Research Article

Multiplicative Isometries on $F$-Algebras of Holomorphic Functions

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We study multiplicative isometries on the following $F$-algebras of holomorphic functions: Smirnov class $N^*_X$, Privalov class $N^p_X$, Bergman-Privalov class $AN^p_X$, and Zygmund $F$-algebra $N\log^\beta N_X$, where $X$ is the open unit ball $B_n$ or the open unit polydisk $D^n$ in $\mathbb{C}^n$.

1. Introduction

Complex-linear isometries on function spaces of holomorphic functions have been studied for almost five decades by many mathematicians. In this paper we study multiplicative isometries on certain $F$-algebras of holomorphic functions. Recall that an $F$-algebra is a topological algebra in which the topology arises from a complete metric. For a positive integer $n$ let $B_n$ denote the open unit ball in the $n$-dimensional complex vector space $\mathbb{C}^n$ and $D^n$ the unit polydisk in $\mathbb{C}^n$. We characterize multiplicative isometries on the Smirnov class, the Privalov class, the Bergman-Privalov class and the Zygmund $F$-algebras on $B_n$ or $D^n$. Surjective multiplicative maps on the Smirnov class, and the Bergman-Privalov class have already been correspondingly characterized in [1, 2].

2. Preliminaries

In studying surjective isometries in [1, 2] we applied the Mazur-Ulam theorem for surjective maps on certain subspaces, which themselves are Banach spaces, of the given $F$-algebras.
Generally we do not assume surjectivity of the isometries in this paper, so instead of the Mazur-Ulam theorem we use Lemma 2.1. Recall that a normed real-linear space \( L \) is uniformly convex if for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that the inequality \( \|a + b\| \leq 2 - \delta \) holds for every pair of \( a, b \in L \) with \( \|a\| \leq 1, \|b\| \leq 1, \) and \( \|a - b\| \geq \varepsilon \). It is well known that Hilbert spaces and \( L^p \)-spaces for \( 1 < p < \infty \) are uniformly convex.

**Lemma 2.1.** Let \( L_1 \) and \( L_2 \) be normed real-linear spaces with \( L_2 \) uniformly convex. Let \( S \) be an isometry from \( L_1 \) into \( L_2 \) such that \( S(0) = 0 \). Then \( S \) is real-linear.

The lemma might be well known, but we give a sketch of the proof for the completeness and the benefit of the reader.

**Proof of Lemma 2.1.** Let \( a, b \) be arbitrary elements of \( L_1 \). Put \( 2r = \|a - b\| \). Then since \( S \) is an isometry, \( \|S(a) - S(b)\| = 2r \) and \( \|S(a) - (S(a) + S(b))/2\| = \|S(b) - (S(a) + S(b))/2\| = r \). We also have \( \|S(a) - (S(a) + S(b))/2\| = \|S(b) - (S(a) + S(b))/2\| = r \).

Suppose that \( (S(a) + S(b))/2 \neq (S(a) + S(b))/2 \). Set

\[
\varepsilon = \left\| S\left( \frac{a + b}{2} \right) - \frac{S(a) + S(b)}{2} \right\|.
\]

Since \( L_2 \) is uniformly convex and \( \varepsilon \) is positive there exists a \( \delta > 0 \) such that

\[
\left\| S\left( \frac{a + b}{2} \right) - \frac{S(a) + S(b)}{2} \right\| + \left\| S\left( \frac{a + b}{2} \right) - \frac{S(a) + S(b)}{2} \right\| \leq 2r - \delta,
\]

\[
\left\| S\left( \frac{a + b}{2} \right) - \frac{S(a) + S(b)}{2} \right\| + \left\| S\left( \frac{a + b}{2} \right) - \frac{S(a) + S(b)}{2} \right\| \leq 2r - \delta.
\]

Then by the triangle inequality

\[
\|2S(a) - 2S(b)\| \leq 4r - 2\delta
\]

holds, which contradicts to \( \|S(a) - S(b)\| = 2r \). Thus we get \( S((a + b)/2) = (S(a) + S(b))/2 \), from which for \( b = 0 \) we obtain \( S(a/2) = S(a)/2 \). Substituting \( a \) by \( a + b \) in the last equality we get

\[
S\left( \frac{a + b}{2} \right) = S\left( \frac{a + b}{2} \right) = \frac{S(a) + S(b)}{2},
\]

so that \( S(a + b) = S(a) + S(b) \). A routine argument yields \( S(ta) = tS(a), t \in \mathbb{R} \). \( \square \)

For \( X \in \{ \mathbb{B}_n, \mathbb{D}_n \} \), we denote by \( \partial X \) its distinguished boundary. For \( X = \mathbb{B}_n \), this is the topological boundary \( \partial \mathbb{B}_n \), and for the polydisk \( \mathbb{D}_n \), it is the torus \( \mathbb{T}^n \). Denote the normalized Lebesgue measure on \( \partial X \) by \( \sigma \). A holomorphic map \( \varphi \) is inner if \( \lim_{r \to 1^-} \varphi(rz) \) exists and lies in \( \partial X \) for almost all \( z \in \partial X \) with respect to \( \sigma \). We say that \( \lim_{r \to 1^-} \varphi(rz) \) is the boundary map of \( \varphi \) and denote it by \( \varphi^* \). We say that \( \varphi^* \) is measure preserving if \( \sigma((\varphi^*)^{-1}(E)) = \sigma(E) \) for every Borel set \( E \subset \partial X \).
Now we recall definitions and some properties of the Smirnov class, the Privalov class, the Bergman-Privalov class, and the Zygmund $F$-algebra on $B_\mathbf{n}$ or $D^n$. The space of all holomorphic functions on $X = B_\mathbf{n}$ or $D^n$ is denoted by $H(X)$. For each $0 < p \leq \infty$, the Hardy space is denoted by $H^p(X)$ with the norm $\| \cdot \|_p$.

### 2.1. Smirnov Class $N_*(X)$

Let $X \in \{ B_\mathbf{n}, D^n \}$. The Nevanlinna class $N(X)$ on $X$ is defined as the set of all holomorphic functions $f$ on $X$ such that

$$
\sup_{0 < r < 1} \int_{\partial X} \ln(1 + |f(r\zeta)|) d\sigma(\zeta) < \infty \quad \text{(2.5)}
$$

holds. It is known that every $f \in N(X)$ has a finite nontangential limit, denoted by $f^*$, almost everywhere on $\partial X$.

The Smirnov class $N_*(X)$ is defined as

$$
N_*(X) = \left\{ f \in N(X) : \sup_{0 < r < 1} \int_{\partial X} \ln(1 + |f(r\zeta)|) d\sigma(\zeta) = \int_{\partial X} \ln(1 + |f^*(\zeta)|) d\sigma(\zeta) \right\}. \quad \text{(2.6)}
$$

Define a metric

$$
d_{N_*(X)}(f, g) = \int_{\partial X} \ln(1 + |f^*(\zeta) - g^*(\zeta)|) d\sigma(\zeta) \quad \text{(2.7)}
$$

for $f, g \in N_*(X)$. With the metric $d_{N_*(X)}(\cdot, \cdot)$ the Smirnov class $N_*(X)$ becomes an $F$-algebra and

$$
\bigcup_{q > 0} H^q(X) \subset N_*(X), \quad \text{(2.8)}
$$

in particular, $H^\infty(X)$ is a dense subalgebra of $N_*(X)$. The convergence in the metric is stronger than uniform convergence on compact subsets of $X$.

Complex-linear isometries on the Smirnov class were characterized by Stephenson in [3].

### 2.2. Privalov Class $N^p(X)$

Let $X \in \{ B_\mathbf{n}, D^n \}$. The Privalov class $N^p(X)$, $1 < p < \infty$, is defined as (for the original source see [4, 5])

$$
N^p(X) = \left\{ f \in H(X) : \sup_{0 < r < 1} \int_{\partial X} (\ln(1 + |f(r\zeta)|))^p d\sigma(\zeta) < \infty \right\}. \quad \text{(2.9)}
$$
It is well known that $N^p(X)$ is a subalgebra of $N(X)$, hence every $f \in N^p(X)$ has a finite nontangential limit almost everywhere on $\partial X$. Define a metric

$$d_p(f, g) = \left( \int_{\partial X} \left( \ln(1 + |f^*(\zeta) - g^*(\zeta)|) \right)^p d\sigma(\zeta) \right)^{1/p}$$

(2.10)

for $f, g \in N^p(X)$. With this metric $N^p(X)$ is an $F$-algebra (cf. [6, 7]) and

$$\bigcup_{q > 0} H^q(X) \subset N^p(X) \subset N(X).$$

(2.11)

The Hardy algebra $H^\infty(X)$ is dense in $N^p(X)$. The convergence on the metric is stronger than uniform convergence on compacts of $X$.

Complex-linear isometries on $N^p(X)$ are investigated by Iida and Mochizuki [8] for one-dimensional case, and by Subbotin [7] for a general case.

### 2.3. Bergman-Privalov Class $AN^p_\alpha(X)$

Let $1 \leq p < \infty$ and $\alpha > -1$. The Bergman-Privalov class on the unit ball $\mathbb{B}_n$ and the polydisk $\mathbb{D}_n$ are defined, respectively, as

$$AN^p_\alpha(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \|f\|_{AN^p_\alpha(\mathbb{B}_n)}^p = \int_{\mathbb{B}_n} (\ln(1 + |f(z)|))^p dV_{\alpha,n}(z) < \infty \right\},$$

$$AN^p_\alpha(\mathbb{D}_n) = \left\{ f \in H(\mathbb{D}_n) : \|f\|_{AN^p_\alpha(\mathbb{D}_n)}^p = \int_{\mathbb{D}_n} (\ln(1 + |f(z)|))^p \prod_{j=1}^n dV_{\alpha,1}(z_j) < \infty \right\},$$

(2.12)

where $dV_{\alpha,n}(z) = c_{\alpha,n}(1 - |z|^2)^\alpha dV(z)$ for the normalized Lebesgue volume measure $dV$ on $\mathbb{B}_n$ and $c_{\alpha,n}$ is a normalization constant, that is $V_{\alpha,n}(\mathbb{B}_n) = 1$. Let $X \in \{\mathbb{B}_n, \mathbb{D}_n\}$. In what follows $dV_{\alpha,n}(z)$ denotes $dV_{\alpha,n}(z)$ for $X = \mathbb{B}_n$ and $\prod_{j=1}^n dV_{\alpha,1}(z_j)$ for $X = \mathbb{D}_n$, respectively. The Bergman-Privalov class $AN^p_\alpha(X)$ is an $F$-algebra with respect to the metric

$$d_{AN^p_\alpha(X)}(f, g) = \|f - g\|_{AN^p_\alpha(X)}$$

(2.13)

for $f, g \in AN^p_\alpha(X)$. For some results in the case $p = 1$ see [9].

The weighted Bergman space for $q > 0$ and $\alpha > -1$ on the unit ball $\mathbb{B}_n$ and the polydisk $\mathbb{D}_n$ are defined, respectively, as

$$A^q_{\alpha}(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \|f\|_{A^q_{\alpha}(\mathbb{B}_n)}^q = \int_{\mathbb{B}_n} |f(z)|^q dV_{\alpha,n}(z) < \infty \right\},$$

$$A^q_{\alpha}(\mathbb{D}_n) = \left\{ f \in H(\mathbb{D}_n) : \|f\|_{A^q_{\alpha}(\mathbb{D}_n)}^q = \int_{\mathbb{D}_n} |f(z)|^q \prod_{j=1}^n dV_{\alpha,1}(z_j) < \infty \right\}.$$
It is known that
\[
\bigcup_{q>0} A_q^\alpha(X) \subset A_{N_\alpha}^X. \tag{2.15}
\]

Complex-linear isometries on the Bergman-Privalov class on the unit ball were characterized by Matsugu and Ueki in [10] and on the polydisk by Stević in [2].

### 2.4. Zygmund F-Algebra $N_{\log^\beta N}(X)$

Let $\beta > 0$ and $\varphi^\beta(t) = t(\ln(\gamma + t))^{\beta}$, where $\gamma = \max\{e, e^\beta\}$. Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. The Zygmund F-algebra $N_{\log^\beta N}(X)$ on $X$ is defined as

\[
N_{\log^\beta N}(X) = \left\{ f \in H(X) : \sup_{0 \leq r < 1} \int_{\partial X} \varphi^\beta(\ln(1 + |f(\zeta)|))d\sigma(\zeta) < \infty \right\}. \tag{2.16}
\]

It is known that

\[
N_{\log^\beta N}(X) = \left\{ f \in H(X) : \sup_{0 \leq r < 1} \int_{\partial X} \varphi^\beta(\ln^* |f(\zeta)|)d\sigma(\zeta) < \infty \right\}, \tag{2.17}
\]

\[
\bigcup_{p>0} H_p^\alpha(X) \subset N_{\log^\beta N}(X) \subset N_\alpha(X). \tag{2.18}
\]

This implies that the finite nontangential limit $f^*$ exists almost everywhere on $\partial X$, for any $f \in N_{\log^\beta N}$. For $f, g \in N_{\log^\beta N}$

\[
d_{N_{\log^\beta N}(X)}(f, g) = \int_{\partial X} \varphi^\beta(\ln(1 + |f^*(\zeta) - g^*(\zeta)|))d\sigma(\zeta) \tag{2.19}
\]

defines a complete metric on $N_{\log^\beta N}(X)$ and $N_{\log^\beta N}(X)$ is an $F$-algebra with this metric (cf. [11]).

Ueki [12] characterized the complex-linear isometries on the Zygmund F-algebra on the balls.

### 3. Main Results

In this section we formulate and prove the main results in this paper.

#### 3.1. Multiplicative Isometries on $N_\alpha(X)$

Our first result concerns the Smirnov class.
Theorem 3.1. Let $X \in \{\mathbb{B}_n, \mathbb{D}_n\}$. Suppose that $T : N_* (X) \to N_* (X)$ is a (not necessarily linear) multiplicative isometry. Then there is an inner map $\varphi$ on $X$ whose boundary map $\varphi^*$ is measure preserving and such that either of the following formulas holds:

$$
T(f) = f \circ \varphi \quad \text{for every } f \in N_* (X),
$$

(3.1)


$$
T(f) = \overline{f \circ \varphi} \quad \text{for every } f \in N_* (X).
$$

Proof. First we claim that $T(1) = 1$. Since $T(1) = T(1)^2$ and $T(1)$ is a holomorphic function on the connected open set $X$ we get $T(1) = 0$ or $T(1) = 1$. But $T(1) = 0$ is impossible because if it were $T(1) = 0$, then $0 = T(f)T(1) = T(f)$, for each $f \in N_* (X)$, which contradicts with the assumption that $T$ is an isometry. As $T(0) = T(0)^2$ and $T$ is injective, we obtain $T(0) = 0$. Similarly $T(-1) = -1$ is also observed by making use of $T(-1)^2 = T(1) = 1$. Then $T(i^2) = T(1^2) = -1$ assert that $T(i) = i$ or $T(i) = -i$. If $T(i) = i$, then the first formula of the conclusion will follow and the second one will follow from $T(i) = -i$.

Next we show $T(1/2) = 1/2$. Put $r = 1/2$. Suppose that $|T(r)| > r$ on a set of positive measure on $\partial X$. Then there exists a subset $E$ of positive measure and $\varepsilon > 0$ with $|T(r)^*| \geq (1 + \varepsilon)r$ on $E$. Since

$$
\lim_{n \to \infty} \frac{\ln(1 + (1 + \varepsilon)^n r^n)}{\ln(1 + r^n)} = \infty,
$$

(3.2)

there is a positive integer $n_0$ such that

$$
\int_{E} \ln(1 + (1 + \varepsilon)^{n_0} r^{n_0}) d\sigma > \int_{\partial X} \ln(1 + r^{n_0}) d\sigma.
$$

(3.3)

From this and since $T$ is a multiplicative isometry on $N_* (X)$ we have that

$$
\int_{\partial X} \ln(1 + r^{n_0}) d\sigma = \int_{\partial X} \ln(1 + |T(r)^*|^{n_0}) d\sigma
$$

$$
\geq \int_{E} \ln(1 + (1 + \varepsilon)^{n_0} r^{n_0}) d\sigma > \int_{\partial X} \ln(1 + r^{n_0}) d\sigma,
$$

(3.4)

which is a contradiction proving $|T(r)^*| \leq r$ almost everywhere on $\partial X$. Hence $|T(1/r)^*| \geq 1/r$ holds almost everywhere on $\partial X$ as $T(r)T(1/r) = T(1) = 1$ almost everywhere on $\partial X$. Since

$$
\ln\left(1 + \frac{1}{r}\right) = \int_{\partial X} \ln\left(1 + \frac{1}{r}\right) d\sigma = \int_{\partial X} \ln\left(1 + \left|T\left(\frac{1}{r}\right)^*\right|\right) d\sigma,
$$

(3.5)

we have that $|T(1/r)^*| = 1/r$ and $|T(r)^*| = r$ almost everywhere on $\partial X$.

Since $\ln(1 + (1 - r)) = d(r, 1) = d(T(r), 1)$ and

$$
d(T(r), 1) = \int_{\partial X} \ln(1 + |1 - T(r)^*|) d\sigma,
$$

(3.6)
it is easy to check that $T(1/2)^* = 1/2$ almost everywhere on $\partial X$. Hence $T(1/2) = 1/2$ holds. As $T$ is multiplicative, $T$ is 1/2-homogeneous in the sense that $T(f/2) = T(f)/2$ holds for every $f \in N_+(X)$.

Let $f, g \in H^1(X)$. It requires only elementary calculation applying the 1/2-homogeneity of $T$ to check that

$$
\int_{\partial X} \ln \left(1 + \frac{|f^* - g^*|}{2m} \right) d\sigma = \int_{\partial X} \ln \left(1 + \frac{|T(f)^* - T(g)^*|}{2m} \right) d\sigma
$$

(3.7)

holds. Multiplying (3.7) by $2^m$ and then letting $m \to \infty$ we get

$$
\int_{\partial X} |f^* - g^*| d\sigma = \int_{\partial X} |T(f)^* - T(g)^*| d\sigma
$$

(3.8)

by the monotone convergence theorem, since $2^m \ln(1 + (t/2^m))$ nondecreases monotonically to $t$ as $m \to \infty$ for any $t \geq 0$, which can be easily proved by considering the function $g_t(x) = x \ln(1 + (t/x))$. From (3.8) for $g = 0$, we obtain $T(H^2(X)) \subseteq H^1(X)$ and the restricted map $T|_{H^2(X)}$ is an isometry with respect to the metric induced by the $H^1$-norm $\| \cdot \|_1$.

Let the function $\theta$ on the interval $[0, \infty)$ be defined as

$$
\theta(x) = \begin{cases} 
\frac{1}{2}, & x = 0 \\
\frac{x - \ln(1 + x)}{x^2}, & x > 0.
\end{cases}
$$

(3.9)

It is easy to check that $\theta$ is positive and continuous on $[0, \infty)$ and $\lim_{x \to \infty} \theta(x) = 0$. Hence $\theta$ is bounded on $[0, \infty)$, so that

$$
M_{\theta} := \sup_{x \geq 0} \theta(x) < \infty.
$$

(3.10)

We claim that the inclusion $T(H^2(X)) \subseteq H^2(X)$ and $T|_{H^2(X)}$ is isometric with respect to the metric induced by the $H^2$-norm. For this purpose let $f, g \in H^2(X)$. Now note that since $H^2(X) \subseteq H^1(X)$, equality (3.7) holds and as well as the next equality

$$
\int_{\partial X} \left| \frac{f^* - g^*}{2m} \right| d\sigma = \int_{\partial X} \left| \frac{T(f)^* - T(g)^*}{2m} \right| d\sigma.
$$

(3.11)

By subtracting (3.7) from (3.11) and then multiplying such obtained equation by $2^m$ we obtain

$$
\int_{\partial X} |f^* - g^*|^2 \theta \left( \frac{|f^* - g^*|}{2m} \right) d\sigma = \int_{\partial X} |T(f)^* - T(g)^*|^2 \theta \left( \frac{|T(f)^* - T(g)^*|}{2m} \right) d\sigma.
$$

(3.12)

As $\theta$ is bounded the function $M_{\theta}|f^* - g^*|^2$ is an integrable function dominating the integrand in the left-hand side integral in (3.12). Letting $m \to \infty$ and applying the Lebesgue theorem
on dominated convergence to the left-hand side and Fatou’s lemma to the right-hand side (as \( \theta \) is positive on \([0, \infty)\)) we obtain

\[
\int_{\partial X} |f^* - g^*|^2 \theta(0) d\sigma \geq \int_{\partial X} |T(f)^* - T(g)^*|^2 \theta(0) d\sigma.
\]

From this and since \( \theta(0) = 1/2 \) we get that the function \( |T(f)^* - T(g)^*|^2 \) is integrable. Letting again \( m \to \infty \) in (3.12) we have that

\[
\int_{\partial X} |f^* - g^*|^2 d\sigma = \int_{\partial X} |T(f)^* - T(g)^*|^2 d\sigma
\]

by the Lebesgue theorem on dominated convergence now applied to both integrals in (3.12). Hence \( \|f - g\|_2 = \|T(f) - T(g)\|_2 \) for every pair of \( f, g \in H^2(X) \). For \( g = 0 \), we get \( \|f\|_2 = \|T(f)\|_2 \) and consequently \( T(H^2(X)) \subseteq H^2(X) \), as claimed.

Since \( H^2(X) \) is a Hilbert space, it is uniformly convex. Hence by Lemma 2.1 the restriction \( T|_{H^2(X)} \) is real-linear. Since the operations of scalar multiplication and addition on \( N_\ast \) are continuous and \( H^2(X) \) is dense in \( N_\ast(X) \) we see that \( T \) is real-linear on \( N_\ast(X) \).

First assume \( T(i) = i \). As \( T \) is real-linear and multiplicative, \( T \) is complex-linear in this case. Then by [3, Theorem 2.2] and since \( T(1) = 1 \), there is an inner map \( \psi \) such that \( T(f) = f \circ \psi \) for every \( f \in N_\ast(X) \).

Now assume \( T(i) = -i \). Let \( \tilde{T} : N_\ast(X) \to N_\ast(X) \) be defined as \( \tilde{T}(f) = T(f) \) for every \( f \in N_\ast(X) \), where

\[
\tilde{T}(z_1, \ldots, z_n) = \overline{f(z_1, \ldots, z_n)}
\]

for \( f \in N_\ast(X) \). Then \( \tilde{T} \) is well defined and a complex-linear isometry from \( N_\ast(X) \) into itself. Again by [3, Theorem 2.2] we have that there is an inner map \( \psi \) on \( X \) whose boundary map \( \psi^\ast \) is measure preserving such that \( \tilde{T}(f) = f \circ \psi \) for every \( f \in N_\ast \). This implies that \( T(f) = \tilde{T}(f) = f \circ \overline{\psi} \) for every \( f \in N_\ast(X) \).

**Corollary 3.2** (see [1]). Let \( X = \{\mathbb{B}_n, \mathbb{D}^n\} \). Suppose that \( T : N_\ast(X) \to N_\ast(X) \) is a (not necessarily linear) surjective multiplicative isometry. Then there is a holomorphic automorphism \( \psi \) on \( X \) such that either of the following formulas holds:

\[
T(f) = f \circ \psi \quad \text{for every } f \in N_\ast(X),
\]

\[
T(f) = \overline{f \circ \overline{\psi}} \quad \text{for every } f \in N_\ast(X),
\]

where \( \psi \) is a unitary transformation for \( X = \mathbb{B}_n \) while for \( X = \mathbb{D}^n \), \( \psi(z_1, \ldots, z_n) = (e^{i\theta_j} z_{j_1}, \ldots, e^{i\theta_j} z_{j_n}) \) for some real numbers \( \theta_j \) for \( j = 1, \ldots, n \) and a permutation \( (j_1, \ldots, j_n) \) of the integers from 1 to \( n \).

**Proof.** By Theorem 3.1, \( T \) is complex-linear or conjugate linear. If \( T \) is complex-linear, then the result holds by [3, Corollary 2.3]. If \( T \) is conjugate linear, then put \( \tilde{T}(f) = T(f) \) for \( f \in N_\ast(X) \), where \( \tilde{T} \) is defined as in (3.15). Then \( \tilde{T}(f) = f \circ \psi \), for every \( f \in N_\ast(X) \), and for an inner
map \( \varphi \) on \( X \) whose boundary map \( \varphi^* \) is measure preserving. Since \( \widetilde{T} \) is a surjective isometry, the desired property of \( \varphi \) again follows from [3, Corollary 2.3].

### 3.2. Multiplicative Isometries on \( N^p(X) \)

The next result concerns the Privalov class.

**Theorem 3.3.** Let \( X \in \{ \mathbb{B}_n, \mathbb{D}^n \} \) and \( 1 < p < \infty \). Suppose that \( T : N^p(X) \to N^p(X) \) is a (not necessarily linear) multiplicative isometry. Then there is an inner map \( \varphi \) on \( X \) whose boundary map \( \varphi^* \) is measure preserving and such that either of the following formulas holds:

\[
T(f) = f \circ \varphi \quad \text{for every } f \in N^p(X), \\
T(f) = \overline{f} \circ \overline{\varphi} \quad \text{for every } f \in N^p(X).
\]

**Proof.** Since \( T \) is multiplicative we see by the same way as in the proof of Theorem 3.1 that \( T(0) = 0, T(1) = 1 \) and \( T(i) = i \) or \( T(i) = -i \). Also we see that \( T(1/2) = 1/2 \). It follows by the proof of Theorem 3.1 that for every pair \( f \) and \( g \) in \( H^p(X) \),

\[
\int_{\partial X} \left( \ln \left( 1 + \frac{|f^* - g^*|^2}{2m} \right) \right)^{p} d\sigma = \int_{\partial X} \left( \ln \left( 1 + \frac{|T(f)^* - T(g)^*|^2}{2m} \right) \right)^{p} d\sigma
\]

holds. Multiplying (3.18) by \( 2^{mp} \) and then letting \( m \to \infty \) we get

\[
\int_{\partial X} |f^* - g^*|^p d\sigma = \int_{\partial X} |T(f)^* - T(g)^*|^p d\sigma.
\]

Thus \( T(H^p(X)) \subseteq H^p(X) \). The Hardy space \( H^p(X) \) can be seen as a subspace of \( L^p(\partial X) \). Since \( L^p(\partial X) \) is uniformly convex, so is \( H^p(X) \) for \( 1 < p < \infty \). Then by Lemma 2.1 the operator \( T \) is real-linear on \( H^p(X) \). Since \( H^p(X) \) is a dense subspace of \( N^p(X) \) we see that \( T \) is real-linear on \( N^p(X) \). As we have already learnt that \( T(i) = i \) or \( T(i) = -i \), we obtain that \( T \) is complex-linear or conjugate linear on \( N^p(X) \). The rest of the proof is similar to the last part of the proof of Theorem 3.1 applying [7, Theorem 1] instead of [3, Theorem 2.2]. We omit the details.

**Corollary 3.4.** Let \( X \in \{ \mathbb{B}_n, \mathbb{D}^n \} \) and \( 1 < p < \infty \). Suppose that \( T : N^p(X) \to N^p(X) \) is a (not necessarily linear) surjective multiplicative isometry. Then there is a holomorphic automorphism \( \varphi \) on \( X \) such that either of the following formulas holds:

\[
T(f) = f \circ \varphi \quad \text{for every } f \in N^p(X), \\
T(f) = \overline{f} \circ \overline{\varphi} \quad \text{for every } f \in N^p(X),
\]

where \( \varphi \) is a unitary transformation for \( X = \mathbb{B}_n \), while for \( X = \mathbb{D}^n \), \( \varphi(z_1, \ldots, z_n) = (e^{i\theta_1}z_{j_1}, \ldots, e^{i\theta_n}z_{j_n}) \) for some real numbers \( \theta_j \) for \( j = 1, \ldots, n \) and a permutation \( (j_1, \ldots, j_n) \) of the integers from 1 to \( n \).
Proof. By Theorem 3.3, $T$ is complex-linear or conjugate linear. If $T$ is complex-linear, then the result follows directly from [7, Corollary and Remark 3]. If $T$ is conjugate linear, then put $\tilde{T}(f) = T(\tilde{f})$ for $f \in N^p(X)$, where $\tilde{f}$ is defined as in (3.15). Then $\tilde{T}$ is a complex-linear isometric surjection from $N^p(X)$ onto itself. Hence by [7, Corollary and Remark 3] there is a desired automorphism on $X$ such that $T(f) = \tilde{f} \circ \overline{\varphi}$ for every $f \in N^p(X)$. \hfill \Box

### 3.3. Multiplicative Isometries on $AN^p_a(X)$

The next result concerns the Bergman-Privalov class.

**Theorem 3.5.** Let $X \in \{B_n, \mathbb{D}^n\}, 1 \leq p < \infty$ and $\alpha > -1$. Suppose that $T : AN^p_a(X) \to AN^p_a(X)$ is a (not necessarily linear) multiplicative isometry. Then there is a holomorphic self-map $\varphi$ on $X$ with the property that

$$\int_X h \circ \varphi(z) dV_\alpha(z) = \int_X h(z) dV_\alpha(z)$$

(3.21)

for every bounded or positive Borel function $h$ on $X$ such that either of the following formulas holds:

$$T(f) = f \circ \varphi \quad \text{for every } f \in AN^p_a(X),$$

$$T(f) = \overline{f} \circ \overline{\varphi} \quad \text{for every } f \in AN^p_a(X).$$

(3.22)

Proof. We can prove the theorem in a way similar to that in the proofs of Theorem 3.1 for $p = 1$ and Theorem 3.3 for $1 < p < \infty$. For the case of $p = 1$, instead of using the Hardy spaces $H^1(X)$ and $H^2(X)$ we make use of the weighted Bergman spaces $A^1_a(X)$ and $A^2_a(X)$. For the case of $1 < p < \infty$, instead of using the Hardy space $H^p(X)$ we make use of the weighted Bergman space $A^p_a(X)$. We also apply [10, Theorem 1] for $X = B_n$ and [2, Theorem 2] for $X = \mathbb{D}^n$ to represent complex-linear isometries instead of [3, Theorem 2.2]. \hfill \Box

**Corollary 3.6** (see [2]). Let $X \in \{B_n, \mathbb{D}^n\}, 1 \leq p < \infty$ and $\alpha > -1$. Suppose that $T : AN^p_a(X) \to AN^p_a(X)$ is a (not necessarily linear) surjective multiplicative isometry. Then there is a holomorphic automorphism $\varphi$ on $X$ such that either of the following formulas holds:

$$T(f) = f \circ \varphi \quad \text{for every } f \in AN^p_a(X),$$

$$T(f) = \overline{f} \circ \overline{\varphi} \quad \text{for every } f \in AN^p_a(X),$$

(3.23)

where $\varphi$ is a unitary transformation for $X = B_n$, while for $X = \mathbb{D}^n$, $\varphi(z_1, \ldots, z_n) = (e^{i\theta_1}z_{j_1}, \ldots, e^{i\theta_n}z_{j_n})$ for some real numbers $\theta_j$ for $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to $n$.

Proof. By Theorem 3.5, $T$ is complex-linear or conjugate linear. Suppose that $T$ is complex-linear. If $X = B_n$, then the conclusion follows by [10, Theorem 2], while for $X = \mathbb{D}^n$ the conclusion follows similar to the corresponding part of the proof of [2, Theorem 3]. If $T$ is conjugate linear, then the conclusion follows from the similar argument in the proof of Corollary 3.2. \hfill \Box
\[ \textbf{3.4. Isometries on } \text{Nlog}^\beta \text{N}(X) \]

In [12] Ueki characterized complex-linear isometries on the Zygmund F-algebra on $B_n$. For $D^n$ the following result is proved similar to [12, Theorem 1]. Hence it is omitted.

**Theorem 3.7.** Let $\beta > 0$. If $T$ is a complex-linear isometry of $\text{Nlog}^\beta \text{N}(D^n)$ into itself, then there exist an inner function $\Psi$ and an inner map $\varphi$ on $D^n$ whose boundary map $\varphi^*$ is measure preserving on $\mathbb{T}^n$ such that

\[ T(f) = \Psi C_\varphi(f) = \Psi(f \circ \varphi) \quad \text{for every } f \in \text{Nlog}^\beta \text{N}(D^n). \quad (3.24) \]

Conversely, for given such $\Psi$ and $\varphi$, the weighted composition operator $\Psi C_\varphi$ is an injective linear isometry of $\text{Nlog}^\beta \text{N}(D^n)$.

For the surjective isometries the result is as follows.

**Corollary 3.8.** An isometry $T$ of $\text{Nlog}^\beta \text{N}(D^n)$ is surjective if and only if $T = aC_\varphi$ where $a \in \mathbb{C}$ with $|a| = 1$ and $\mathcal{H}(z_1, \ldots, z_n) = (e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n)$ for some real numbers $\theta_j$, $j = 1, \ldots, n$ and a permutation $(j_1, \ldots, j_n)$ of the integers from 1 to $n$.

To prove Corollary 3.8 we need the next auxiliary result.

**Lemma 3.9.** For any function $f \in \text{N}(D^n)$, $f \in \text{Nlog}^\beta \text{N}(D^n)$ if and only if $\varphi_\beta(\text{ln}^+|f^*|) \in L^1(\mathbb{T}^n)$ and

\[ \varphi_\beta(\text{ln}^+|f(z)|) \leq \int_{\mathbb{T}^n} P(z, \zeta) \varphi_\beta(\text{ln}^+|f^*(\zeta)|) d\sigma(\zeta) \quad \text{for } z \in D^n, \quad (3.25) \]

where $P(z, \zeta)$ denotes the Poisson kernel for $D^n$;

\[ P(z, \zeta) = P_{r_1}(\theta_1 - \phi_1) \cdots P_{r_n}(\theta_n - \phi_n) \quad (3.26) \]

for $z = (r_1e^{i\phi_1}, \ldots, r_ne^{i\phi_n})$, $\zeta = (e^{i\phi_1}, \ldots, e^{i\phi_n})$ and

\[ P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2} \quad (3.27) \]

is the Poisson kernel for the unit disk $D$.

**Proof.** If $f \in \text{Nlog}^\beta \text{N}(D^n)$, then Fatou’s lemma shows that $\varphi_\beta(\text{ln}^+|f^*|) \in L^1(\mathbb{T}^n)$. The inclusion (2.18) implies $f \in \text{N}_1(D^n)$, and so we see that $\text{ln}^+|f|$ has the least $n$-harmonic majorant. Since the least $n$-harmonic majorant of $\text{ln}^+|f|$ is the Poisson integral $P[\text{ln}^+|f^*|]$, we obtain the following inequality:

\[ \text{ln}^+|f(z)| \leq \int_{\mathbb{T}^n} P(z, \zeta) \text{ln}^+|f^*(\zeta)| d\sigma(\zeta) \quad \text{for } z \in D^n. \quad (3.28) \]
Note that \( \varphi_\beta(t) \) is strictly increasing and convex on \([0, \infty)\), and the measures \( d\mu_z(\zeta) = P(z, \zeta)d\sigma(\zeta) \) are normalized on \( \mathbb{T}^n \), which follows from the well-known equality

\[
\int_{\mathbb{T}^n} P(z, \zeta)d\sigma(\zeta) = 1. \tag{3.29}
\]

Applying Jensen’s inequality to (3.28), we obtain the desired inequality (3.25).

Conversely we put \( z = r\eta(0 \leq r < 1, \eta \in \mathbb{T}^n) \) in (3.25). By integrating with respect to \( \eta \) and applying Fubini’s theorem, we have that

\[
\int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f(r\eta)||)d\sigma(\eta) \leq \int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f^*(\zeta)||)d\sigma(\zeta) \int_{\mathbb{T}^n} P(r\eta, \zeta)d\sigma(\eta). \tag{3.30}
\]

By the symmetric property \( P(r\eta, \zeta) = P(r\zeta, \eta) \) and the normalization property of the Poisson kernel, we obtain that

\[
\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f(r\eta)||)d\sigma(\eta) \leq \int_{\mathbb{T}^n} \varphi_\beta(\ln^+|f^*(\zeta)||)d\sigma(\zeta). \tag{3.31}
\]

Hence the condition \( \varphi_\beta(\ln^+|f^*|) \in L^1(\mathbb{T}^n) \) implies that \( f \in N\log^\theta N(\mathbb{D}^n) \). \( \square \)

Now we give a proof of Corollary 3.8.

**Proof of Corollary 3.8.** Suppose that \( T \) is surjective. Then Theorem 3.7 gives that \( T = \Psi \mathbb{C}_\varphi \). A standard argument shows that \( \varphi \) is an automorphism of \( \mathbb{D}^n \). So there are conformal maps \( \varphi_j \) \((j = 1, \ldots, n)\) of \( \mathbb{D} \) onto \( \mathbb{D} \) and there is a permutation \((j_1, \ldots, j_n)\) of the integers from 1 to \( n \) such that

\[
\varphi(z_1, \ldots, z_n) = (\varphi_1(z_{j_1}), \ldots, \varphi_n(z_{j_n})). \tag{3.32}
\]

The mean value theorem shows that

\[
\int_{\mathbb{T}^n} \varphi_k(\zeta_{j_k})d\sigma(\zeta) = \int_{\mathbb{T}} \varphi_k(\zeta_{j_k})d\sigma_1(\zeta_{j_k}) = \varphi_k(0) \tag{3.33}
\]

for each \( k \in \{1, \ldots, n\} \). Here \( d\sigma_1 \) denotes the one-dimensional normalized Lebesgue measure on the unit circle \( \mathbb{T} \).

On the other hand, the measure-preserving property of \( \varphi^* \) gives that

\[
\int_{\mathbb{T}^n} \varphi_k(\zeta_{j_k})d\sigma(\zeta) = \int_{\mathbb{T}^n} \langle \varphi^*(\zeta), e_k \rangle d\sigma(\zeta) = \int_{\mathbb{T}^n} \langle \varphi^*(\zeta), e_k \rangle d\sigma_1(\zeta_{j_k}) = \int_{\mathbb{T}^n} \zeta_{k}d\sigma(\zeta) = 0. \tag{3.34}
\]

By (3.33) and (3.34) we see that \( \varphi \) fixes the origin, and so each \( \varphi_k \) is the rotation transform.
Next we prove that $\Psi$ is a unimodular constant. If $f \in N\log^\beta N(\mathbb{D}^n)$ is such that $1 = T(f) = \Psi C_\varphi(f)$, then $1/\Psi = f \circ \varphi \in N\log^\beta N(\mathbb{D}^n)$. Inequality (3.25) in Lemma 3.9 gives that
\[
\varphi^\beta\left(\ln^+ \frac{1}{|\Psi(z)|}\right) \leq \int_{T^n} P(z, \zeta) \varphi^\beta\left(\ln^+ \frac{1}{|\Psi^* (\zeta)|}\right) d\sigma(\zeta) = 0,
\]
and so we have $1/|\Psi| \leq 1$ on $\mathbb{D}^n$. Since $\Psi$ is inner, $\Psi$ is a unimodular constant.

Now we show results on multiplicative isometries on the Zygmund $F$-algebras on $\mathbb{B}_n$ and $\mathbb{D}^n$.

**Theorem 3.10.** Let $X \in \{\mathbb{B}_n, \mathbb{D}^n\}$. Suppose that $T : N\log^\beta N(X) \to N\log^\beta N(X)$ is a (not necessarily linear) multiplicative isometry. Then there exists an inner map $\varphi$ on $X$ whose boundary map $\varphi^*$ is measure preserving on $\partial X$, such that either of the following formulas holds:
\[
T(f) = f \circ \varphi \quad \text{for every } f \in N\log^\beta N(X),
\]
\[
T(f) = \overline{f \circ \varphi} \quad \text{for every } f \in N\log^\beta N(X).
\]

Note that multiplicative isometries of the Privalov class and the Zygmund $F$-algebra have the same form as multiplicative isometries of the Smirnov class.

**Proof of Theorem 3.10.** As $T$ is multiplicative we obtain $T(1) = 1$, $T(0) = 0$, $T(-1) = -1$ and $T(i) = i$ or $T(i) = -i$. Since
\[
\lim_{n \to \infty} \frac{\varphi^\beta\left(\{(1 + \epsilon)/2\}^n\right)}{\varphi^\beta\left(\{1/2\}^n\right)} = \infty
\]
holds for every $\epsilon > 0$, the equation $T(1/2) = 1/2$ is proved similarly as in Theorem 3.1.

Let $f, g \in H^1(X)$. Then we can prove that
\[
\int_{\partial X} 2^n \varphi^\beta\left(\ln\left(1 + \left|\frac{f^*-g^*}{2m}\right|\right)\right) d\sigma = \int_{\partial X} 2^n \varphi^\beta\left(\ln\left(1 + \left|\frac{T(f)^*-T(g)^*}{2m}\right|\right)\right) d\sigma,
\]
following the lines of the corresponding part of the proof in Theorem 3.1. By some calculation we see that
\[
\varphi^\beta(\ln(1 + x)) \leq (\ln \varphi^\beta)^\beta x
\]
holds for every $x \geq 0$. Hence we get
\[
2^n \varphi^\beta\left(\ln\left(1 + \left|\frac{f^*-g^*}{2m}\right|\right)\right) \leq (\ln \varphi^\beta)^\beta |f^* - g^*|,
\]
almost everywhere on $\partial X$ and $(\ln \gamma_\beta)^\beta |f^* - g^*|$ is an integrable function dominating $2^m \varphi_\beta (\ln(1 + |(f^* / 2^m) - (g^* / 2^m)|))$. We get

$$
\lim_{m \to \infty} \int_{\partial X} 2^m \varphi_\beta \left( \ln \left( 1 + \left| \frac{f^*}{2^m} - \frac{g^*}{2^m} \right| \right) \right) d\sigma = (\ln \gamma_\beta)^\beta \int_{\partial X} |f^* - g^*| d\sigma
$$

(3.41)

by the Lebesgue dominated convergence theorem since

$$
\lim_{m \to \infty} 2^m \varphi_\beta \left( \ln \left( 1 + \left| \frac{f^*}{2^m} - \frac{g^*}{2^m} \right| \right) \right) = (\ln \gamma_\beta)^\beta |f^* - g^*|.
$$

(3.42)

On the other hand, applying Fatou’s lemma we get

$$
(\ln \gamma_\beta)^\beta \int_{\partial X} |T(f)^* - T(g)^*| d\sigma
$$

\[ \leq \lim \inf_{m \to \infty} \int_{\partial X} 2^m \varphi_\beta \left( \ln \left( 1 + \left| \frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m} \right| \right) \right) d\sigma \\
= \lim \inf_{m \to \infty} \int_{\partial X} 2^m \varphi_\beta \left( \ln \left( 1 + \left| \frac{f^*}{2^m} - \frac{g^*}{2^m} \right| \right) \right) d\sigma \\
= (\ln \gamma_\beta)^\beta \int_{\partial X} |f^* - g^*| d\sigma < \infty,
$$

(3.43)

from which for $g = 0$ we get $T(H^1(X)) \subseteq H^1(X)$. Since

$$
2^m \varphi_\beta \left( \ln \left( 1 + \left| \frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m} \right| \right) \right) \leq (\ln \gamma_\beta)^\beta |T(f)^* - T(g)^*|
$$

(3.44)

follows from (3.40), the function $(\ln \gamma_\beta)^\beta |T(f)^* - T(g)^*|$ is an integrable function dominating $2^m \varphi_\beta \left( \ln \left( 1 + \left| \frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m} \right| \right) \right)$. Hence

$$
(\ln \gamma_\beta)^\beta \int_{\partial X} |T(f)^* - T(g)^*| d\sigma = \lim_{m \to \infty} \int_{\partial X} 2^m \varphi_\beta \left( \ln \left( 1 + \left| \frac{T(f)^*}{2^m} - \frac{T(g)^*}{2^m} \right| \right) \right) d\sigma
$$

(3.45)

holds by the Lebesgue dominated convergence theorem. Consequently

$$
\int_{\partial X} |f^* - g^*| d\sigma = \int_{\partial X} |T(f)^* - T(g)^*| d\sigma
$$

(3.46)

holds. As $f$ and $g$ are arbitrary elements of $H^1(X)$ we obtain that $T|_{H^1(X)}$ is isometric on $H^1(X)$ with respect to the metric induced by the $H^1$-norm.
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We also obtain that there exists a bounded positive continuous function \( \theta_1 \) on \([0, \infty)\) such that \( \theta_1(0) \neq 0 \) and

\[
x^2 \theta_1(x) = \{ \ln \gamma \}_x^\beta x - \varphi_\beta(\ln(1 + x)).
\]

(3.47)

Applying this equality we obtain that \( T(H^2(X)) \subseteq H^2(X) \) and \( T|_{H^2(X)} \) is a real-linear isometry on \( H^2(X) \), hence \( T \) is a complex-linear (if \( T(i) = i \)) or conjugate linear isometry (if \( T(i) = -i \)) on \( N\log^\beta \mathcal{N}(X) \), similar as in the proof of Theorem 3.1. The rest of the proof is similar to the last part of the proof of Theorem 3.1 applying [12, Theorem 1] for \( X = \mathbb{B}_n \) and Theorem 3.7 for \( X = \mathbb{D}^n \) instead of [3, Theorem 2.2]. We omit the details.

\[
\square
\]

Corollary 3.11. Let \( X \in \{ \mathbb{B}_n, \mathbb{D}^n \} \). Suppose that \( T : N\log^\beta \mathcal{N}(X) \to N\log^\beta \mathcal{N}(X) \) is a (not necessarily linear) surjective multiplicative isometry. Then there exists a holomorphic automorphism \( \varphi \) on \( X \) such that either of the following formulas holds:

\[
T(f) = f \circ \varphi \quad \text{for every } f \in N\log^\beta \mathcal{N}(X),
\]

(3.48)

\[
T(f) = f \circ \overline{\varphi} \quad \text{for every } f \in N\log^\beta \mathcal{N}(X),
\]

where \( \varphi \) is a unitary transformation for \( X = \mathbb{B}_n \), while for \( X = \mathbb{D}^n \), \( \varphi(z_1, \ldots, z_n) = (e^{i\theta_1}z_{j_1}, \ldots, e^{i\theta_n}z_{j_n}) \) for some real numbers \( \theta_j \), \( j = 1, \ldots, n \) and a permutation \( (j_1, \ldots, j_n) \) of the integers from 1 to \( n \).

Note that surjective multiplicative isometries of the Privalov class, the Bergman-Privalov class, and the Zygmund \( F \)-algebra have the same form as surjective multiplicative isometries of the Smirnov class.

Proof of Corollary 3.11. By Theorem 3.10, \( T \) is complex-linear or conjugate linear. Suppose that \( T \) is complex-linear. Applying [12, Corollary 1] for \( X = \mathbb{B}_n \) and Corollary 3.8 for \( X = \mathbb{D}^n \) the result follows in this case. If \( T \) is conjugate linear, then the result follows by similar arguments as in the proof of Corollary 3.2.

\[
\square
\]

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