}} = x_{2n-2k+i}, \quad n \geq \left\lceil \frac{k-i+1}{2} \right\rceil, \quad i \in \{0, 1\}, \] (3.20)

that is, the sequences \( x_{2n} \) and \( x_{2n+1} \) are 2k-periodic from which the result easily follows. \( \square \)

**Theorem 3.7.** Let \( a = -1, b \neq 0, k \) be odd in (1.2), and \( (x_n)_{n \geq k} \) be a well-defined solution of equation (1.2). Then the following statements are true.

(a) If \( x_0 x_{-k} = 2/b \), then the solution is 4k-periodic.

(b) If \( |bx_0 x_{-k} - 1| < 1 \), then \( x_{2n} \to 0 \) and \( x_{2n+1} \to \infty \), as \( n \to \infty \).

(c) If \( |bx_0 x_{-k} - 1| > 1 \), then \( x_{2n+1} \to 0 \) and \( x_{2n} \to \infty \), as \( n \to \infty \).

**Proof.** As in Theorem 3.6, we obtain (3.18) and consequently (3.19) holds. If \( k = 2l + 1 \) for some \( l \in \mathbb{N}_0 \), then from (1.2) and (3.19) we have
\[
\begin{align*}
x_{2n} x_{2n-2l-1} &= x_0 x_{-(2l+1)}, \\
x_{2n+1} x_{2n-2l} &= x_{1} x_{-2l} = \frac{x_0 x_{-(2l+1)}}{bx_0 x_{-(2l+1)} - 1}.
\end{align*}
\] (3.21)

From (3.21) we obtain
\[
\begin{align*}
x_{2n} &= \frac{x_0 x_{-(2l+1)}}{x_{2n-(2l+1)}} = \left( bx_0 x_{-(2l+1)} - 1 \right) x_{2n-(4l+2)}, \\
x_{2n+1} &= \frac{x_1 x_{-2l}}{x_{2n-2l}} = \frac{x_{2n+1-(4l+2)}}{bx_0 x_{-(2l+1)} - 1}.
\end{align*}
\] (3.22)

From relation (3.22), the statements in this theorem easily follow. \( \square \)

**Remark 3.8.** Note that the case \( bx_0 x_{-k} - 1 = -1 \) is not possible in Theorem 3.7. Namely, in this case \( x_0 x_{-k} = 0 \), due to the assumption \( b \neq 0 \) and, as we can see from (1.2), the solution is not well-defined.

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