Research Article

α-Well-Posedness for Quasivariational Inequality Problems

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We introduce and study the concepts of α-well-posedness and L-α-well-posedness for quasivariational inequality problems having a unique solution and the concepts of α-well-posedness in the generalized sense and L-α-well-posedness in the generalized sense for quasivariational inequality problems having more than one solution. We present some necessary and/or sufficient conditions for the various kinds of well-posedness to occur. Our results generalize and strengthen previously known results for quasivariational inequality problems.

1. Introduction

Let $E$ be a reflexive real Banach space and let $K$ be a nonempty closed convex subset of $E$. Let $S$ be a set-valued mapping from $K$ to $K$ and let $A$ be an operator from $E$ to the dual space $E^*$. Bensoussan and Lions [1], Baiocchi and Capelo [2], and Mosco [3] considered the following quasivariational inequality (in short, (QVIP)), which is to find a point $u_0 \in K$ such that

$$u_0 \in S(u_0), \quad \langle Au_0, u_0 - v \rangle \leq 0, \quad \forall v \in S(u_0). \quad (1.1)$$

The interest in quasivariational inequality problems lies in the fact that many economic or engineering problems are modeled through them, as explained in [4, 5] where a wide bibliography on variational inequalities, quasivariational inequality problems, and related problems is contained. Moreover, under suitable assumptions, a quasivariational inequality is equivalent to a generalized Nash equilibrium problem [3].

On the other hand, well-posedness plays a crucial role in the stability theory for optimization problems, which guarantees that, for an approximating solution sequence, there exists a subsequence which converges to a solution [6]. The study of well-posedness for
scalar minimization problems started from Tikhonov [7] and Levitin and Polyak [8]. Since the convergence of numerical methods for quasivariational inequality Problems can be obtained also with the aid of well-posedness theory, Lignola [9] introduced and investigated the concepts of well-posedness and L-well-posedness for quasivariational inequalities having a unique solution and the concepts of well-posedness and L-well-posedness in the generalized sense for quasivariational inequality problems having more than one solution.

In this paper, inspired by the above concepts of well-posedness for (QVIP), we introduce and study the concepts of $\alpha$-well-posedness and $L\alpha$-well-posedness for quasivariational inequality Problems having a unique solution and the concepts of $\alpha$-well-posedness in the generalized sense and $L\alpha$-well-posedness in the generalized sense for quasivariational inequality Problems having more than one solution. The results in this paper generalize and improve the results in [9, 10].

2. Preliminaries

Denote by $\Gamma$ the solution set of (QVIP). Let $\alpha > 0$. In order to investigate the $\alpha$-well-posed for (QVIP), we need the following definitions.

First we recall the notion of Mosco convergence [11]. A sequence $(H_n)_n$ of subsets of $E$ Mosco converges to a set $H$ if

$$H = \liminf_n H_n = w - \limsup_n H_n,$$

(2.1)

where $\liminf_n H_n$ and $w - \limsup_n H_n$ are, respectively, the Painlevé-Kuratowski strong limit inferior and weak limit superior of a sequence $(H_n)_n$, that is,

$$\liminf_n H_n = \{y \in E: \exists y_n \in H_n, n \in N, \text{ with } y_n \rightharpoonup y\},$$

$$w - \limsup_n H_n = \{y \in E: \exists n_k \uparrow +\infty, n_k \in N, \exists y_{n_k} \in H_{n_k}, k \in N, \text{ with } y_{n_k} \rightharpoonup y\},$$

(2.2)

where “$\rightharpoonup$” means weak convergence, “$\rightarrow$” means strong convergence.

If $H = \liminf_n H_n$, we call the sequence $(H_n)_n$ of subsets of $E$ Lower Semi-Mosco which converges to a set $H$.

It is easy to see that a sequence $(H_n)_n$ of subsets of $E$ Mosco converges to a set $H$ which implies that the sequence $(H_n)_n$, also Lower Semi-Mosco, converges to the set $H$, but the converse is not true in general.

We will use the usual abbreviations usc and lsc for “upper semicontinuous” and “lower semicontinuous,” respectively. Let $E$ be a reflexive real Banach space with dual $E^*$. An operator $A : E \rightarrow E^*$ will be called hemicontinuous if it is continuous from every segment of $E$ to $E^*$ endowed with the weak topology. $A : E \rightarrow E^*$ will be called monotone if $\langle Au - Av, u - v \rangle \geq 0$ for every $u$ and $v \in E$. $A : E \rightarrow E^*$ will be called pseudomonotone if $\langle Au, u - v \rangle \leq 0 \Rightarrow \langle Av, u - v \rangle \leq 0$ for every $u$ and $v \in E$. 

Definition 2.1. A sequence \((u_n)_n\) is an \(\alpha\)-approximating sequence for (QVIP) if

(i) \((u_n) \in K, \text{ for all } n \in \mathbb{N}\);

(ii) there exists a sequence \((\varepsilon_n)_n\), \(\varepsilon_n > 0\), decreasing to 0 such that
\[
d(u_n, S(u_n)) \leq \varepsilon_n, \quad \text{that is, } u_n \in B(S(u_n), \varepsilon_n), \quad \forall n \in \mathbb{N},
\]
\[
\langle Av, u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n, \quad \forall v \in S(u_n), \quad \forall n \in \mathbb{N}.
\] (2.3)

Definition 2.2. A quasivariational inequality (QVIP) is said to be \(\alpha\)-well-posed (resp., \(\alpha\)-well-posed in the generalized sense) if it has a unique solution \(u_0\) and every \(\alpha\)-approximating sequence \((u_n)_n\) strongly converges to \(u_0\) (resp., if the solution set \(\Gamma\) of (QVIP) is nonempty and for every \(\alpha\)-approximating sequence \((u_n)_n\) has a subsequence which strongly converges to a point of \(\Gamma\)).

Definition 2.3. A sequence \((u_n)_n\) is an \(L\)-\(\alpha\)-approximating sequence for (QVIP) if:

(i) \((u_n) \in K, \text{ for all } n \in \mathbb{N}\);

(ii) there exists a sequence \((\varepsilon_n)_n\), \(\varepsilon_n > 0\), decreasing to 0 such that \(d(u_n, S(u_n)) \leq \varepsilon_n\), and
\[
\langle Av, u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n, \quad \forall v \in S(u_n), \quad \forall n \in \mathbb{N}.
\] (2.4)

Definition 2.4. A quasivariational inequality (QVIP) is said to be \(L\)-\(\alpha\)-well-posed (resp., \(L\)-\(\alpha\)-well-posed in the generalized sense) if it has a unique solution \(u_0\) and every \(L\)-\(\alpha\)-approximating sequence \((u_n)_n\) strongly converges to \(u_0\) (resp., if the solution set \(\Gamma\) of (QVIP) is nonempty and for every \(L\)-\(\alpha\)-approximating sequence \((u_n)_n\) has a subsequence which strongly converges to a point of \(\Gamma\)).

It is worth noting that if \(\alpha = 0\), then the definitions of \(\alpha\)-well-posedness, \(\alpha\)-well-posedness in the generalized sense, \(L\)-\(\alpha\)-well-posedness, and \(L\)-\(\alpha\)-well-posedness in the generalized sense for (QVIP), respectively, reduce to those of the well-posedness, well-posedness in the generalized sense, \(L\)-well-posedness, and \(L\)-well-posedness in the generalized sense for (QVIP) in [9]. We also note that Definition 2.2 generalizes and extends \(\alpha\)-well-posedness and \(\alpha\)-well-posedness in the generalized sense of variational inequalities in [10] which are related to the continuously differentiable gap function of variational inequality Problems introduced by Fukushima [12].

We recall some lemmas which will be needed in the rest of this paper.

**Lemma 2.5** (see [13]). Let \((H_n)_n\) be a sequence of nonempty subsets of the space \(E\) such that

(i) \(H_n\) is convex for every \(n \in \mathbb{N}\);

(ii) \(H_0 \subseteq \lim_n \inf H_n\);

(iii) there exists \(m \in \mathbb{N}\) such that \(\text{int} \bigcap_{n \geq m} H_n \neq \emptyset\).

Then, for every \(u_0 \in \text{int} H_0\), there exists a positive real number \(\delta\) such that \(B(u_0, \delta) \subseteq H_n\), for all \(n \geq m\).

If \(E\) is a finite dimensional space, then assumption (iii) can be replaced by

(iii)' \(\text{int} H_0 \neq \emptyset\).
The following Lemmas 2.6 and 2.7 play important roles in this paper. Now we present a Minty type lemma for quasivariational inequalities as follows.

**Lemma 2.6.** Suppose that set-valued mapping $S$ is nonempty convex-valued, the operator $A$ is hemicontinuous and monotone, $u_0 \in S(u_0)$. Then the following conditions are equivalent:

1. $(Au_0, u_0 - v) - (\alpha/2)\|u_0 - v\|^2 \leq 0$, for all $v \in S(u_0)$,
2. $(Av, u_0 - v) - (\alpha/2)\|u_0 - v\|^2 \leq 0$, for all $v \in S(u_0)$.

**Proof.** We first prove that (ii) implies (i). Let $v$ be a arbitrary point of $S(u_0)$. For every number $t \in [0, 1]$, since the set-valued mapping $S$ is convex-valued and $u_0 \in S(u_0)$, the point $v_t = tv + (1 - t)u_0$ belongs to $S(u_0)$. It follows from (ii) that

$$
(Av_t, u_0 - v_t) - \frac{\alpha}{2}\|u_0 - v_t\|^2 \leq 0.
$$

From the definition of $v_t$, one has

$$
\lim_{t \to 0} \left( (Av_t, u_0 - v) - \frac{\alpha}{2}t\|u_0 - v\|^2 \right) \leq 0,
$$

and it follows from the hemicontinuity of $A$ that

$$
(Au_0, u_0 - v) \leq 0,
$$

then

$$
(Au_0, u_0 - v) - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0, \quad \forall v \in S(u_0).
$$

The converse is an easy consequence of monotonicity of $A$. □

**Lemma 2.7.** Assume that set-valued mapping $S$ is nonempty convex-valued, then $u_0 \in \Gamma$ if and only if the following conditions hold:

$$
0 \in S(u_0), \quad (Au_0, u_0 - v) - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0, \quad \forall v \in S(u_0).
$$

**Proof.** The necessity is clearly held. Now we prove the sufficiency. Let for all $v \in S(u_0)$, for all $t \in [0, 1]$, $v_t = tv + (1 - t)u_0$. Since $S$ is convex-valued, $v_t \in S(u_0)$, one has

$$
(Au_0, u_0 - v_t) - \frac{\alpha}{2}\|u_0 - v_t\|^2 \leq 0, \quad \forall t \in (0, 1],
$$

which implies that

$$
(Au_0, u_0 - v) - t\frac{\alpha}{2}\|u_0 - v\|^2 \leq 0, \quad \forall t \in (0, 1], \forall v \in S(u_0).
$$
The above inequality implies, for $t$ converging to zero, that $u_0$ is a solution of (QVIP). This completes the proof. \hfill \Box

### 3. Case of a Unique Solution

In this section, we investigate some metric characterizations of $\alpha$-well-posedness and $L$-$\alpha$-well-posedness for (QVIP).

For any $\varepsilon > 0$, we consider the set

$$Q_\varepsilon = \left\{ u \in K : u \in B(S(u), \varepsilon), \ (Au, u - v) - \frac{\alpha}{2} \| u - v \|^2 \leq \varepsilon, \ \forall v \in S(u) \right\},$$

$$L_\varepsilon = \left\{ u \in K : u \in B(S(u), \varepsilon), \ (Av, u - v) - \frac{\alpha}{2} \| u - v \|^2 \leq \varepsilon, \ \forall v \in S(u) \right\}. \tag{3.1}$$

**Theorem 3.1.** Let the same assumptions be as in Lemma 2.7. Then, one has

(a) (QVIP) is $\alpha$-well-posed if and only if the solution set $\Gamma$ of (QVIP) is nonempty and $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon = 0$;

(b) moreover, if $A : E \to E^*$ is pseudomonotone, then (QVIP) is $L$-$\alpha$-well-posed if and only if the solution set $\Gamma$ of (QVIP) is nonempty and $\lim_{\varepsilon \to 0} \text{diam } L_\varepsilon = 0$.

**Proof.** We only prove (a). The proof of (b) is similar and is omitted here. Suppose that (QVIP) is $\alpha$-well-posed, then $\Gamma \neq \emptyset$. It follows from Lemma 2.7 that $Q_\varepsilon \neq \emptyset$. Suppose by contradiction that there exists a real number $\beta$, such that $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon > \beta > 0$, then there exists $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, and $(w_n), (z_n) \in Q_\varepsilon$, such that $\| w_n - z_n \| > \beta$, for all $n \in N$. Since the sequences $(w_n)_n, (z_n)_n$ are both $\alpha$-approximating sequences for (QVIP), $(w_n)_n$ and $(z_n)_n$ strongly converge to the unique solution $u_0$, and this gives a contradiction. Therefore, $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon = 0$.

Conversely, let $(u_n)_n, u_n \in K$, be an $\alpha$-approximating sequence for (QVIP). Then there exists a sequence $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, such that

$$d(u_n, S(u_n)) \leq \varepsilon_n, \quad \forall n \in N,$$

$$(Au_n, u_n - v) - \frac{\alpha}{2} \| u_n - v \|^2 \leq \varepsilon_n, \quad \forall v \in S(u_n), \quad \forall n \in N. \tag{3.2}$$

that is, $u_n \in Q_{\varepsilon_n}$, for all $n \in N$. It is easy to see $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon = 0$ and $\Gamma \neq \emptyset$ implying that $\Gamma$ is a singleton point set. Indeed, if there exist two different solutions $z_1, z_2$, then from Lemma 2.7, we know that $z_1, z_2 \in Q_\varepsilon$, for all $\varepsilon > 0$. Thus, $\lim_{\varepsilon \to 0} \text{diam } Q_\varepsilon \geq \| z_1 - z_2 \| \neq 0$, a contraction. Let $u_0$ be the unique solution of (QVIP). It follows from Lemma 2.7 that $u_0 \in Q_{\varepsilon_0}$. Thus, $\lim_{n \to 0} \| u_n - u_0 \| \leq \lim_{n \to 0} \text{diam } Q_{\varepsilon_n} = 0$. So $(u_n)_n$ strongly converge to $u_0$. Therefore, (QVIP) is $\alpha$-well-posed. \hfill \Box

**Theorem 3.2.** Let $\alpha > 0$ and the following assumptions hold:

(i) the set-valued mapping $S$ is nonempty convex-valued, and, for each sequence $(u_n)_n$ in $K$ converges to $u_0$, the sequence $(S(u_n))_n$ Lower Semi-Mosco converging to $S(u_0)$;
(ii) for every converging sequence \((h_n)_n\), there exists \(m \in N\), such that
\[
\text{int} \bigcap_{n \geq m} S(h_n) \neq \emptyset;
\]
(3.3)

(iii) the operator \(A\) is hemicontinuous and monotone on \(K\).

Then, (QVIP) is \(\alpha\)-well-posed if and only if
\[
Q_\epsilon \neq \emptyset, \quad \forall \epsilon > 0, \quad \lim_{\epsilon \to 0} \text{diam } Q_\epsilon = 0.
\]
(3.4)

**Proof.** The necessity has been proved in Theorem 3.1(a).

Conversely, assume that (3.4) holds. It is easy to see that (3.4) implies that the solution set \(\Gamma\) of (QVIP) is a singleton point set. Let \((u_n)_n\) be an \(\alpha\)-approximating sequence for (QVIP), that is, there exists a sequence \(\epsilon_n > 0\), with \(\epsilon_n \to 0\), such that
\[
d(u_n, S(u_n)) \leq \epsilon_n, \quad \forall n \in N,
\]
\[
\langle Au_n, u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \epsilon_n, \quad \forall v \in S(u_n), \forall n \in N.
\]
(3.5)

Therefore, \(u_n \in Q_{\epsilon_n}\) for all \(n \in N\). In light of (3.4), \((u_n)_n\) is a Cauchy sequence and strongly converges to a point \(u_0 \in K\). In order to obtain that \(u_0\) solves (QVIP), we start to prove that \(u_0 \in S(u_0)\). For each \(n \in N\), choose \(u_n' \in S(u_n)\), such that \(\|u_n - u_n'\| < d(u_n, S(u_n)) + \epsilon_n \leq 2\epsilon_n\).

It follows from \(u_n \to u_0\) and \(\epsilon_n \to 0\) that \(u_n' \to u_0\). It follows from the assumption (i) that \(\lim_n \inf S(u_n) = S(u_0)\). Thus, \(u_0 \in S(u_0)\).

To complete the proof, consider an arbitrary point \(v \in S(u_0)\). By Lower Semi-Mosco convergence again, we have \(S(u_0) \subseteq \lim_n \inf S(u_n)\). Also observe that assumption (ii) applied to the constant sequence \(h_n = u_0\), for all \(n \in N\), implies that \(\text{int } S(u_0) \neq \emptyset\). From Lemma 2.5, it follows that if \(v \in \text{int } S(u_0)\), then there exist \(m \in N\) and \(\delta > 0\) such that \(\text{int } B(v, \delta) \subseteq S(u_n)\), for all \(n > m\). Thus, \(v \in S(u_n)\) for \(n\) sufficiently large. Notice the \(A\) is monotone and the sequence \((u_n)_n\) is an \(\alpha\)-approximating sequence for (QVIP), then we have
\[
\langle Av, u_0 - v \rangle = \lim_n \langle Av, u_n - v \rangle \leq \lim_n \inf \langle Au_n, u_n - v \rangle \leq \lim_n \left(\epsilon_n + \frac{\alpha}{2} \|u_n - v\|^2\right) = \frac{\alpha}{2} \|u_0 - v\|^2.
\]
(3.6)

If \(v \in S(u_0) \setminus \text{int } S(u_0)\), let \((v_n)_n\) be a sequence converging to \(v\), whose point belongs to a segment contained in \(\text{int } S(u_0)\). Since \(v_n \in S(u_0)\), for all \(n \in N\), one has
\[
\langle Av_n, u_0 - v_n \rangle \leq \frac{\alpha}{2} \|u_0 - v_n\|^2.
\]
(3.7)

Since the hemicontinuity of \(A\),
\[
\langle Av, u_0 - v \rangle \leq \frac{\alpha}{2} \|u_0 - v\|^2, \quad \forall v \in S(u_0).
\]
(3.8)
Proof. The necessity follows from Theorem 3.1 and Lemma 2.7. Then, by Lemma 2.7, we obtain that $u_0$ solves (QVIP). This completes the proof. 

Now, we present a result in which assumption (ii) of above theorem is dropped, while the continuity assumption on the operator $A$ is strengthened.

**Theorem 3.3.** Let the following assumptions hold:

(i) the set-valued mapping $S$ is nonempty convex-valued, and, for each sequence $(u_n)_n$ in $K$ converging to $u_0$, the sequence $(S(u_n))_n$ Lower Semi-Mosco converges to $S(u_0)$;

(ii) the operator $A$ is $(s, w)$-continuous on $K$.

Then, (QVIP) is $\alpha$-well-posed if and only if (3.4) holds.

**Proof.** The necessity follows from Theorem 3.1 and Lemma 2.7.

Conversely, let $(u_n)_n$ be an $\alpha$-approximating sequence for (QVIP) and (3.4) holds. From (3.4) and the proof of Theorem 3.2, we can obtain that $(u_n)_n$ strongly converges to $u_0$, with $u_0 \in S(u_0)$. Since Lower Semi-Mosco convergence implies for every $v \in S(u_0)$, there exists sequence $(v_n)_n$ strongly converging to $v$ such that $v_n \in S(u_n)$. Since the operator $A$ is $(s, w)$-continuous and $(u_n)_n$ is an $\alpha$-approximating sequence for (QVIP), we have

$$
\langle Au_0, u_0 - v \rangle = \lim_n \langle Au_n, u_n - v_n \rangle \leq \lim_n \left( \varepsilon_n + \frac{\alpha}{2} \| u_n - v_n \|^2 \right) = \frac{\alpha}{2} \| u_0 - v \|^2.
$$

By Lemma 2.7, we obtain that $u_0$ solves (QVIP). This completes the proof. 

**Theorem 3.4.** Let the following assumptions hold:

(i) the set-valued mapping $S$ is nonempty convex-valued, and, for each sequence $(u_n)_n$ in $K$ converges to $u_0$, the sequence $(S(u_n))_n$ Lower Semi-Mosco converging to $S(u_0)$;

(ii) for every converging sequence $(h_n)_n$, there exists $m \in N$, such that

$$
\text{int} \bigcap_{n \geq m} S(h_n) \neq \emptyset;
$$

(iii) the operator $A$ is hemicontinuous and monotone on $K$.

Then, (QVIP) is $L-\alpha$-well-posed if and only if

$$
L_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0} \text{diam } L_\varepsilon = 0.
$$

**Proof.** Assume that (QVIP) is $L-\alpha$-well-posed, then it follows from the monotonicity of $A$ that $\emptyset \neq \Gamma \neq L_\varepsilon$, for all $\varepsilon > 0$. It follows from Theorem 3.1(b) that the necessity can be completed.

Assume that (3.12) holds. Let $(u_n)_n$ be an $L-\alpha$-approximating sequence for (QVIP), then there exists a sequence $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, such that $u_n \in L_{\varepsilon_n}$, for all $n \in N$. Following
the same argument as the proof of Theorem 3.1, it is easy to see \( \lim_{n \to 0} \text{diam } L \varepsilon = 0 \) and \( \pi \neq \emptyset \) imply that \( \Gamma \) is a singleton point set. In light of the assumption, \( (u_n)_n \) is a Cauchy sequence and strongly converges to a point \( u_0 \in K \) and \( u_0 \in S(u_0) \). Let \( v \in \text{int } S(u_0) \) and using Lemma 2.5, one has \( v \in S(u_n) \), for \( n \) sufficiently large. Then, we get

\[
(Av, u_0 - v) - \frac{\alpha}{2} \|u_0 - v\|^2 = \lim_n \left( (Av, u_n - v) - \frac{\alpha}{2} \|u_n - v\|^2 \right) \leq \lim_n \varepsilon_n = 0.
\]

(3.13)

If \( v \in S(u_0) \setminus \text{int } S(u_0) \), let a sequence \( v_n \) converges to \( v \), whose points belong to a segment contained in \( \text{int } S(u_0) \). Since

\[
(Av_n, u_0 - v_n) - \frac{\alpha}{2} \|u_0 - v_n\|^2 \leq 0
\]

(3.14)

and the operator \( A \) is hemicontinuous, one gets

\[
(Av, u_0 - v) - \frac{\alpha}{2} \|u_0 - v\|^2 \leq 0.
\]

(3.15)

According to Lemmas 2.6 and 2.7, \( u_0 \) is the solution of (QVIP).

\[\square\]

**Theorem 3.5.** Let the following assumptions hold:

(i) the set-valued mapping \( S \) is nonempty convex-valued, and, for each sequence \( (u_n)_n \) in \( K \) converging to \( u_0 \), the sequence \( (S(u_n))_n \) Lower Semi-Mosco converges to \( S(u_0) \);

(ii) the operator \( A \) is \((s, w)\)-continuous and monotone on \( K \).

Then, (QVIP) is L-\( \alpha \)-well-posed if and only if (3.12) holds.

**Proof.** Assume (3.12) holds. Let \( (u_n)_n \) be an L-\( \alpha \)-approximating sequence for (QVIP), then there exists a sequence \( \varepsilon_n > 0 \), with \( \varepsilon_n \to 0 \), such that \( (u_n)_n \subset L_{\varepsilon_n} \), for all \( n \in \mathbb{N} \). Since \( \lim_{\varepsilon \to 0} \text{diam } L \varepsilon = 0 \), \( (u_n)_n \) is a Cauchy sequence and converges to \( u_0 \). As the similar proof of Theorem 3.2, \( u_0 \in S(u_0) \). Let \( v \in S(u_0) \). Since Lower Semi-Mosco convergence implies for every \( v \in S(u_0) \), there exists a sequence \( (v_n)_n \) converging to \( v \), such that \( v_n \in S(u_n) \). Since \( A \) is \((s, w)\)-continuous and \( (u_n)_n \) is an L-\( \alpha \)-approximating sequence for (QVIP), one has

\[
(Av, u_0 - v) - \frac{\alpha}{2} \|u_0 - v\|^2 = \lim_n \left( (Av_n, u_n - v_n) - \frac{\alpha}{2} \|u_n - v_n\|^2 \right) \leq \lim_n \varepsilon_n = 0.
\]

(3.16)

Applying Lemmas 2.6 and 2.7, we have that (QVIP) is L-\( \alpha \)-well-posed.

The necessity can be completed as Theorem 3.3.

\[\square\]

**4. \( \alpha \)-Well-Posedness in the Generalized Sense**

In this section, we introduce and investigate some metric characterizations of \( \alpha \)-well-posedness in the generalized sense and L-\( \alpha \)-well-posedness in the generalized sense for (QVI).
**Definition 4.1** (see [11]). Let \((X, d)\) be a metric space and let \(A, B\) be nonempty subsets of \(X\). The Hausdorff distance \(H(\cdot, \cdot)\) between \(A\) and \(B\) is defined by

\[ H(A, B) = \max\{e(A, B), e(B, A)\}, \tag{4.1} \]

where \(e(A, B) = \sup_{a \in A} d(a, B)\) with \(d(a, B) = \inf_{b \in B} \|a - b\|\).

**Definition 4.2** (see [11]). Let \(A\) be a nonempty subset of \(X\). The measure of non compactness \(\mu\) of the set \(A\) is defined by

\[ \mu(A) = \inf \left\{ \varepsilon > 0 : A \subseteq \bigcup_{i=1}^{n} A_i, \text{ diam } A_i < \varepsilon, \ i = 1, 2, \ldots, n \right\}, \tag{4.2} \]

where diam means the diameter of a set.

**Theorem 4.3.** Let the same assumptions be as in Lemma 2.7. Then, one has the following.

(a) \((QVIP)\) is \(\alpha\)-well-posed in the generalized sense if and only if the solution set \(\Gamma\) of \((QVIP)\) is nonempty compact and \(e(Q_{\varepsilon}, \Gamma) \to 0\), as \(\varepsilon \to 0\).

(b) Moreover, if \(A\) is pseudomonotone, then \((QVIP)\) is \(L\)-\(\alpha\)-well-posed in the generalized sense if and only if the solution set \(\Gamma\) of \((QVIP)\) is nonempty compact and \(e(L_{\varepsilon}, \Gamma) \to 0\), as \(\varepsilon \to 0\).

**Proof.** We only prove (a), the proof of (b) is similar and is omitted here. Assume that \((QVIP)\) is \(\alpha\)-well-posed in the generalized sense, then the \(\Gamma\) is nonempty and compact. It follows from Lemma 2.7 that \(Q_{\varepsilon} \neq \emptyset\). Now we prove \(e(Q_{\varepsilon}, \Gamma) \to 0\), as \(\varepsilon \to 0\). Suppose by contradiction that there exists \(\beta > 0\), \(\varepsilon_n \to 0\), and \(w_n \in Q_{\varepsilon_n}\), such that \(d(w_n, \Gamma) \geq \beta\). It follows from \(w_n \in Q_{\varepsilon_n}\) that \((w_n)_n\) is an \(\alpha\)-approximating sequence for \((QVIP)\). \((QVIP)\) is \(\alpha\)-well-posedness in the generalized sense, then there exists a subsequence \((w_{n_k})_k\) of \((w_n)_n\) strongly converging to a point of \(\Gamma\). This contradicts \(d(w_{n_k}, \Gamma) \geq \beta\). Thus, \(e(Q_{\varepsilon_n}, \Gamma) \to 0\), as \(\varepsilon \to 0\).

For the converse, let \((u_n)_n\) be an \(\alpha\)-approximating sequence for \((QVIP)\), then \(u_n \in Q_{\varepsilon_n}\). It follows from \(e(Q_{\varepsilon_n}, \Gamma) \to 0\) that there exists a sequence \(z_n \in \Gamma\), such that \(d(u_n, z_n) \to 0\). Since \(\Gamma\) is compact, there exists a subsequence \((z_{n_k})_k\) of \((z_n)_n\) strongly converging to \(u_0 \in \Gamma\). Thus there exists the corresponding subsequence \((u_{n_k})_k\) of \((u_n)_n\) strongly converging to \(u_0\). Therefore, \((QVIP)\) is \(\alpha\)-well-posed in the generalized sense. \(\square\)

**Theorem 4.4.** (a) If \((QVIP)\) is \(\alpha\)-well-posed in the generalized sense, then

\[ Q_{\varepsilon} \neq \emptyset, \quad \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0} \mu(Q_{\varepsilon}) = 0. \tag{4.3} \]

(b) If (4.3) and the following assumptions hold:

(i) the set-valued mapping \(S\) is nonempty convex-valued, and, for each sequence \((u_n)_n\) in \(K\) converges to \(u_0\), the sequence \((S(u_n))_n\) Lower Semi-Mosco converging to \(S(u_0)\);

(ii) the operator \(A\) is \((s, w)\)-continuous on \(K\),

then, \((QVIP)\) is \(\alpha\)-well-posed in the generalized sense.
Proof. (a) Suppose that (QVIP) is $\alpha$-well-posed in the generalized sense. So $Q_\varepsilon \neq \emptyset$, for all $\varepsilon > 0$. By Theorem 4.3(a), $\Gamma$ is nonempty compact and $e(Q_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$. For any $\varepsilon > 0$, we have

$$H(Q_\varepsilon, \Gamma) = \max\{e(Q_\varepsilon, \Gamma), e(\Gamma, Q_\varepsilon)\} = e(Q_\varepsilon, \Gamma),$$

(4.4)

and since $\Gamma$ is compact, $\mu(\Gamma) = 0$. For every $n \in N$, the following relation holds [14]:

$$\mu(Q_\varepsilon) \leq 2H(Q_\varepsilon, \Gamma) + \mu(\Gamma) = 2H(Q_\varepsilon, \Gamma) = 2e(Q_\varepsilon, \Gamma).$$

(4.5)

It follows from $e(Q_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$, that $\lim_{\varepsilon \to 0} \mu(Q_\varepsilon) = 0$.

(b) Assume that (4.3) holds. Then, for any $\varepsilon > 0$, $cl(Q_\varepsilon)$ is nonempty closed and increasing with $\varepsilon > 0$. By (4.3), $\lim_{\varepsilon \to 0} \mu(cl(Q_\varepsilon)) = \lim_{\varepsilon \to 0} \mu(Q_\varepsilon) = 0$, where $cl(Q_\varepsilon)$ is the closure of $Q_\varepsilon$. By the generalized Cantor theorem [11, page 412], we know that

$$\lim_{\varepsilon \to 0} H(cl(Q_\varepsilon), \Delta) = 0, \text{ as } \varepsilon \to 0,$$

(4.6)

where $\Delta = \bigcap_{\varepsilon > 0} cl(Q_\varepsilon)$ is nonempty compact.

Now we show that

$$\Gamma = \Delta.$$

(4.7)

It follows from Lemma 2.7 that $\Gamma \subseteq \Delta$. So we need to prove that $\Delta \subseteq \Gamma$. Indeed, let $u_0 \in \Delta$. Then, $d(u_0, Q_\varepsilon) = 0$ for every $\varepsilon > 0$. Given $\varepsilon_n > 0$, $\varepsilon_n \to 0$, for every $n$, there exists $u_n \in Q_{\varepsilon_n}$ such that $d(u_0, u_n) < \varepsilon_n$. Hence, $u_n \to u_0$ and

$$d(u_n, S(u_n)) \leq \varepsilon_n,$$

(4.8)

$$\langle Au_n, u_n - v \rangle \leq \varepsilon_n + \frac{\alpha}{2}\|u_n - v\|^2, \quad \forall v \in S(u_n).$$

(4.9)

It follows from (4.8), $u_n \to u_0$, and the proof of Theorem 3.2 that $u_0 \in S(u_0)$.

Since Lower Semi-Mosco convergence implies that, for every $v \in S(u_0)$, there exists a sequence $v_n \in S(u_n)$, for all $n \in N$, such that $\lim_{n} v_n = v$ in the strongly topology.

Since the operator $A$ is $(s, w)$-continuous on $K$, hence

$$\langle Au_0, u_0 - v \rangle - \frac{\alpha}{2}\|u_0 - v\|^2 = \lim_{n} \left[\langle Au_n, u_n - v_n \rangle - \frac{\alpha}{2}\|u_n - v_n\|^2 \right] \leq \lim_{n} \varepsilon_n = 0.$$

(4.10)

By Lemma 2.7, we know $u_0 \in \Gamma$. Thus, $\Delta \subseteq \Gamma$. It follows from (4.6) and (4.7) that $\lim_{\varepsilon \to 0} e(Q_\varepsilon, \Gamma) = 0$. It follows from the compactness of $\Gamma$ and Theorem 4.3(a) that (QVIP) is $\alpha$-well-posed in the generalized sense. The proof is completed. \hfill \Box

**Theorem 4.5.** Let $K$ be a nonempty, compact, and convex subset of $E$, let the set-valued mapping $S$ be nonempty convex-valued, and, for each sequence $(u_n)_n$ in $K$ converging to $u_0$, the sequence $(S(u_n))_n$ Lower Semi-Mosco converges to $S(u_0)$, and the operator $A$ is $(s, w)$-continuous on $K$. Then, (QVIP) is $\alpha$-well-posed in the generalized sense.
Proof. Let \((u_n)_n\) be an \(\alpha\)-approximating sequence for (QVIP). Since the set \(K\) is compact, there exists subsequence \((u_{n_k})_k\) of \((u_n)_n\) strongly converging to a point \(u_0 \in K\). Reasoning as in Theorem 3.3, we get \(u_0 \in S(u_0)\) and \(u_0\) solves (QVIP). Therefore, (QVIP) is \(\alpha\)-well-posed in the generalized sense.

**Theorem 4.6.** Let the following assumptions hold:

(i) the set-valued mapping \(S\) is nonempty convex-valued, and, for each sequence \((u_n)_n\) in \(K\) converging to \(u_0\), the sequence \((S(u_n))_n\) Lower Semi-Mosco converges to \(S(u_0)\);

(ii) the operator \(A\) is \((s, \omega)\)-continuous and monotone on \(K\).

Then, (QVIP) is \(L-\alpha\)-well-posed in the generalized sense if and only if

\[
L_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0} \mu(L_\varepsilon) = 0. \tag{4.11}
\]

Proof. Assume that (QVIP) is \(L-\alpha\)-well-posed in the generalized sense. It follows from Lemma 2.7 and the monotonicity of \(A\) that \(\Gamma \subseteq L_\varepsilon\), for all \(\varepsilon > 0\). And so \(L_\varepsilon \neq \emptyset\), for each \(\varepsilon > 0\). By Theorem 4.3(b), we can get \(e(L_\varepsilon, \Gamma) \to 0\) as \(\varepsilon \to 0\). From the proof of Theorem 4.4, we also obtain

\[
\mu(L_\varepsilon) \leq 2H(L_\varepsilon, \Gamma) + \mu(\Gamma) = 2H(L_\varepsilon, \Gamma) = 2e(L_\varepsilon, \Gamma). \tag{4.12}
\]

Thus, \(\lim_{\varepsilon \to 0} \mu(L_\varepsilon) = 0\).

Conversely, assume (4.11) holds. Then, for any \(\varepsilon > 0\), \(\text{cl}(L_\varepsilon)\) is nonempty closed and increasing with \(\varepsilon > 0\). By (4.11), \(\lim_{\varepsilon \to 0} \mu(\text{cl}(L_\varepsilon)) = \lim_{\varepsilon \to 0} \mu(L_\varepsilon) = 0\), where \(\text{cl}(L_\varepsilon)\) is the closure of \(L_\varepsilon\). By the generalized Cantor theorem [11, page 412, we know that

\[
\lim_{\varepsilon \to 0} H(\text{cl}(L_\varepsilon), \Delta) = 0, \quad \text{as } \varepsilon \to 0, \tag{4.13}
\]

where \(\Delta = \bigcap_{\varepsilon > 0} \text{cl}(L_\varepsilon)\) is nonempty compact.

Now we show that

\[
\Gamma = \Delta. \tag{4.14}
\]

It follow from Lemma 2.7 and the monotonicity of \(A\) that \(\Gamma \subseteq \Delta\). So we need to prove that \(\Delta \subseteq \Gamma\). Indeed, let \(u_0 \in \Delta\). Then \(d(u_0, L_\varepsilon) = 0\) for every \(\varepsilon > 0\). Given \(\varepsilon_n > 0\), \(\varepsilon_n \to 0\), for every \(n\), there exists \(u_n \in L_{\varepsilon_n}\) such that \(d(u_n, u_0) < \varepsilon_n\). Hence, \(u_n \to u_0\) and

\[
d(u_n, S(u_n)) \leq \varepsilon_n, \tag{4.15}
\]

\[
(Av, u_n - v) \leq \varepsilon_n + \frac{\alpha}{2} \|u_n - v\|^2, \quad \forall v \in S(u_n). \tag{4.16}
\]

It follows from (4.15), \(x_n \to x_0\), and the proof of Theorem 3.2 that \(u_0 \in S(u_0)\).

Since \(S(u_n)\) Lower Semi-Mosco converges to \(S(u_0)\), for every \(v \in S(u_0)\), there exists a sequence \(v_n \in S(u_n)\), for all \(n \in N\), such that \(\lim_n v_n = v\) in the strong topology.
Since the operator $A$ is $(s, w)$-continuous on $K$, hence
\[
\langle Av, u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 = \lim_n \left( \langle Av_n, u_n - v_n \rangle - \frac{\alpha}{2} \|u_n - v_n\|^2 \right) \leq \lim_{n} \epsilon_n = 0.
\]
(4.17)

By Lemma 2.6 we know that $u_0 \in S(u_0)$, such that
\[
\langle Au_0, u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 \leq 0, \quad \forall v \in S(u_0).
\]
(4.18)

It follow from Lemma 2.7 that $u_0 \in \Gamma$. Thus, $\Delta \subseteq \Gamma$. It follows from (4.13) and (4.14) that $\lim_{\epsilon \to 0} e(L_{\epsilon}, \Gamma) = 0$. It follows from the compactness of $\Gamma$ and Theorem 4.3(b) that (QVIP) is $L$-$\alpha$-well-posed in the generalized sense. The problem is completed.

**Remark 4.7.** It is easy to see that if $\alpha = 0$, then by the main results in our paper, we can recover the corresponding results in [9] with the weaker condition that $S(x_n)$ Lower Semi-Mosco converges to $S(x_0)$ instead of the condition that $S$ is $(s, w)$-closed and $(s, w)$-subcontinuous, and $(s, s)$-lower semicontinuous.

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