Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 121795, 11 pages
doi:10.1155/2012/121795

Research Article

A Set of Mathematical Constants Arising Naturally in the Theory of the Multiple Gamma Functions

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Received 7 August 2012; Accepted 10 September 2012

Academic Editor: Sung G. Kim

We introduce a set of mathematical constants which is involved naturally in the theory of multiple Gamma functions. Then we present general asymptotic inequalities for these constants whose special cases are seen to contain all results very recently given in Chen 2011.

1. Introduction and Preliminaries

The double Gamma function $\Gamma_2 = 1/G$ and the multiple Gamma functions Γ_n were defined and studied systematically by Barnes [1–4] in about 1900. Before their investigation by Barnes, these functions had been introduced in a different form by, for example, Hölder [5], Alexeiewsky [6] and Kinkelin [7]. In about the middle of the 1980s, these functions were revived in the study of the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ (see [8–13]). Since then the multiple Gamma functions have attracted many authors' concern and have been used in various ways. It is seen that a set of constants $\{A_q \mid q \in \mathbb{N} := \{1, 2, 3, \ldots\}\}$ given in (1.11) involves naturally in the theory of the multiple Gamma functions Γ_n (see [14–20] and references therein). For example, for sufficiently large real x and $a \in \mathbb{C}$, we have the Stirling formula for the G-function (see [1]; see also [21, page 26, equation (7)]):

$$\begin{aligned}\log G(x+a+1) = {} & \frac{x+a}{2}\log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax \\ & + \left(\frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax\right)\log x + O\left(x^{-1}\right) \quad (x \longrightarrow \infty),\end{aligned} \tag{1.1}$$

where A is the Glaisher-Kinkelin constant (see [7, 22–24]) given in (1.16) below. The Glaisher-Kinkelin constant A, the constants B and C below introduced by Choi and Srivastava have been used, among other things, in the closed-form evaluation of certain series involving zeta functions and in calculation of some integrals of multiple Gamma functions. So trying to give asymptotic formulas for these constants A, B, and C are significant. Very recently Chen [25] presented nice asymptotic inequalities for these constants A, B, and C by mainly using the Euler-Maclaurin summation formulas. Here, we aim at presenting asymptotic inequalities for a set of the mathematical constants A_q $(q \in \mathbb{N})$ given in (1.11) some of whose special cases are seen to yield all results in [25].

For this purpose, we begin by summarizing some differential and integral formulas of the function $f(x)$ in (1.2).

Lemma 1.1. *Differentiating the function*

$$f(x) := x^q \log x \quad (q \in \mathbb{N}; x > 0) \tag{1.2}$$

ℓ times, we obtain

$$f^{(\ell)}(x) = x^{q-\ell} \left\{ \prod_{j=1}^{\ell} (q-j+1) \log x + P_\ell(q) \right\} \quad (\ell \in \mathbb{N}; 1 \leqq \ell \leqq q), \tag{1.3}$$

where $P_\ell(q)$ is a polynomial of degree $\ell - 1$ in q satisfying the following recurrence relation:

$$P_\ell(q) = \begin{cases} (q-\ell+1)P_{\ell-1}(q) + \prod_{j=1}^{\ell-1} (q-j+1) & (\ell \in \mathbb{N} \setminus \{1\}; 2 \leqq \ell \leqq q), \\ 1 & (\ell = 1). \end{cases} \tag{1.4}$$

In fact, by mathematical induction on $\ell \in \mathbb{N}$, we can give an explicit expression for $P_\ell(q)$ as follows:

$$P_\ell(q) = \prod_{j=1}^{\ell} (q-j+1) \cdot \sum_{j=1}^{\ell} \frac{1}{q-j+1} \quad (\ell \in \mathbb{N}; 1 \leqq \ell \leqq q). \tag{1.5}$$

Setting $\ell = q$ in (1.3) *and* (1.5)*, respectively, we get*

$$f^{(q)}(x) = q!(\log x + H_q) \quad (q \in \mathbb{N}), \tag{1.6}$$

where H_n are the harmonic numbers defined by

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}). \tag{1.7}$$

Differentiating $f^{(q)}(x)$ in (1.6) *r times, we obtain*

$$f^{(q+r)}(x) = (-1)^{r+1} q!(r-1)! \, \frac{1}{x^r} \quad (r \in \mathbb{N}). \tag{1.8}$$

Integrating the function $f(x)$ in (1.2) *from* 1 *to n, we get*

$$\int_1^n f(x)dx = \frac{n^{q+1}}{q+1} \log n + \frac{1-n^{q+1}}{(q+1)^2} \quad (q, n \in \mathbb{N}). \tag{1.9}$$

For each $q \in \mathbb{N}$, define a sequence $\{A_q(n)\}_{n=1}^{\infty}$ by

$$\begin{aligned} \log A_q(n) := {} & \sum_{k=1}^{n} k^q \log k \\ & - \left(\frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \sum_{r=1}^{[(q+1)/2]} \frac{B_{2r}}{(2r)!} \cdot \prod_{j=1}^{2r-1} (q-j+1) \cdot n^{q+1-2r} \right) \log n \\ & + \frac{n^{q+1}}{(q+1)^2} - \sum_{r=1}^{[(q+1)/2]+((-1)^q-1)/2} \frac{B_{2r}}{(2r)!} P_{2r-1}(q) n^{q+1-2r} \quad (n, q \in \mathbb{N}), \end{aligned} \tag{1.10}$$

where B_r are Bernoulli numbers given in (1.12), $P_r(q)$ are given in (1.5), and $[x]$ denotes (as usual) the greatest integer $\leqq x$. Define a set of mathematical constants $A_q (q \in \mathbb{N})$ by

$$\log A_q := \lim_{n \to \infty} \log A_q(n) \quad (q \in \mathbb{N}). \tag{1.11}$$

The Bernoulli numbers B_r are defined by the generating function (see [21, Section 1.6]; see also, [26, Section 1.7]):

$$\frac{z}{e^z - 1} = \sum_{r=0}^{\infty} B_r \frac{z^r}{r!} \quad (|z| < 2\pi). \tag{1.12}$$

We introduce a well-known formula (see [21, Section 2.3]):

$$B_{2p} = (-1)^{p+1} \frac{2(2p)!}{(2\pi)^{2p}} \zeta(2p) \quad (p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.13}$$

where $\zeta(s)$ is the Riemann Zeta function defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; s \neq 1). \end{cases} \tag{1.14}$$

It is easy to observe from (1.13) that

$$B_{4p} < 0, \quad B_{4p+2} > 0 \quad (p \in \mathbb{N}_0). \tag{1.15}$$

Remark 1.2. We find that the constants A_1, A_2 and A_3 correspond with the Glaisher-Kinkelin constant A, the constants B and C introduced by Choi and Srivastava, respectively:

$$\log A_1 = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k\log k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\log n + \frac{n^2}{4}\right\} = \log A, \tag{1.16}$$

where A denotes the Glaisher-Kinkelin constant whose numerical value is

$$\begin{gathered} A \cong 1.282427130\cdots, \\ \log A_2 = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k^2\log k - \left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\log n + \frac{n^3}{9} - \frac{n}{12}\right\} = \log B, \\ \log A_3 = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k^3\log k - \left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}\right)\log n + \frac{n^4}{16} - \frac{n^2}{12}\right\} = \log C. \end{gathered} \tag{1.17}$$

Here B and C are constants whose approximate numerical values are given by

$$B \cong 1.03091\ 675\ \cdots, \qquad C \cong 0.97955\ 746\ \cdots. \tag{1.18}$$

The constants B and C were considered recently by Choi and Srivastava [16, 18]. See also Adamchik [27, page 199]. Bendersky [28] presented a set of constants including B and C.

2. Euler-Maclaurin Summation Formula

We begin by recalling the Euler-Maclaurin summation formula (*cf.* Hardy ([29, 30], page 318)):

$$\sum_{k=1}^{n} f(k) \sim C_0 + \int_a^n f(x)dx + \frac{1}{2}f(n) + \sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}f^{(2r-1)}(n), \tag{2.1}$$

where C_0 is an arbitrary constant to be determined in each special case and B_r are the Bernoulli numbers given in (1.12). For another useful summation formula, see Edwards [31, page 117].

Let f be a function of class $C^{2p+2}([a,b])$, and let the interval $[a,b]$ be partitioned into m subintervals of the same length $h = (b-a)/m$. Then we have another useful form of the Euler-Maclaurin summation formula (see, e.g., [32]): There exists $0 < \theta < 1$ such that

$$\begin{aligned}\sum_{k=0}^{m} f(a+kh) &= \frac{1}{h}\int_a^b f(x)dx + \frac{f(a)+f(b)}{2} \\ &\quad + \sum_{k=1}^{p} \frac{h^{2k-1}}{(2k)!} B_{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) \\ &\quad + \frac{h^{2p+2}}{(2p+2)!} B_{2p+2} \sum_{k=0}^{m-1} f^{(2p+2)}(a+kh+\theta h),\end{aligned} \tag{2.2}$$

where $m, p \in \mathbb{N}$. Under the same conditions in (2.2), setting $m = n-1$, $a = 1$, $b = n$, and $h = 1$ in (2.2), we obtain a simple summation formula (see [25]):

$$\sum_{k=1}^{m} f(k) = \int_1^n f(x)\ dx + \frac{f(1)+f(n)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\ \left(f^{(2k-1)}(n) - f^{(2k-1)}(1)\right) + R_n(f,p), \tag{2.3}$$

where, for convenience, the remainder term $R_n(f,p)$ is given by

$$R_n(f,p) := \frac{B_{2p+2}}{(2p+2)!} \sum_{k=1}^{n-1} f^{(2p+2)}(k+\theta) \tag{2.4}$$

which is seen to be bounded by

$$\left|R_n(f,p)\right| \leqq \frac{2}{(2\pi)^{2p}} \int_1^n \left|f^{(2p+1)}(x)\right| dx. \tag{2.5}$$

Zhu and Yang [33] established some useful formulas originated from the Euler-Maclaurin summation formula (2.1) (see also [25]) asserted by the following lemma.

Lemma 2.1. *Let $\ell \in \mathbb{N}$ and let f have its first $2p+2$ derivatives on an interval $[\ell, \infty)$ such that $f^{(2p)}(x) > 0$ and $f^{(2p+2)}(x) > 0$ (or $f^{(2p)}(x) < 0$ and $f^{(2p+2)}(x) < 0$) and $f^{(2p-1)}(\infty) = 0$. Then the following results hold true:*

(i) *The sequence*

$$\mathbf{a_n} := \sum_{k=\ell}^{n} f(k) - \int_\ell^n f(x)dx - \frac{1}{2}f(n) - \sum_{k=1}^{p-1} \frac{B_{2k}}{(2k)!}\ f^{(2k-1)}(n) \quad (n \geqq \ell) \tag{2.6}$$

converges. Let $\mathbf{a} := \lim_{n\to\infty} \mathbf{a_n}$.

(ii) *For* $f^{(2p)}(x) > 0$ *and* $f^{(2p+2)}(x) > 0$, *we have*

$$0 < (-1)^{p-1}\ (\mathbf{a} - \mathbf{a_n}) < (-1)^p\ \frac{B_{2p}}{(2p)!}\ f^{(2p-1)}(n) \quad (n \geqq \ell). \tag{2.7}$$

For $f^{(2p)}(x) < 0$ *and* $f^{(2p+2)}(x) < 0$, *we have*

$$0 > (-1)^{p-1}(\mathbf{a} - \mathbf{a_n}) > (-1)^p\ \frac{B_{2p}}{(2p)!}\ f^{(2p-1)}(n) \quad (n \geqq \ell). \tag{2.8}$$

(iii) *There exists* $\theta \in (0,1)$ *such that*

$$\sum_{k=\ell}^{n} f(k) = \mathbf{a} + \int_{\ell}^{n} f(x)dx + \frac{1}{2}\ f(n) + \sum_{k=1}^{p-1} \frac{B_{2k}}{(2k)!}\ f^{(2k-1)}(n) + \theta \cdot \frac{B_{2p}}{(2p)!}\ f^{(2p-1)}(n). \tag{2.9}$$

3. Asymptotic Formulas and Inequalities for A_q

Applying the function $f(x)$ in (1.2) to the Euler-Maclaurin summation formula (2.1) with $a = 1$ and using the results presented in Lemma 1.1, we obtain an asymptotic formula for the sequence $A_q(n)$ as in the following theorem.

Theorem 3.1. *The following asymptotic formulas for the constants* $A_q(n)$ *and* A_q *hold true:*

$$\begin{aligned} \log A_q(n) \sim C_q &+ \frac{1}{(q+1)^2} + \frac{1-(-1)^q}{2}\ \frac{B_{q+1}H_q}{q+1} \\ &+ (-1)^q q! \sum_{r=[(q+1)/2]+1}^{\infty} \frac{B_{2r}}{(2r)!}\ \frac{(2r-q-2)!}{n^{2r-q-1}}, \end{aligned} \tag{3.1}$$

where C_q*'s are constants dependent on each* q *and an empty sum is understood (as usual) to be nil. And*

$$\log A_q = \lim_{n\to\infty} \log A_q(n) = C_q + \frac{1}{(q+1)^2} + \frac{1-(-1)^q}{2}\ \frac{B_{q+1}\ H_q}{q+1}. \tag{3.2}$$

Proof. We only note that

(i) $1 \leqq r \leqq [(q+1)/2]$

$$f^{(2r-1)}(n) = n^{q+1-2r} \cdot \prod_{j=1}^{2r-1}(q-j+1) \cdot \log n + n^{q+1-2r} P_{2r-1}(q). \tag{3.3}$$

(ii) $r \geqq [(q+1)/2]+1$

$$f^{(2r-1)}(n) = (-1)^q q! \frac{(2r-q-2)!}{n^{2r-q-1}}. \tag{3.4}$$

□

Applying the function $f(x)$ in (1.2) to the formula (2.9) with $\ell = 1$, and using the results presented in Lemma 1.1, we get two sided inequalities for the difference of $\log A_q(n)$ and $\log A_q$ asserted by Theorem 3.2.

Theorem 3.2. *The following inequalities hold true:*

$$\begin{aligned} &q! \sum_{r=[(q+1)/2]+1}^{2p} \frac{(2r-q-2)!}{(2r)!} \frac{B_{2r}}{n^{2r-q-1}} \\ &\quad < (-1)^q (\log A_q(n) - \log A_q) \\ &\quad < q! \sum_{r=[(q+1)/2]+1}^{2p+1} \frac{(2r-q-2)!}{(2r)!} \frac{B_{2r}}{n^{2r-q-1}} \quad (p, q, n \in \mathbb{N}). \end{aligned} \tag{3.5}$$

Proof. Setting the function $f(x)$ in (1.2) in the formula (2.9) with $\ell = 1$, and using the results presented in Lemma 1.1, we get

$$\begin{aligned} \log A_q(n) = \ &\log A_q + (-1)^q q! \sum_{r=[(q+1)/2]+1}^{p-1} \frac{(2r-q-2)!}{(2r)!} \frac{B_{2r}}{n^{2r-q-1}} \\ &+ (-1)^q q! \frac{(2p-q-2)!}{(2p)!} \frac{B_{2p}}{n^{2p-q-1}} \theta, \end{aligned} \tag{3.6}$$

for some $\theta \in (0,1)$.

Replacing p by $2p+1$ and $2p+2$ in (3.6), respectively, we obtain

$$\begin{aligned} \log A_q(n) - \log A_q &= (-1)^q q! \sum_{r=[(q+1)/2]+1}^{2p} \frac{(2r-q-2)!}{(2r)!} \frac{B_{2r}}{n^{2r-q-1}} \\ &\quad + (-1)^q q! \frac{(4p-q)!}{(4p+2)!} \frac{B_{4p+2}}{n^{4p+1-q}} \theta. \\ \log A_q(n) - \log A_q &= (-1)^q q! \sum_{r=[(q+1)/2]+1}^{2p+1} \frac{(2r-q-2)!}{(2r)!} \frac{B_{2r}}{n^{2r-q-1}} \\ &\quad + (-1)^q q! \frac{(4p+2-q)!}{(4p+4)!} \frac{B_{4p+4}}{n^{4p+3-q}} \theta. \end{aligned} \tag{3.7}$$

In view of (1.15), we find the following inequalities:

$$\begin{aligned}
&q!\sum_{r=[(q+1)/2]+1}^{2p}\frac{(2r-q-2)!}{(2r)!}\frac{B_{2r}}{n^{2r-q-1}}\\
&\quad<\log A_q(n)-\log A_q\\
&\quad<q!\sum_{r=[(q+1)/2]+1}^{2p+1}\frac{(2r-q-2)!}{(2r)!}\frac{B_{2r}}{n^{2r-q-1}}\quad (q \text{ is even}),\\
&q!\sum_{r=[(q+1)/2]+1}^{2p}\frac{(2r-q-2)!}{(2r)!}\frac{B_{2r}}{n^{2r-q-1}}\\
&\quad<\log A_q-\log A_q(n)\\
&\quad<q!\sum_{r=[(q+1)/2]+1}^{2p+1}\frac{(2r-q-2)!}{(2r)!}\frac{B_{2r}}{n^{2r-q-1}}\quad (q \text{ is odd}).
\end{aligned}\tag{3.8}$$

Finally it is easily seen that the two-sided inequalities (3.8) can be expressed in a single form (3.5). □

Remark 3.3. The special cases of (3.5) when $q=1$, $q=2$, and $q=3$ are easily seen to correspond with Equations (8), (31), and (32) in Chen's work [25], respectively.

Applying the function $f(x)$ in (1.2) to the formula (2.3) and using the results presented in Lemma 1.1, we get two-sided inequalities for the $\log A_q$ asserted by Theorem 3.4.

Theorem 3.4. *The following inequalities hold true:*

$$\begin{aligned}
&\alpha_q+q!\sum_{k=[(q+1)/2]+1}^{2p}\frac{(2k-q-2)!}{(2k)!}B_{2k}\\
&\quad<\log A_q\\
&\quad<\alpha_q+q!\sum_{k=[(q+1)/2]+1}^{2p+1}\frac{(2k-q-2)!}{(2k)!}B_{2k}\quad (q \text{ is odd}),
\end{aligned}\tag{3.9}$$

$$\begin{aligned}
&\alpha_q-q!\sum_{k=[(q+1)/2]+1}^{2p+1}\frac{(2k-q-2)!}{(2k)!}B_{2k}\\
&\quad<\log A_q\\
&\quad<\alpha_q-q!\sum_{k=[(q+1)/2]+1}^{2p}\frac{(2k-q-2)!}{(2k)!}B_{2k}\quad (q \text{ is even}),
\end{aligned}\tag{3.10}$$

where, for convenience,

$$\alpha_q := \frac{1}{(q+1)^2} + \frac{1-(-1)^q}{2}\frac{B_{q+1}H_q}{q+1} - \sum_{k=1}^{[(q+1)/2]} \frac{B_{2k}}{(2k)!}P_{2k-1}(q) \quad (q \in \mathbb{N}). \tag{3.11}$$

Proof. Setting the function $f(x)$ in (1.2) in the formula (2.3), and using the results presented in Lemma 1.1, we have, for some $\theta \in (0,1)$,

$$\begin{aligned} \log A_q(n) = \alpha_q &+ (-1)^q q! \sum_{k=[(q+1)/2]+1}^{p} \frac{(2k-2-q)!}{(2k)!} B_{2k}\left(\frac{1}{n^{2k-1-q}} - 1\right) \\ &+ (-1)^{q+1} q! \frac{(2p+1-q)!}{(2p+2)!} B_{2p+2} \sum_{k=1}^{n-1} \frac{1}{(k+\theta)^{2p+2-q}}. \end{aligned} \tag{3.12}$$

Replacing p by $2p$ and $2p+1$ in (3.12), respectively, we obtain

$$\begin{aligned} \log A_q(n) = \alpha_q &+ (-1)^q q! \sum_{k=[(q+1)/2]+1}^{2p} \frac{(2k-2-q)!}{(2k)!} B_{2k}\left(\frac{1}{n^{2k-1-q}} - 1\right) \\ &+ (-1)^{q+1} q! \frac{(4p+1-q)!}{(4p+2)!} B_{4p+2} \sum_{k=1}^{n-1} \frac{1}{(k+\theta)^{4p+2-q}}, \\ \log A_q(n) = \alpha_q &+ (-1)^q q! \sum_{k=[(q+1)/2]+1}^{2p+1} \frac{(2k-2-q)!}{(2k)!} B_{2k}\left(\frac{1}{n^{2k-1-q}} - 1\right) \\ &+ (-1)^{q+1} q! \frac{(4p+3-q)!}{(4p+4)!} B_{4p+4} \sum_{k=1}^{n-1} \frac{1}{(k+\theta)^{4p+4-q}}. \end{aligned} \tag{3.13}$$

In view of (1.15), we find from (3.13) that

$$\begin{aligned} &\alpha_q - q! \sum_{k=[(q+1)/2]+1}^{2p} \frac{(2k-2-q)!}{(2k)!} B_{2k}\left(\frac{1}{n^{2k-1-q}} - 1\right) \\ &\quad < \log A_q(n) \\ &\quad < \alpha_q - q! \sum_{k=[(q+1)/2]+1}^{2p+1} \frac{(2k-2-q)!}{(2k)!} B_{2k}\left(\frac{1}{n^{2k-1-q}} - 1\right) \quad (q \text{ is odd}), \end{aligned}$$

$$\begin{aligned}
&\alpha_q + q! \sum_{k=[(q+1)/2]+1}^{2p+1} \frac{(2k-2-q)!}{(2k)!} B_{2k}\left(\frac{1}{n^{2k-1-q}} - 1\right) \\
&\quad < \log A_q(n) \\
&\quad < \alpha_q + q! \sum_{k=[(q+1)/2]+1}^{2p} \frac{(2k-2-q)!}{(2k)!} B_{2k}\left(\frac{1}{n^{2k-1-q}} - 1\right) \quad (q \text{ is even}).
\end{aligned} \tag{3.14}$$

Now, taking the limit on each side of the inequalities in (3.14) as $n \to \infty$, we obtain the results in Theorem 3.4. □

Remark 3.5. It is easily seen that the specialized inequalities of (3.9) when $q = 1$ and $q = 3$ and (3.10) when $q = 2$ correspond with those inequalities of Equations (9), (34), and (33) in Chen's work [25], respectively.

Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (2012-0002957).

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