Research Article

Iterative Algorithms with Perturbations for Solving the Systems of Generalized Equilibrium Problems and the Fixed Point Problems of Two Quasi-Nonexpansive Mappings

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We introduce new iterative algorithms with perturbations for finding a common element of the set of solutions of the system of generalized equilibrium problems and the set of common fixed points of two quasi-nonexpansive mappings in a Hilbert space. Under suitable conditions, strong convergence theorems are obtained. Furthermore, we also consider the iterative algorithms with perturbations for finding a common element of the solution set of the systems of generalized equilibrium problems and the common fixed point set of the super hybrid mappings in Hilbert spaces.

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and $C$ a nonempty closed convex subset of $H$ and let $T$ be a mapping of $C$ into $H$. Then, $T : C \to H$ is said to be nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$. A mapping $T : C \to H$ is said to be quasi-nonexpansive if $\| Tx - y \| \leq \| x - y \|$ for all $x \in C$ and $y \in F(T) := \{ x \in C : Tx = x \}$. It is well known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping $T$ is closed and convex; see Itoh and Takahashi [1]. A mapping $T : C \to H$ is called nonspreading [2] if

$$2 \| Tx - Ty \|^2 \leq \| Tx - y \|^2 + \| Ty - x \|^2, \quad (1.1)$$

for all $x, y \in C$. We remark that nonlinear every nonspreading mappings are quasi-nonexpansive mappings if the set of fixed points is nonempty.
Recall that a mapping $\Psi : C \to H$ is said to be $\mu$-inverse strongly monotone if there exists a positive real number $\mu$ such that

$$\langle \Psi x - \Psi y, \; x - y \rangle \geq \mu \| \Psi x - \Psi y \|^2, \quad \forall x, y \in C. \quad (1.2)$$

If $\Psi$ is a $\mu$-inverse strongly monotone mapping of $C$ into $H$, then it is obvious that $\Psi$ is $1/\mu$-Lipschitz continuous.

Let $G : C \times C \to \mathbb{R}$ be a bifuction and $\Psi : C \to H$ be $\mu$-inverse strongly monotone mapping. The generalized equilibrium problem (for short, GEP) for $F$ and $\Psi$ is to find $z \in C$ such that

$$G(z, y) + \langle \Psi z, y - z \rangle \geq 0, \quad \forall y \in C. \quad (1.3)$$

The problem (1.3) was studied by Moudafi [3]. The set of solutions for problem (1.3) is denoted by $\text{GEP}(F, \Psi)$, that is,

$$\text{GEP}(F, \Psi) = \{ z \in C : G(z, y) + \langle \Psi z, y - z \rangle \geq 0, \; \forall y \in C \}. \quad (1.4)$$

If $\Psi \equiv 0$ in (1.3), then GEP reduces to the classical equilibrium problem and $\text{GEP}(G, 0)$ is denoted by $\text{EP}(G)$, that is,

$$\text{EP}(G) = \{ z \in C : G(z, y) \geq 0, \; \forall y \in C \}. \quad (1.5)$$

If $G \equiv 0$ in (1.3), then GEP reduces to the classical variational inequality and $\text{GEP}(0, \Psi)$ is denoted by $\text{VI}(\Psi, C)$, that is,

$$\text{VI}(\Psi, C) = \{ z \in C : \langle \Psi z, y - z \rangle \geq 0, \; \forall y \in C \}. \quad (1.6)$$

The problem (1.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, Min–Max problems, the Nash equilibrium problems in noncooperative games, and others; see, for example, Blum and Oettli [4] and Moudafi [3].

nonexpansive mappings and the set of solutions of a general system of variational inequalities for a cocoercive mapping in a real Hilbert space. Their results extend and improve many results in the literature.

In 1967, Wittmann [11] (see also [12]) proved the strong convergence theorem of Halpern’s type [13] \(\{x_n\}\) defined by, for any \(x_1 = x \in C\),

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \tag{1.7}
\]

where \(\{\alpha_n\} \subset (0, 1)\) satisfies \(\lim_{n \to \infty} \alpha_n = 0\), \(\sum_{n=1}^{\infty} \alpha_n = \infty\), and \(\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty\). In [14], Kurokawa and Takahashi also studied the following Halpern’s type for nonspreading mappings in a Hilbert space; see also Hojo and Takahashi [15]. Let \(T\) be a nonspreading mapping of \(C\) into itself. Let \(u \in C\) and define two sequences \(\{x_n\}\) and \(\{z_n\}\) in \(C\) as follows: \(x_1 = x \in C\) and

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \quad \text{where} \quad z_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n \tag{1.8}
\]

for all \(n = 1, 2, \ldots\), where \(\{\alpha_n\} \subset (0, 1)\), \(\lim_{n \to \infty} \alpha_n = 0\) and \(\sum_{n=1}^{\infty} \alpha_n = \infty\). If \(F(T)\) is nonempty, they proved that \(\{x_n\}\) and \(\{z_n\}\) converge strongly to \(P_{F(T)}u\), where \(P_{F(T)}\) is the metric projection of \(H\) onto \(F(T)\). Recently, Yao and Shahzad [16] gave the following iteration process for nonexpansive mappings with perturbation: \(x_1 \in C\) and

\[
x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(\alpha_n u_n + (1 - \alpha_n)Tx_n), \quad \forall n \in \mathbb{N}, \tag{1.9}
\]

where \(\{\alpha_n\}\) and \(\{\beta_n\}\) are sequences in \([0, 1]\), and the sequence \(\{u_n\} \subseteq H\) is a small perturbation for the \(n\)-step iteration satisfying \(\|u_n\| \to 0\) as \(n \to \infty\). In fact, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate.

On the other hand, very recently, Chuang et al. [17] considered the following iteration process for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points for a quasi-nonexpansive mapping \(T : C \to H\) with perturbation

\[
q_1 \in H,
\]

\[
x_n \in C, \quad \text{such that} \quad G(x_n, y) + \frac{1}{r_n}(y - x_n, x_n - q_n) \geq 0, \quad \forall y \in C, \tag{1.10}
\]

\[
y_n = \beta_n x_n + (1 - \beta_n)Tx_n,
\]

\[
q_{n+1} = \alpha_n u_n + (1 - \alpha_n)y_n, \quad \forall n \in \mathbb{N},
\]

where \(C\) is a nonempty closed convex subset of \(H\), \(G : C \times C \to \mathbb{R}\) is a function, \(\{\alpha_n\}\) and \(\{\beta_n\}\) are real sequences in \((0, 1)\), and \(\{u_n\} \subset H\) is a convergent sequence and \(\{r_n\} \subset [a, \infty)\) for some \(a > 0\). They obtained a strong convergence theorem for such iterations.

In this paper, inspired and motivated by Yao and Shahzad [16], S. Takahashi and W. Takahashi [18] and Chuang et al. [17], we introduce a new iterative algorithms with perturbations for finding a common element of the set of solutions of the system of generalized equilibrium problems and the set of common fixed points of two quasi-nonexpansive
mappings in a Hilbert space. Under suitable conditions, strong convergence theorems are obtained. Furthermore, we also consider the iterative algorithms with perturbations for finding a common element of the solution set of the system of generalized equilibrium problems and the common fixed point set of the super hybrid mappings in a Hilbert space.

2. Preliminaries

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). We denote the strongly convergence and the weak convergence of \( \{ x_n \} \) to \( x \in H \) by \( x_n \rightharpoonup x \) and \( x_n \rightarrow x \), respectively. In a Hilbert space, it is known that

\[
\| \lambda x + (1 - \lambda)y \| = \lambda \| x \| + (1 - \lambda) \| y \| - \lambda(1 - \lambda) \| x - y \|, \tag{2.1}
\]

for all \( x, y \in H \) and \( \lambda \in \mathbb{R} \); see [19]. Furthermore, we have that for any \( x, y, u, v \in H \)

\[
2\langle x - y, u - v \rangle = \| x - v \| + \| y - u \| - \| x - u \| - \| y - v \|. \tag{2.2}
\]

Let \( C \) be a nonempty closed convex subset of \( H \) and \( x \in H \). We know that there exists a unique nearest point \( z \in C \) such that \( \| x - z \| = \inf_{y \in C} \| x - y \| \). We denote such a correspondence by \( z = P_Cx \). The mapping \( P_C \) is called the metric projection of \( H \) onto \( C \). It is known that \( P_C \) is nonexpansive and

\[
\langle x - P_Cx, P_Cx - u \rangle \geq 0; \tag{2.3}
\]

for all \( x \in H \) and \( u \in C \); see [19, 20] for more details.

Let \( C \) be a nonempty, closed and convex subset of \( H \) and let \( G : C \times C \rightarrow \mathbb{R} \) be a bifunction. For solving the generalized equilibrium problem, let us assume that the bifunction \( G : C \times C \rightarrow \mathbb{R} \) satisfies the following conditions:

(A1) \( G(x, x) = 0 \) for all \( x \in C \);

(A2) \( G \) is monotone, that is, \( G(x, y) + G(y, x) \leq 0 \) for any \( x, y \in C \);

(A3) for each \( x, y, z \in C \)

\[
\lim_{t \downarrow 0} G(tz + (1 - t)x, y) \leq G(x, y); \tag{2.4}
\]

(A4) for each \( x \in C, G(x, \cdot) \) is convex and lower semicontinuous.

We know the following lemma which appears implicitly in Blum and Oettli [4].

Lemma 2.1 (see [4]). Let \( C \) be a nonempty closed convex subset of \( H \) and let \( G \) be a bifunction of \( C \times C \) into \( \mathbb{R} \) satisfying (A1)–(A4). Let \( r > 0 \) and \( x \in H \). Then, there exists a unique \( z \in C \) such that

\[
G(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \forall y \in C. \tag{2.5}
\]

The following lemma was also given in Combettes and Hirstoaga [5].
Lemma 2.2 (see [5]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( G : C \times C \to \mathbb{R} \) be a bifunction which satisfies conditions (A1)–(A4). For \( r > 0 \) and \( x \in H \), define a mapping \( T_r : H \to C \) as follows:

\[
T_r(x) = \left\{ z \in C : G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}
\]  

(2.6)

for all \( x \in H \). Then the following hold:

(i) \( T_r \) is single-valued;

(ii) \( T_r \) is firmly nonexpansive, that is, for any \( x, y \in C \),

\[
\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;
\]  

(2.7)

(iii) \( (EP) \) is a closed convex subset of \( C \);

(iv) \( F(T_r) = EP(G) \).

Remark 2.3. For any \( x \in H \) and \( r > 0 \), by Lemma 2.2 (i), there exists \( u \in H \) such that

\[
G(u, y) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \quad \forall y \in H.
\]  

(2.8)

Replacing \( x \) with \( x - r\Psi x \in H \) in (2.8), we have

\[
G(u, y) + \langle \Psi x, y - u \rangle + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \quad \forall y \in H,
\]  

(2.9)

where \( \Psi : H \to H \) is an inverse strongly monotone mapping.

Lemma 2.4 (see [21]). Let \( \{\Gamma_n\} \) be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence \( \{\Gamma_{n_i}\} \) of \( \{\Gamma_n\} \) which satisfies \( \Gamma_{n_i} < \Gamma_{n_i+1} \) for all \( i \in \mathbb{N} \). Define the sequence \( \{\tau(n)\}_{n \geq n_0} \) of integers as follows:

\[
\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1} \}, \quad \text{for } n \in \mathbb{N},
\]  

(2.10)

where \( n_0 \in \mathbb{N} \) such that \( \{k \leq n_0 : \Gamma_k < \Gamma_{k+1} \} \neq \emptyset \). Then, the following hold:

(i) \( \tau(1) \leq \tau(2) \leq \cdots \) and \( \tau(n) \to \infty \);

(ii) \( \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \) and \( \Gamma_n \leq \Gamma_{\tau(n)+1} \) for all \( n \in \mathbb{N} \).

Lemma 2.5 (see [22]). Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence of nonnegative real numbers, \( \{\alpha_n\} \) a sequence of real numbers in \([0,1]\) with \( \sum_{n=1}^{\infty} \alpha_n = \infty \), \( \{u_n\} \) a sequence of nonnegative real numbers with \( \sum_{n=1}^{\infty} u_n < \infty \), \( \{t_n\} \) a sequence of real numbers with \( \lim \sup \ t_n \leq 0 \). Suppose that

\[
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n t_n + u_n, \quad \forall n \in \mathbb{N}.
\]  

(2.11)

Then \( \lim_{n \to \infty} a_n = 0 \).
3. Main Results

Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $\Psi_i$ a $\mu_i$-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two mappings. Let $\{x_n\}$ be a sequence generated in the following manner:

\[
x_1 \in H,
\]

\[
G_1(u_{n,1}, y) + \langle \Psi_1 x_n, y - u_{n,1} \rangle + \frac{1}{r_n}(y - u_{n,1}, u_{n,1} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
G_2(u_{n,2}, y) + \langle \Psi_2 x_n, y - u_{n,2} \rangle + \frac{1}{r_n}(y - u_{n,2}, u_{n,2} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
\vdots
\]

\[
G_k(u_{n,k}, y) + \langle \Psi_k x_n, y - u_{n,k} \rangle + \frac{1}{r_n}(y - u_{n,k}, u_{n,k} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
\omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i},
\]

\[
y_n = \gamma_n \omega_n + (1 - \gamma_n) T_1 \omega_n,
\]

\[
z_n = \beta_n y_n + (1 - \beta_n) T_2 \omega_n,
\]

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n, \quad \forall n \in \mathbb{N},
\]

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0, 1)$ and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Under certain appropriate assumptions imposed on the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, the strong convergence theorem of $\{x_n\}$ defined by (3.1) is studied in the following theorem.

**Theorem 3.1.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $\Psi_i$ a $\mu_i$-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two quasi-nonexpansive mappings such that $I - T_j$ are demiclosed at zero with $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} \text{GEP}(G_i, \Psi_i)) \neq \emptyset$. Let the sequences $\{x_n\}, \{y_n\},$ and $\{z_n\}$ be defined by (3.1), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{u_n\}$ satisfy the following conditions:

(C1) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0$;

(C3) $\liminf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0$;

(C4) $\lim_{n \to \infty} \omega_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to $x^*$, where $x^* = P_{\Omega}u$. 
Proof. We first have that for all \( i = 1, 2, \ldots, k, \) \( I - r_n\Psi_i \) is a nonexpansive mapping. Indeed, for all \( x, y \in C \), we obtain

\[
\|(I - r_n\Psi_i)x - (I - r_n\Psi_i)y\|^2 = \|(x - y) - r_n(\Psi_i x - \Psi_i y)\|^2 \\
= \|x - y\|^2 - 2r_n(\Psi_i x - \Psi_i y, x - y) + r_n^2\|\Psi_i x - \Psi_i y\|^2 \\
\leq \|x - y\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x - \Psi_i y\|^2 \\
\leq \|x - y\|^2^2.
\]

(3.2)

Thus \( I - r_n\Psi_i \) is nonexpansive for each \( i \in \{1, 2, \ldots, k\} \). Now, let \( w \in \Omega \) be arbitrary. By (C4), \( \{u_n\} \) is a bounded sequence, there exists \( M \leq 0 \) such that

\[
\sup_{n \in \mathbb{N}} \|u_n - w\| \leq M.
\]

(3.3)

For each \( i = 1, 2, \ldots, k \) and \( n \in \mathbb{N} \), we have from \( u_{n,i} = T_{r_n}(x_n - r_n\Psi_ix_n) \) that

\[
\|u_{n,i} - w\| = \|T_{r_n}(x_n - r_n\Psi_ix_n) - T_{r_n}(w - r_n\Psi_iw)\| \\
\leq \|(x_n - r_n\Psi_ix_n) - (w - r_n\Psi_iw)\| \\
\leq \|x_n - w\|, \tag{3.4}
\]

which gives also that

\[
\|\omega_n - w\| \leq \frac{1}{K} \sum_{i=1}^{k} \|u_{n,i} - w\| \leq \|x_n - w\| \quad \forall w \in \Omega.
\]

(3.5)

Since \( T_1 \) is quasi-nonexpansive, we have

\[
\|y_n - w\| = \|\gamma_n\omega_n + (1 - \gamma_n)T_1\omega_n - w\| \\
\leq \|\gamma_n(\omega_n - w) + (1 - \gamma_n)(T_1\omega_n - w)\| \\
\leq \gamma_n\|\omega_n - w\| + (1 - \gamma_n)\|T_1\omega_n - w\| \leq \|\omega_n - w\|. \tag{3.6}
\]

So, we have from (3.5) and (3.6) and the quasi-nonexpansiveness of \( T_2 \) that

\[
\|x_{n+1} - w\| = \|\alpha_n(u_n - w) + (1 - \alpha_n)(z_n - w)\| \\
\leq \alpha_n\|u_n - w\| + (1 - \alpha_n)\|z_n - w\| \\
\leq \alpha_n\|u_n - w\| + (1 - \alpha_n)\{\beta_n\|y_n - w\| + (1 - \beta_n)\|T_2\omega_n - w\|\} \\
\leq \alpha_n\|u_n - w\| + (1 - \alpha_n)\{\beta_n\|\omega_n - w\| + (1 - \beta_n)\|\omega_n - w\|\} \\
\leq \alpha_n\|u_n - w\| + (1 - \alpha_n)\|\omega_n - w\| \\
\leq \alpha_n\|u_n - w\| + (1 - \alpha_n)\|x_n - w\| \\
\leq \max\{M, \|x_n - w\|\}. \tag{3.7}
\]
By Induction, we have that
\[ \|x_n - w\| \leq \max\{\|x_1 - w\|, M\}, \quad \forall n \in \mathbb{N}. \tag{3.8} \]

Thus we obtain that \(\|x_n - w\|\) is bounded, so also \(\{x_n\}, \{y_n\}, \{z_n\}, \{\omega_n\}, \{T_1\omega_n\}, \) and \(\{T_2\omega_n\}\) are bounded. Since \(\Omega\) is closed and convex, we can take \(x^* = P_{\Omega}u\). It follows that
\[
\|y_n - x^*\|^2 = \|y_n(\omega_n - x^*) + (1 - y_n)(T_1\omega_n - x^*)\|^2 \\
= y_n\|\omega_n - x^*\|^2 + (1 - y_n)\|T_1\omega_n - x^*\|^2 - y_n(1 - y_n)\|\omega_n - T_1\omega_n\|^2 \\
\leq y_n\|\omega_n - x^*\|^2 + (1 - y_n)\|\omega_n - x^*\|^2 - y_n(1 - y_n)\|\omega_n - T_1\omega_n\|^2 \\
= \|\omega_n - x^*\|^2 - y_n(1 - y_n)\|\omega_n - T_1\omega_n\|^2 \\
\leq \|\omega_n - x^*\|^2. \tag{3.9}
\]

From (3.9), we have
\[
\|z_n - x^*\|^2 = \|\beta_n(y_n - x^*) + (1 - \beta_n)(T_2\omega_n - x^*)\|^2 \\
= \beta_n\|y_n - x^*\|^2 + (1 - \beta_n)\|T_2\omega_n - x^*\|^2 - \beta_n(1 - \beta_n)\|y_n - T_2\omega_n\|^2 \\
\leq \beta_n\|\omega_n - x^*\|^2 + (1 - \beta_n)\|\omega_n - x^*\|^2 - \beta_n(1 - \beta_n)\|y_n - T_2\omega_n\|^2 \\
= \|\omega_n - x^*\|^2 - \beta_n(1 - \beta_n)\|y_n - T_2\omega_n\|^2 \\
\leq \|\omega_n - x^*\|^2. \tag{3.10}
\]

Hence we have from (3.5), (3.9), and (3.10) that
\[
\|\omega_{n+1} - x^*\|^2 \leq \|x_{n+1} - x^*\|^2 \\
= \|\alpha_n(u_n - x^*) + (1 - \alpha_n)(z_n - x^*)\|^2 \\
= \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|z_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|u_n - z_n\|^2 \\
\leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|z_n - x^*\|^2 \\
= \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\left\{\beta_n\|y_n - x^*\|^2 + (1 - \beta_n)\|T_2\omega_n - x^*\|^2 \\
- \beta_n(1 - \beta_n)\|y_n - T_2\omega_n\|^2 \right\} \\
\leq \alpha_n\|u_n - x^*\|^2 + \beta_n\left\{\gamma_n\|\omega_n - x^*\|^2 + (1 - \gamma_n)\|T_1\omega_n - x^*\|^2 \\
- \gamma_n(1 - \gamma_n)\|\omega_n - T_1\omega_n\|^2 \right\} + (1 - \beta_n)\|T_2\omega_n - x^*\|^2 \\
- \beta_n(1 - \beta_n)\|y_n - T_2\omega_n\|^2 \]
Abstract and Applied Analysis

\[ \leq \alpha_n \||u_n - x^*||^2 + \beta_n \left( \gamma_n \||\omega_n - x^*||^2 + (1 - \gamma_n) \||\omega_n - x^*||^2 \right) \\
- \gamma_n (1 - \gamma_n) \||T_1 \omega_n||^2 + (1 - \beta_n) \||\omega_n - x^*||^2 \\
- \beta_n (1 - \beta_n) \||y_n - T_2 \omega_n||^2 \]

\[ = \alpha_n \||u_n - x^*||^2 + ||\omega_n - x^*||^2 - \gamma_n (1 - \gamma_n) \||T_1 \omega_n||^2 \\
- \beta_n (1 - \beta_n) \||y_n - T_2 \omega_n||^2 \]

(3.11)

We also have that

\[ \gamma_n (1 - \gamma_n) \||\omega_n - T_1 \omega_n||^2 \leq \alpha_n \||u_n - x^*||^2 + ||T_1 \omega_n||^2 \]

\[ \beta_n (1 - \beta_n) \||y_n - T_2 \omega_n||^2 \leq \alpha_n \||u_n - x^*||^2 + ||T_2 \omega_n||^2 \]

(3.12)

(3.13)

Furthermore, we have from \( y_n = \gamma_n \omega_n + (1 - \gamma_n)T_1 \omega_n \) that

\[ ||\omega_n - T_2 \omega_n|| \leq ||\omega_n - y_n|| + ||y_n - T_2 \omega_n|| \]

\[ = ||\omega_n - y_n - (1 - \gamma_n)T_1 \omega_n|| + ||y_n - T_2 \omega_n|| \]

\[ = (1 - \gamma_n) \||\omega_n - T_1 \omega_n|| + ||y_n - T_2 \omega_n|| \]

(3.14)

On the other hand, since \( x_{n+1} - x^* = \alpha_n (u_n - x^*) + (1 - \alpha_n) (z_n - x^*) \), we have

\[ \||\omega_{n+1} - x^*||^2 \leq \||x_{n+1} - x^*||^2 \]

\[ \leq (1 - \alpha_n) \||z_n - x^*||^2 + 2\alpha_n \langle u_n - x^*, x_{n+1} - x^* \rangle \]

\[ \leq (1 - \alpha_n) \||\omega_n - x^*||^2 + 2\alpha_n \langle u_n - x^*, x_{n+1} - x^* \rangle \]

\[ = (1 - \alpha_n) \||\omega_n - x^*||^2 + 2\alpha_n \langle u_n - u, x_{n+1} - x^* \rangle \]

\[ + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \]

\[ = (1 - \alpha_n) \||\omega_n - x^*||^2 + 2\alpha_n \langle u_n - u, x_{n+1} - x^* \rangle \]

\[ + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle. \]

(3.15)

We also have that

\[ \||x_{n+1} - \omega_n|| \leq \||x_{n+1} - y_n|| + ||y_n - \omega_n|| \]

\[ = ||\alpha_n (u_n - y_n) + (1 - \alpha_n) (z_n - y_n)|| + ||(1 - \gamma_n) \omega_n - T_1 \omega_n|| \]

\[ \leq \alpha_n \||u_n - y_n|| + (1 - \alpha_n) \||\beta_n y_n + (1 - \beta_n) T_2 \omega_n - y_n|| \]

\[ + (1 - \gamma_n) \||\omega_n - T_1 \omega_n|| \]

\[ = \alpha_n \||u_n - y_n|| + (1 - \alpha_n) \||\beta_n y_n + (1 - \beta_n) T_2 \omega_n - y_n|| \]

\[ + (1 - \gamma_n) \||\omega_n - T_1 \omega_n||. \]

(3.16)
Moreover, for any $i \in \{1, 2, \ldots, k\}$, we have from $u_{n,i} = T_{r_n}(x_n - r_n \Psi_i x_n)$ that

\[
\|u_{n,i} - x^*\|^2 \leq \|(x_n - x^*) - r_n(\Psi_i x_n - \Psi_i x^*)\|^2 \\
= \|x_n - x^*\|^2 - 2r_n \langle x_n - x^*, \Psi_i x_n - \Psi_i x^* \rangle + r_n^2 \|\Psi_i x_n - \Psi_i x^*\|^2 \\
\leq \|x_n - x^*\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2.
\]

(3.17)

It follows that

\[
\|\omega_n - x^*\|^2 = \left\| \sum_{i=1}^k \frac{1}{k} (u_{n,i} - x^*) \right\|^2 \\
\leq \frac{1}{k} \sum_{i=1}^k \|u_{n,i} - x^*\|^2 \\
\leq \|x_n - x^*\|^2 - \frac{1}{k} \sum_{i=1}^k r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2.
\]

(3.18)

This implies that

\[
\|x_{n+1} - x^*\|^2 = \|\alpha_n (u_n - x^*) + (1 - \alpha_n)(x_n - x^*)\|^2 \\
\leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 \\
\leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 \\
- (1 - \alpha_n)\frac{1}{k} \sum_{i=1}^k r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2,
\]

(3.19)

and hence

\[
(1 - \alpha_n)\frac{1}{k} \sum_{i=1}^k r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2 \leq \alpha_n\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.
\]

(3.20)

Furthermore, we have from Lemma 2.2 that for any $i \in \{1, 2, \ldots, k\}$, we have

\[
\|u_{n,i} - x^*\|^2 \leq \langle x_n - r_n \Psi_i x_n, (x^* - r_n \Psi_i x^*), u_{n,i} - x^* \rangle \\
= \frac{1}{2} \left\{ \|x_n - r_n \Psi_i x_n - (x^* - r_n \Psi_i x^*)\|^2 + \|u_{n,i} - x^*\|^2 \\
- \|x_n - r_n \Psi_i x_n - (x^* - r_n \Psi_i x^*) - (u_{n,i} - x^*)\|^2 \right\} \\
\leq \frac{1}{2} \left\{ \|x_n - x^*\|^2 + \|u_{n,i} - x^*\|^2 - \|(x_n - u_{n,i}) - r_n(\Psi_i x_n - \Psi_i x^*)\|^2 \right\} \\
= \frac{1}{2} \left\{ \|x_n - x^*\|^2 + \|u_{n,i} - x^*\|^2 - \|x_n - u_{n,i}\|^2 - r_n^2\|\Psi_i x_n - \Psi_i x^*\|^2 \\
+ 2r_n\langle x_n - u_{n,i}, \Psi_i x_n - \Psi_i x^* \rangle \right\}.
\]

(3.21)
Abstract and Applied Analysis

This implies that

\[ \|u_{n,i} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_{n,i}\|^2 + 2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|. \] (3.22)

Then we have from (3.22) that

\[ \|\omega_n - x^*\|^2 \leq \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x^*\|^2 \]

\[ \leq \|x_n - x^*\|^2 - \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2 \]

\[ + \frac{1}{k} \sum_{i=1}^{k} 2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|. \] (3.23)

Hence we have from (3.23) that

\[ \|x_{n+1} - x^*\|^2 \leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n)\|\omega_n - x^*\|^2 \]

\[ \leq \alpha_n\|u_n - x^*\|^2 + (1 - \alpha_n) \left( \|x_n - x^*\|^2 - \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2 \right) \]

\[ + (1 - \alpha_n) \left( \frac{1}{k} \sum_{i=1}^{k} 2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\| \right). \] (3.24)

It follows that

\[ (1 - \alpha_n) \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\|^2 \leq \alpha_n\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \]

\[ + (1 - \alpha_n) \left( \frac{1}{k} \sum_{i=1}^{k} 2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\| \right). \] (3.25)

Next, we will consider the following two cases.

Case A. Put \( \Gamma_n = \|\omega_n - x^*\|^2 \) for all \( n \in \mathbb{N} \). Suppose that \( \Gamma_{n+1} \leq \Gamma_n \) for all \( n \in \mathbb{N} \). In this case \( \lim_{n \to \infty} \Gamma_n \) exists and then \( \lim_{n \to \infty} (\Gamma_{n+1} - \Gamma_n) = 0 \). By (C1), (C3), and (3.12), we have

\[ \lim_{n \to \infty} \|\omega_n - T_1 \omega_n\| = 0. \] (3.26)

Similarly by (C1), (C2), and (3.13), we also have

\[ \lim_{n \to \infty} \|y_n - T_2 \omega_n\| = 0. \] (3.27)
So, we have from (3.14), (3.26), and (3.27) that

$$\lim_{n \to \infty} \|\omega_n - T_2\omega_n\| = 0. \quad (3.28)$$

Since \( \lim_{n \to \infty} \|\omega_n - x^*\| \) exists, we have from (3.11) and (3.26)

$$\lim_{n \to \infty} \|\omega_n - x^*\| = \lim_{n \to \infty} \|x_n - x^*\|. \quad (3.29)$$

We also have from (C1), (3.16), (3.26), and (3.27) that

$$\lim_{n \to \infty} \|x_{n+1} - \omega_n\| = 0. \quad (3.30)$$

Since \( \lim_{n \to \infty} \|x_n - x^*\| \) exists we have from (C1) and (3.20) that

$$\lim_{n \to \infty} \|\Psi_i x_n - \Psi_i x^*\| = 0, \quad \forall i = 1, 2, \ldots, k. \quad (3.31)$$

This together with (3.25) and the existence of \( \lim_{n \to \infty} \|x_n - x^*\| \) implies that

$$\lim_{n \to \infty} \|u_{n,i} - x_n\| = 0, \quad \forall i = 1, 2, \ldots, k, \quad (3.32)$$

which gives that

$$\|\omega_n - x_n\| \leq \frac{1}{k} \sum_{i=1}^{k} \|u_{n,i} - x_n\| \longrightarrow 0 \quad \text{as} \ n \longrightarrow \infty. \quad (3.33)$$

So, from (3.30), \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \) Furthermore, we have from (3.33) that

$$\|\omega_{n+1} - \omega_n\| \leq \|\omega_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - \omega_n\| \longrightarrow 0 \quad \text{as} \ n \longrightarrow \infty; \quad (3.34)$$

that is

$$\lim_{n \to \infty} \|\omega_{n+1} - \omega_n\| = 0. \quad (3.35)$$

Now, since \( \{\omega_n\} \) is a bounded sequence, there exists a subsequence \( \{\omega_{n_j}\} \) of \( \{\omega_n\} \) such that

$$\limsup_{n \to \infty} \langle u - x^*, \omega_n - x^* \rangle = \lim_{j \to \infty} \langle u - x^*, \omega_{n_j} - x^* \rangle. \quad (3.36)$$

Without loss of generality, we may assume that \( \omega_{n_j} \rightharpoonup v. \) Since \( T_1 \) is demiclosed at zero and by (3.26), we conclude that \( v \in F(T_1). \) Similarly, since \( T_2 \) is demiclosed at zero and by (3.28), we have \( v \in F(T_2). \) Therefore, we get that

$$v \in F(T_1) \cap F(T_2). \quad (3.37)$$
Next, we show that \( v \in \bigcap_{i=1}^{k} \text{GEP}(G_i, \Psi_i) \). For each \( i \in \{1, 2, \ldots, k\} \), since \( u_{n,i} = T_{r_n}(x_n - r_n \Psi_i x_n) \), we have

\[
G_i(u_{n,i}, y) + \langle \Psi_i x_n, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, u_{n,i} - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.38}
\]

From (A2), we also have

\[
\langle \Psi_i x_n, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, u_{n,i} - x_n \rangle \geq G_i(y, u_{n,i}). \tag{3.39}
\]

Replacing \( n \) by \( n_j \), we have

\[
\langle \Psi_i x_{n_j}, y - u_{n,j,i} \rangle + \left( y - u_{n,j,i} \frac{u_{n,j,i} - x_n}{r_{n_j}} \right) \geq G_i(y, u_{n,j,i}). \tag{3.40}
\]

Put \( y_t = ty + (1 - t)v \) for all \( t \in (0, 1] \) and \( y \in C \). Since \( v \in C \), then \( y_t \in C \) and

\[
\langle y_t - u_{n,j,i}, \Psi_i y_t \rangle \geq \langle y_t - u_{n,j,i}, \Psi_i y_t \rangle - \langle y_t - u_{n,j,i}, \Psi_i x_n \rangle
- \left( y_t - u_{n,j,i} \frac{u_{n,j,i} - x_n}{r_{n_j}} \right) + G_i(y_t, u_{n,j,i})
= \langle y_t - u_{n,j,i} \Psi_i y_t - \Psi_i u_{n,j,i} \rangle + \langle y_t - u_{n,j,i} \Psi_i u_{n,j,i} - \Psi_i x_n \rangle
- \left( y_t - u_{n,j,i} \frac{u_{n,j,i} - x_n}{r_{n_j}} \right) + G_i(y_t, u_{n,j,i}). \tag{3.41}
\]

Since \( \|u_{n,j,i} - x_n\| \to 0 \) as \( j \to \infty \), we obtain that \( \|\Psi_i u_{n,j,i} - \Psi_i x_n\| \to 0 \) as \( j \to \infty \). Furthermore, by the monotonicity of \( \Psi_i \), we obtain that

\[
\langle y_t - u_{n,j,i}, \Psi_i y_t - \Psi_i u_{n,j,i} \rangle \geq 0. \tag{3.42}
\]

Taking \( j \to \infty \) in (3.41), we have from (A4) that

\[
\langle y_t - v, \Psi_i y_t \rangle \geq G_i(y_t, v). \tag{3.43}
\]

Now, from (A1), (A4), and (3.43), we also have

\[
0 = G_i(y_t, y_t) \leq tG_i(y_t, y) + (1 - t)G_i(y_t, v)
\leq tG_i(y_t, y) + (1 - t)\langle y_t - v, \Psi_i y_t \rangle
= tG_i(y_t, y) + (1 - t)t\langle y - v, \Psi_i y_t \rangle, \tag{3.44}
\]
which yields that
\[ G_i(y_t, y) + (1 - t)(y - v, \Psi_i y_t) \geq 0. \]  
(3.45)

Taking \( t \to 0 \), we have, for each \( y \in \mathcal{C} \)
\[ G_i(v, y) + (y - v, \Psi_i v) \geq 0, \quad \forall i \in \{1, 2, \ldots, k\}. \]  
(3.46)

This shows \( v \in \text{GEP}(G_i, \Psi_i) \), for all \( i = 1, 2, \ldots, k \). Then, \( v \in \cap_{i=1}^{k} \text{GEP}(G_i, \Psi_i) \). Hence we have \( v \in F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} \text{GEP}(G_i, \Psi_i)) = \Omega \). So, we have from (3.36) that
\[ \lim_{n \to \infty} \sup \langle u - x^*, \omega_n - x^* \rangle = \langle u - x^*, v - x^* \rangle \leq 0. \]  
(3.47)

By (C1), (C4), (3.15), (3.30), (3.47), and Lemma 2.5, we obtain that \( \lim_{n \to \infty} \|\omega_n - x^*\| = 0 \). Hence we have from (3.29) that \( \{x_n\} \) converges to \( x^* \), where \( x^* = P_{\Omega} u \).

**Case B.** Assume that there exists a subsequence \( \{\Gamma_{n_i}\}_{i \geq 0} \) of \( \{\Gamma_n\}_{n \geq 0} \) such that \( \Gamma_{n_i} < \Gamma_{n_i+1} \) for all \( i \in \mathbb{N} \). In this case, it follows from Lemma 2.4 that there exists a subsequence \( \{\Gamma_{n_i}\} \) of \( \{\Gamma_n\} \) such that \( \Gamma_{n_i+1} > \Gamma_{n_i} \), where \( \tau : \mathbb{N} \to \mathbb{N} \) is defined by
\[ \tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \quad \forall n \in \mathbb{N}. \]  
(3.48)

So, from (3.12), that
\[ \|\omega_{\tau(n)} - x^*\|^2 - \|\omega_{\tau(n)} - x^*\|^2 + \gamma_{\tau(n)} (1 - \gamma_{\tau(n)}) \|\omega_{\tau(n)} - T_1 \omega_{\tau(n)}\|^2 \leq \alpha_{\tau(n)} \|\omega_{\tau(n)} - x^*\|^2. \]  
(3.49)

Since \( \|\omega_{\tau(n)} - x^*\|^2 := \Gamma_{\tau(n)} < \Gamma_{\tau(n) + 1} := \|\omega_{\tau(n) + 1} - x^*\|^2 \), we have
\[ \gamma_{\tau(n)} (1 - \gamma_{\tau(n)}) \|\omega_{\tau(n)} - T_1 \omega_{\tau(n)}\|^2 \leq \alpha_{\tau(n)} \|\omega_{\tau(n)} - x^*\|^2. \]  
(3.50)

By (C1) and (C3), we have
\[ \lim_{n \to \infty} \|\omega_{\tau(n)} - T_1 \omega_{\tau(n)}\| = 0. \]  
(3.51)

By (3.15), we have
\[ \|\omega_{\tau(n) + 1} - x^*\|^2 \leq (1 - \alpha_{\tau(n)}) \|\omega_{\tau(n)} - x^*\|^2 + 2 \alpha_{\tau(n)} \langle \omega_{\tau(n)} - x^*, \omega_{\tau(n) + 1} - x^* \rangle. \]  
(3.52)
Abstract and Applied Analysis

Now, in view of $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we see that

$$
\|\omega_{\tau(n)} - x^*\|^2 
\leq
2\langle u_{\tau(n)} - x^*, \omega_{\tau(n)+1} - x^* \rangle
$$

$$
= 2\langle u_{\tau(n)} - u, x_{\tau(n)+1} - x^* \rangle + 2\langle u - x^*, x_{\tau(n)+1} - \omega_{\tau(n)} \rangle
$$

$$
+ 2\langle u - x^*, \omega_{\tau(n)} - x^* \rangle. \tag{3.53}
$$

Furthermore, we also have from (3.13) that

$$
\beta_{\tau(n)} \left(1 - \beta_{\tau(n)}\right) \|y_{\tau(n)} - T_2\omega_{\tau(n)}\|^2 
\leq
\alpha_{\tau(n)} \|u_{\tau(n)} - x^*\|^2 + \|\omega_{\tau(n)} - x^*\|^2
$$

$$
- \|\omega_{\tau(n)+1} - x^*\|^2
\leq
\alpha_{\tau(n)} \|u_{\tau(n)} - x^*\|^2. \tag{3.54}
$$

Applying (C1) and (C2) to the last inequality, we get that

$$
\lim_{n \to \infty} \|y_{\tau(n)} - T_2\omega_{\tau(n)}\| = 0. \tag{3.55}
$$

By (C1), (3.16), (3.51), and (3.55), we have

$$
\lim_{n \to \infty} \|x_{\tau(n)+1} - \omega_{\tau(n)}\| = 0. \tag{3.56}
$$

By (3.33), we have

$$
\lim_{n \to \infty} \|\omega_{\tau(n)+1} - x_{\tau(n)+1}\| = 0. \tag{3.57}
$$

It follows from (3.56) and (3.57) that

$$
\lim_{n \to \infty} \|\omega_{\tau(n)+1} - \omega_{\tau(n)}\| = 0. \tag{3.58}
$$

Since $\{\omega_{\tau(n)}\}$ is a bounded sequence, there exists a subsequence $\{\omega_{\tau(n)}\}$ such that

$$
\limsup_{n \to \infty} \langle u - x^*, \omega_{\tau(n)} - x^* \rangle = \lim_{j \to \infty} \langle u - x^*, \omega_{\tau(n)} - x^* \rangle. \tag{3.59}
$$

Following the same argument as the proof of Case A for $\{\omega_{\tau(n)}\}$, we have that

$$
\limsup_{n \to \infty} \langle u - x^*, \omega_{\tau(n)} - x^* \rangle \leq 0. \tag{3.60}
$$

Using (C4), (3.53), (3.56), and (3.60), we have that

$$
\lim_{n \to \infty} \|\omega_{\tau(n)} - x^*\| = 0. \tag{3.61}
$$
By (3.58) and (3.61), we have that

$$\lim_{n \to \infty} \|\omega_{\tau(n) + 1} - x^*\| = 0. \quad (3.62)$$

By Lemma 2.4 (ii), we get \(\lim_{n \to \infty} \Gamma_n = 0\); that is \(\lim_{n \to \infty} \|\omega_n - x^*\| = 0\). We observe that

$$\|x_{n+1} - x^*\|^2 \leq \alpha_n \|u_n - x^*\|^2 + (1 - \alpha_n)\|\omega_n - x^*\|^2. \quad (3.63)$$

Applying (C1), (C4), and \(\lim_{n \to \infty} \|\omega_n - x^*\|^2 = 0\), we have immediately

$$\lim_{n \to \infty} \|x_n - x^*\| = 0; \quad (3.64)$$

that is, \(\{x_n\}\) converges strongly to \(x^*\), where \(x^* = P_{\Omega} u\). This completes the proof. \(\Box\)

Setting \(\Psi_i \equiv 0\) for all \(i = 1, 2, \ldots, k\) in Theorem 3.1, we obtain the following result.

**Corollary 3.2.** Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\). For each \(i = 1, 2, \ldots, k\), let \(G_i : C \times C \to \mathbb{R}\) be a bifunction satisfying (A1)–(A4). For each \(j = 1, 2\), let \(T_j : C \to H\) be two quasi-nonexpansive mappings such that \(I - T_j\) are demiclosed at zero with \(\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} EP(G_i)) \neq \emptyset\). Let the sequences \(\{x_n\}, \{y_n\}, \text{ and } \{z_n\}\) be defined by

\[
\begin{align*}
x_1 & \in H, \\
G_1(u_{n,1}, y) + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle & \geq 0, \quad \forall y \in C, \\
G_2(u_{n,2}, y) + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle & \geq 0, \quad \forall y \in C, \\
& \vdots \\
G_k(u_{n,k}, y) + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle & \geq 0, \quad \forall y \in C, \\
\omega_n & = \frac{1}{k} \sum_{i=1}^{k} u_{n,i}, \\
y_n & = \gamma_n \omega_n + (1 - \gamma_n) T_1 \omega_n, \\
z_n & = \beta_n y_n + (1 - \beta_n) T_2 \omega_n, \\
x_{n+1} & = \alpha_n u_n + (1 - \alpha_n) z_n, \quad \forall n \in \mathbb{N},
\end{align*}
\]

where \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) satisfy the following conditions.

- (C1) \(\lim_{n \to \infty} \alpha_n = 0\) and \(\sum_{n=1}^{\infty} \alpha_n = \infty\);
- (C2) \(\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0\);
- (C3) \(\liminf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0\);
- (C4) \(\lim_{n \to \infty} u_n = u\) for some \(u \in H\).
\[ \{ x_n \} \text{ converges strongly to } x^*, \text{ where } x^* = P_{\Omega}u. \]

In the next results, using Theorem 3.1, we have new strong convergence theorems for two nonexpansive mappings in a Hilbert space.

**Corollary 3.3.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). For each \( i = 1, 2, \ldots, k \), let \( G_i : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)–(A4) and \( \Psi_i \) a \( \mu_i \)-inverse strongly monotone mapping. For each \( j = 1, 2 \), let \( T_j : C \to H \) be two nonexpansive mappings such that \( \Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^k \text{GEP}(G_i, \Psi_i)) \neq \emptyset \). Let the sequences \( \{ x_n \}, \{ y_n \}, \text{ and } \{ z_n \} \) be defined by (3.1), where \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \} \) satisfy the following conditions.

\[
\begin{align*}
(C1) \lim_{n \to \infty} \alpha_n &= 0 \text{ and } \sum_{n=1}^\infty \alpha_n = \infty; \\
(C2) \liminf_{n \to \infty} \beta_n (1 - \beta_n) &> 0; \\
(C3) \liminf_{n \to \infty} \gamma_n (1 - \gamma_n) &> 0; \\
(C4) \lim_{n \to \infty} u_n &= u \text{ for some } u \in H.
\end{align*}
\]

Then \( \{ x_n \} \) converges strongly to \( x^* \), where \( x^* = P_{\Omega}u \).

**4. Applications**

In this section, we present some convergence theorems deduced from the results in the previous section. Recall that a mapping \( T : C \to H \) is said to be nonspreading [2] if

\[ 2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \]  

(4.1)

for all \( x, y \in C \). Further, a mapping \( T : C \to H \) is said to be hybrid [23] if

\[ 3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \]  

(4.2)

for all \( x, y \in C \). These mappings are deduced from a firmly nonexpansive mapping in a Hilbert space.

A mapping \( F : C \to H \) is said to be firmly nonexpansive if

\[ \|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle \]  

(4.3)

for all \( x, y \in C \); see, for instance, Browder [24] and Goebel and Kirk [25]. We also know that a firmly nonexpansive mapping \( F \) can be deduced from an equilibrium problem in a Hilbert space.

Recently, Kocourek et al. [26] introduced a more broad class of nonlinear mappings call generalized hybrid if there are \( \alpha, \beta \in \mathbb{R} \) such that

\[ \alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \]  

(4.4)
Lemma 4.2. Let \( S : C \rightarrow H \) be a nonempty closed convex subset of \( H \). A mapping \( S \) is called super hybrid if and only if
\[
\alpha \| Sx - Sy \|^2 + (1 - \alpha + \gamma) \| x - Sy \|^2 \leq (\beta + (\beta - \alpha) \gamma) \| Sx - y \|^2 + (1 - \beta - (\beta - \alpha - 1) \gamma) \| x - y \|^2 + (\alpha - \beta) \gamma \| x - Sx \|^2 + \gamma \| y - Sy \|^2,
\]
for all \( x, y \in C \). We call such a mapping an \((\alpha, \beta, \gamma)\)-super hybrid mapping. We notice that an \((\alpha, \beta, 0)\)-super hybrid mapping is \((\alpha, \beta)\)-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. A super hybrid mapping is not quasi-nonexpansive generally. For more details, see [27]. Before proving, we need the following lemmas.

**Lemma 4.1** (see [27]). Let \( C \) be a nonempty subset of a Hilbert space \( H \) and let \( \alpha, \beta \) and \( \gamma \) be real numbers with \( \gamma \neq -1 \). Let \( S \) and \( T \) be mappings of \( C \) into \( H \) such that \( S = (1/(1+\gamma)) T + (\gamma/(1+\gamma)) I \). Then, \( T \) is \((\alpha, \beta, \gamma)\)-super hybrid if and only if \( S \) is \((\alpha, \beta)\)-generalized hybrid. In this case, \( F(S) = F(T) \).

**Lemma 4.2** (see [27]). Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( S : C \rightarrow H \) be a generalized hybrid mapping. Then \( S \) is demiclosed on \( C \).

Theorem 4.3. Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). For each \( i = 1, 2, \ldots, k \), let \( G_i : C \times C \rightarrow \mathbb{R} \) be a bifunction satisfying (A1)-(A4) and \( \Psi_i \) a \( \mu_i \)-inverse strongly monotone mapping. For each \( j = 1, 2 \), let \( T_j : C \rightarrow H \) be \((\alpha_j, \beta_j, \gamma_j)\)-super hybrid mappings such that \( \Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^k GEP(G_i, \Psi_i)) \neq \emptyset \). Let the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) be defined by

\[
x_1 \in H,
\]

\[
G_1(u_{n,1}, y) + (\Psi_1 x_n, y - u_{n,1}) + \frac{1}{r_n}(y - u_{n,1}, u_{n,1} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
G_2(u_{n,2}, y) + (\Psi_2 x_n, y - u_{n,2}) + \frac{1}{r_n}(y - u_{n,2}, u_{n,2} - x_n) \geq 0, \quad \forall y \in C,
\]

\vdots

\[
G_k(u_{n,k}, y) + (\Psi_k x_n, y - u_{n,k}) + \frac{1}{r_n}(y - u_{n,k}, u_{n,k} - x_n) \geq 0, \quad \forall y \in C,
\]

\[
\omega_n = \frac{1}{k} \sum_{i=1}^k u_{n,i},
\]

\[
y_n = y_n \omega_n + (1 - \gamma_n) \left( \frac{1}{1 + \gamma_1} T_1 \omega_n + \frac{\gamma_1}{1 + \gamma_1} \omega_n \right),
\]

\[
z_n = \beta_n y_n + (1 - \beta_n) \left( \frac{1}{1 + \gamma_2} T_2 \omega_n + \frac{\gamma_2}{1 + \gamma_2} \omega_n \right),
\]

\[
x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \quad \forall n \in \mathbb{N},
\]
where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \((0,1)\) and \(\{u_n\} \subset H\) is a sequence and \(\{r_n\} \subset [a,2\mu_i)\) for some \(a > 0\) and for all \(i \in \{1,2,\ldots, k\}\). Suppose the following conditions are satisfied.

(C1) \(\lim_{n \to \infty} \alpha_n = 0\) and \(\sum_{n=1}^{\infty} \alpha_n = \infty\);

(C2) \(\lim \inf_{n \to \infty} \beta_n (1-\beta_n) > 0\);

(C3) \(\lim \inf_{n \to \infty} \gamma_n (1-\gamma_n) > 0\);

(C4) \(\lim_{n \to \infty} u_n = u\) for some \(u \in H\).

Then \(\{x_n\}\) converges strongly to \(x^*\), where \(x^* = P_{\Omega} u\).

**Proof.** For each \(j = 1,2\), setting

\[
S_j = \frac{1}{1+y_j} T_j + \frac{y_j}{1+y_j} I,
\]

we have from Lemma 4.1 that each \(S_j\) is a generalized hybrid mapping and \(F(S_j) = F(T_j)\). Since \(F(S_j) \neq \emptyset\), we have that each \(S_j\) is quasi-nonexpansive. Following the proof of Theorem 3.1 and applying Lemma 4.2, we have the desired result. This completes the proof. \(\square\)

Setting \(\Psi \equiv 0\) in Theorem 4.3, we obtains the following result.

**Corollary 4.4.** Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\). For each \(i = 1,2,\ldots, k\), let \(G_i : C \times C \to \mathbb{R}\) be a bifunction satisfying (A1)–(A4). For each \(j = 1,2\), let \(T_j : C \to H\) be \((\alpha_j,\beta_j,\gamma_j)\)-super hybrid mappings such that \(\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} EP(G_i)) \neq \emptyset\). Let the sequences \(\{x_n\}, \{y_n\}\), and \(\{z_n\}\) be defined by

\[
x_1 \in H,
G_1 (u_{n,1},y) + \frac{1}{r_n} (y - u_{n,1},u_{n,1} - x_n) \geq 0, \quad \forall y \in C,
G_2 (u_{n,2},y) + \frac{1}{r_n} (y - u_{n,2},u_{n,2} - x_n) \geq 0, \quad \forall y \in C,
\]

\[\vdots\]

\[G_k (u_{n,k},y) + \frac{1}{r_n} (y - u_{n,k},u_{n,k} - x_n) \geq 0, \quad \forall y \in C,\]

\[
\omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i},
\]

\[y_n = \gamma_n \omega_n + (1-\gamma_n) \left( \frac{1}{1+\gamma_1} T_1 \omega_n + \frac{\gamma_1}{1+\gamma_1} \omega_n \right),\]

\[z_n = \beta_n y_n + (1-\beta_n) \left( \frac{1}{1+\gamma_2} T_2 \omega_n + \frac{\gamma_2}{1+\gamma_2} \omega_n \right),\]

\[x_{n+1} = \alpha_n u_n + (1-\alpha_n) z_n, \quad \forall n \in \mathbb{N},\]
where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \((0, 1)\) and \( \{u_n\} \subset H \) is a sequence and \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \). Suppose the following conditions are satisfied.

1. \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \); \hspace{3cm} (C1)
2. \( \liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0 \); \hspace{3cm} (C2)
3. \( \liminf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0 \); \hspace{3cm} (C3)
4. \( \lim_{n \to \infty} u_n = u \) for some \( u \in H \). \hspace{3cm} (C4)

Then \( \{x_n\} \) converges strongly to \( x^* \), where \( x^* = P_{\Omega^*} u \).

In Corollary 4.4, put \( G_i(x, y) = 0 \) for all \( x, y \in C \) and \( r_n = 1 \) for all \( n \in \mathbb{N} \). Then we have that \( u_{nj} = x_n \) for all \( i = 1, 2, \ldots, k \), which gives that \( \omega_n = (1/k) \sum_{i=1}^{k} u_{n,i} = x_n \). Thus we obtain the following results from Corollary 4.4.

**Corollary 4.5.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). For each \( j = 1, 2 \), let \( T_j : C \to H \) be \((\alpha_j, \beta_j, \gamma_j)\)-super hybrid mappings such that \( F(T_1) \cap F(T_2) \neq \emptyset \). Let the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) be defined by

\[
x_1 \in H,

y_n = \gamma_n x_n + (1 - \gamma_n) \left( \frac{1}{1 + \gamma_1} T_1 x_n + \frac{\gamma_1}{1 + \gamma_1} x_n \right),

z_n = \beta_n y_n + (1 - \beta_n) \left( \frac{1}{1 + \gamma_2} T_2 x_n + \frac{\gamma_2}{1 + \gamma_2} x_n \right),

x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \quad \forall n \in \mathbb{N},
\]

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \((0, 1)\) and \( \{u_n\} \subset H \) is a sequence. Suppose the following conditions are satisfied.

1. \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \); \hspace{3cm} (C1)
2. \( \liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0 \); \hspace{3cm} (C2)
3. \( \liminf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0 \); \hspace{3cm} (C3)
4. \( \lim_{n \to \infty} u_n = u \) for some \( u \in H \). \hspace{3cm} (C4)

Then \( \{x_n\} \) converges strongly to \( x^* \), where \( x^* = P_{F(T_1)\cap F(T_2)} u \).

In Corollary 4.5, put \( T_1 = I \), the identity mapping, and \( T_2 := T \), an \((\alpha, \beta, \gamma)\)-super hybrid mapping. Thus we obtain the following results.

**Corollary 4.6.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). Let \( T \) be an \((\alpha, \beta, \gamma)\)-super hybrid mapping such that \( F(T) \neq \emptyset \). Let the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) be defined by

\[
x_1 \in H,

z_n = \beta_n x_n + (1 - \beta_n) \left( \frac{1}{1 + \gamma} T x_n + \frac{\gamma}{1 + \gamma} x_n \right),

x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \quad \forall n \in \mathbb{N},
\]

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \((0, 1)\) and \( \{u_n\} \subset H \) is a sequence and \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \). Suppose the following conditions are satisfied.

1. \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \); \hspace{3cm} (C1)
2. \( \liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0 \); \hspace{3cm} (C2)
3. \( \liminf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0 \); \hspace{3cm} (C3)
4. \( \lim_{n \to \infty} u_n = u \) for some \( u \in H \). \hspace{3cm} (C4)
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \((0,1)\) and \( \{u_n\} \subset H \) is a sequence. Suppose the following conditions are satisfied.

\[
\begin{align*}
(C1) \quad & \lim_{n \to \infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty; \\
(C2) \quad & \liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0; \\
(C3) \quad & \lim_{n \to \infty} u_n = u \text{ for some } u \in H.
\end{align*}
\]

Then \( \{x_n\} \) converges strongly to \( x^* \), where \( x^* = P_{F(T)} u \).

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**References**


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