Research Article

Some Slater's Type Inequalities for Convex Functions Defined on Linear Spaces and Applications

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Some inequalities of the Slater type for convex functions defined on general linear spaces are given. Applications for norm inequalities and f -divergence measures are also provided.

1. Introduction

Suppose that I is an interval of real numbers with interior $\overset{\circ}{I}$, and $f : I \rightarrow \mathbb{R}$ is a convex function on I . Then f is continuous on $\overset{\circ}{I}$ and has finite left and right derivatives at each point of $\overset{\circ}{I}$. Moreover, if $x, y \in \overset{\circ}{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing functions on $\overset{\circ}{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \rightarrow \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \rightarrow [-\infty, \infty]$ such that $\varphi(\overset{\circ}{I}) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x - a)\varphi(a), \quad \text{for any } x, a \in I. \quad (1.1)$$

It is also well known that if f is convex on I , then ∂f is nonempty, $f'_-, f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x), \quad \text{for any } x \in \overset{\circ}{I}. \quad (1.2)$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\overset{\circ}{I}$, then $\partial f = \{f'\}$.

The following result is well known in the literature as *the Slater inequality*.

Theorem 1.1 (Slater, 1981, [1]). *If $f : I \rightarrow \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $x_i \in I$, $p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$ and $\sum_{i=1}^n p_i \varphi(x_i) \neq 0$, where $\varphi \in \partial f$, then*

$$\frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) \leq f\left(\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)}\right). \quad (1.3)$$

As pointed out in [2] (see also [3, p. 64] and [4, p. 208]), the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)} \in I, \quad (1.4)$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

For recent works on Slater's inequality, see [5–7].

The main aim of the present paper is to extend Slater's inequality for convex functions defined on general linear spaces. A reverse of the Slater's inequality is also obtained. Natural applications for norm inequalities and f -divergence measures are provided as well.

2. Slater's Inequality for Functions Defined on Linear Spaces

Assume that $f : X \rightarrow \mathbb{R}$ is a *convex function* on the real linear space X . Since for any vectors $x, y \in X$ the function $g_{x,y} : \mathbb{R} \rightarrow \mathbb{R}$, $g_{x,y}(t) := f(x + ty)$ is convex, it follows that the following limits exist

$$\nabla_{+(-)} f(x)(y) := \lim_{t \rightarrow 0+(-)} \frac{f(x + ty) - f(x)}{t}, \quad (2.1)$$

and they are called the *right (left) Gâteaux derivatives* of the function f in the point x over the direction y .

It is obvious that for any $t > 0 > s$ we have

$$\begin{aligned} \frac{f(x+ty) - f(x)}{t} &\geq \nabla_+ f(x)(y) = \inf_{t>0} \left[\frac{f(x+ty) - f(x)}{t} \right] \\ &\geq \sup_{s<0} \left[\frac{f(x+sy) - f(x)}{s} \right] = \nabla_- f(x)(y) \geq \frac{f(x+sy) - f(x)}{s}, \end{aligned} \quad (2.2)$$

for any $x, y \in X$ and, in particular,

$$\nabla_- f(u)(u-v) \geq f(u) - f(v) \geq \nabla_+ f(v)(u-v), \quad (2.3)$$

for any $u, v \in X$. We call this *the gradient inequality* for the convex function f . It will be used frequently in the sequel in order to obtain various results related to Slater's inequality.

The following properties are also of importance:

$$\nabla_+ f(x)(-y) = -\nabla_- f(x)(y), \quad (2.4)$$

$$\nabla_{+(-)} f(x)(\alpha y) = \alpha \nabla_{+(-)} f(x)(y), \quad (2.5)$$

for any $x, y \in X$ and $\alpha \geq 0$.

The right Gâteaux derivative is *subadditive* while the left one is *superadditive*, that is,

$$\begin{aligned} \nabla_+ f(x)(y+z) &\leq \nabla_+ f(x)(y) + \nabla_+ f(x)(z), \\ \nabla_- f(x)(y+z) &\geq \nabla_- f(x)(y) + \nabla_- f(x)(z), \end{aligned} \quad (2.6)$$

for any $x, y, z \in X$.

Some natural examples can be provided by the use of normed spaces.

Assume that $(X, \|\cdot\|)$ is a real normed linear space. The function $f : X \rightarrow \mathbb{R}$, $f(x) := (1/2)\|x\|^2$ is a convex function which generates *the superior* and *the inferior semi-inner products*

$$\langle y, x \rangle_{s(i)} := \lim_{t \rightarrow 0+(-)} \frac{\|x+ty\|^2 - \|x\|^2}{2t}. \quad (2.7)$$

For a comprehensive study of the properties of these mappings in the Geometry of Banach Spaces, see the monograph [8].

For the convex function $f_p : X \rightarrow \mathbb{R}$, $f_p(x) := \|x\|^p$ with $p > 1$, we have

$$\nabla_{+(-)} f_p(x)(y) = \begin{cases} p\|x\|^{p-2} \langle y, x \rangle_{s(i)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \quad (2.8)$$

for any $y \in X$.

If $p = 1$, then we have

$$\nabla_{+(-)}f_1(x)(y) = \begin{cases} \|x\|^{-1}\langle y, x \rangle_{s(i)} & \text{if } x \neq 0, \\ +(-)\|y\| & \text{if } x = 0, \end{cases} \quad (2.9)$$

for any $y \in X$.

For a given convex function $f : X \rightarrow \mathbb{R}$ and a given n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, we consider the sets

$$\begin{aligned} \text{Sla}_{+(-)}(f, \mathbf{x}) &:= \{v \in X \mid \nabla_{+(-)}f(x_i)(v - x_i) \geq 0 \ \forall i \in \{1, \dots, n\}\}, \\ \text{Sla}_{+(-)}(f, \mathbf{x}, \mathbf{p}) &:= \left\{v \in X \mid \sum_{i=1}^n p_i \nabla_{+(-)}f(x_i)(v - x_i) \geq 0\right\}, \end{aligned} \quad (2.10)$$

where $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{P}^n$ is a given probability distribution, that is, $p_i \geq 0$ for $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n p_i = 1$.

The following properties of these sets hold.

Lemma 2.1. *For a given convex function $f : X \rightarrow \mathbb{R}$, a given n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, and a given probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{P}^n$, one has*

- (i) $\text{Sla}_-(f, \mathbf{x}) \subset \text{Sla}_+(f, \mathbf{x})$ and $\text{Sla}_-(f, \mathbf{x}, \mathbf{p}) \subset \text{Sla}_+(f, \mathbf{x}, \mathbf{p})$;
- (ii) $\text{Sla}_-(f, \mathbf{x}) \subset \text{Sla}_-(f, \mathbf{x}, \mathbf{p})$ and $\text{Sla}_+(f, \mathbf{x}) \subset \text{Sla}_+(f, \mathbf{x}, \mathbf{p})$ for all $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{P}^n$;
- (iii) the sets $\text{Sla}_-(f, \mathbf{x})$ and $\text{Sla}_-(f, \mathbf{x}, \mathbf{p})$ are convex.

Proof. The properties (i) and (ii) follow from the definition and the fact that $\nabla_+f(x)(y) \geq \nabla_-f(x)(y)$ for any x, y .

(iii) Let us only prove that $\text{Sla}_-(f, \mathbf{x})$ is convex.

If we assume that $y_1, y_2 \in \text{Sla}_-(f, \mathbf{x})$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, then by the superadditivity and positive homogeneity of the Gâteaux derivative $\nabla_-f(\cdot)(\cdot)$ in the second variable we have

$$\begin{aligned} \nabla_-f(x_i)(\alpha y_1 + \beta y_2 - x_i) &= \nabla_-f(x_i)[\alpha(y_1 - x_i) + \beta(y_2 - x_i)] \\ &\geq \alpha \nabla_-f(x_i)(y_1 - x_i) + \beta \nabla_-f(x_i)(y_2 - x_i) \geq 0, \end{aligned} \quad (2.11)$$

for all $i \in \{1, \dots, n\}$, which shows that $\alpha y_1 + \beta y_2 \in \text{Sla}_-(f, \mathbf{x})$

The proof for the convexity of $\text{Sla}_-(f, \mathbf{x}, \mathbf{p})$ is similar and the details are omitted. $\square$

For the convex function $f_p : X \rightarrow \mathbb{R}$, $f_p(x) := \|x\|^p$ with $p \geq 1$, defined on the normed linear space $(X, \|\cdot\|)$ and for the n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in X^n \setminus \{(0, \dots, 0)\}$ we have, by the well-known property of the semi-inner products,

$$\langle y + \alpha x, x \rangle_{s(i)} = \langle y, x \rangle_{s(i)} + \alpha \|x\|^2, \quad \text{for any } x, y \in X, \alpha \in \mathbb{R}, \quad (2.12)$$

that

$$\text{Sla}_{+(-)}(\|\cdot\|^p, \mathbf{x}) = \text{Sla}_{+(-)}(\|\cdot\|, \mathbf{x}) := \left\{ v \in X \mid \langle v, x_j \rangle_{s(i)} \geq \|x_j\|^2 \forall j \in \{1, \dots, n\} \right\} \quad (2.13)$$

which, as can be seen, does not depend on p . We observe, by the continuity of the semi-inner products in the first variable, that $\text{Sla}_{+(-)}(\|\cdot\|, \mathbf{x})$ is closed in $(X, \|\cdot\|)$. Also, we should remark that if $v \in \text{Sla}_{+(-)}(\|\cdot\|, \mathbf{x})$, then for any $\gamma \geq 1$ we also have that $\gamma v \in \text{Sla}_{+(-)}(\|\cdot\|, \mathbf{x})$.

The larger classes, which are dependent on the probability distribution $\mathbf{p} \in \mathbb{P}^n$, are described by

$$\text{Sla}_{+(-)}(\|\cdot\|^p, \mathbf{x}, \mathbf{p}) := \left\{ v \in X \mid \sum_{j=1}^n p_j \|x_j\|^{p-2} \langle v, x_j \rangle_{s(i)} \geq \sum_{j=1}^n p_j \|x_j\|^p \right\}. \quad (2.14)$$

If the normed space is smooth, that is, the norm is Gâteaux differentiable in any nonzero point, then the superior and inferior semi-inner products coincide with the Lumer-Giles semi-inner product $[\cdot, \cdot]$ that generates the norm and is linear in the first variable (see for instance [8]). In this situation,

$$\begin{aligned} \text{Sla}(\|\cdot\|, \mathbf{x}) &= \left\{ v \in X \mid [v, x_j] \geq \|x_j\|^2 \forall j \in \{1, \dots, n\} \right\}, \\ \text{Sla}(\|\cdot\|^p, \mathbf{x}, \mathbf{p}) &= \left\{ v \in X \mid \sum_{j=1}^n p_j \|x_j\|^{p-2} [v, x_j] \geq \sum_{j=1}^n p_j \|x_j\|^p \right\}. \end{aligned} \quad (2.15)$$

If $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, then $\text{Sla}(\|\cdot\|^p, \mathbf{x}, \mathbf{p})$ can be described by

$$\text{Sla}(\|\cdot\|^p, \mathbf{x}, \mathbf{p}) = \left\{ v \in X \mid \left\langle v, \sum_{j=1}^n p_j \|x_j\|^{p-2} x_j \right\rangle \geq \sum_{j=1}^n p_j \|x_j\|^p \right\}, \quad (2.16)$$

and if the family $\{x_j\}_{j=1, \dots, n}$ is orthogonal, then obviously, by the Pythagoras theorem, we have that the sum $\sum_{j=1}^n x_j$ belongs to $\text{Sla}(\|\cdot\|, \mathbf{x})$ and therefore to $\text{Sla}(\|\cdot\|^p, \mathbf{x}, \mathbf{p})$ for any $p \geq 1$ and any probability distribution $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{P}^n$.

We can state now the following results that provide a generalization of Slater's inequality as well as a counterpart for it.

Theorem 2.2. *Let $f : X \rightarrow \mathbb{R}$ be a convex function on the real linear space X , $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ an n -tuple of vectors, and $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{P}^n$ a probability distribution. Then for any $v \in \text{Sla}_+(f, \mathbf{x}, \mathbf{p})$, one has the inequalities*

$$\nabla_- f(v)(v) - \sum_{i=1}^n p_i \nabla_- f(v)(x_i) \geq f(v) - \sum_{i=1}^n p_i f(x_i) \geq 0. \quad (2.17)$$

Proof. If we write the gradient inequality for $v \in \text{Sla}_+(f, \mathbf{x}, \mathbf{p})$ and x_i , then we have that

$$\nabla_- f(v)(v - x_i) \geq f(v) - f(x_i) \geq \nabla_+ f(x_i)(v - x_i), \quad (2.18)$$

for any $i \in \{1, \dots, n\}$.

By multiplying (2.18) with $p_i \geq 0$ and summing over i from 1 to n , we get

$$\sum_{i=1}^n p_i \nabla_- f(v)(v - x_i) \geq f(v) - \sum_{i=1}^n p_i f(x_i) \geq \sum_{i=1}^n p_i \nabla_+ f(x_i)(v - x_i). \quad (2.19)$$

Now, since $v \in \text{Sla}_+(f, \mathbf{x}, \mathbf{p})$, then the right hand side of (2.19) is nonnegative, which proves the second inequality in (2.17).

By the superadditivity of the Gâteaux derivative $\nabla_- f(\cdot)(\cdot)$ in the second variable, we have

$$\nabla_- f(v)(v) - \nabla_- f(v)(x_i) \geq \nabla_- f(v)(v - x_i), \quad (2.20)$$

which, by multiplying with $p_i \geq 0$ and summing over i from 1 to n , produces the inequality

$$\nabla_- f(v)(v) - \sum_{i=1}^n p_i \nabla_- f(v)(x_i) \geq \sum_{i=1}^n p_i \nabla_- f(v)(v - x_i). \quad (2.21)$$

Utilising (2.19) and (2.21), we deduce the desired result (2.17). $\square$

Remark 2.3. The above result has the following form for normed linear spaces. Let $(X, \|\cdot\|)$ be a normed linear space, $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ an n -tuple of vectors from X , and $p = (p_1, \dots, p_n) \in \mathbb{P}^n$ a probability distribution. Then for any vector $v \in X$ with the property

$$\sum_{j=1}^n p_j \|x_j\|^{p-2} \langle v, x_j \rangle_s \geq \sum_{j=1}^n p_j \|x_j\|^p, \quad p \geq 1, \quad (2.22)$$

we have the inequalities

$$p \left[\|v\|^p - \sum_{j=1}^n p_j \|x_j\|^{p-2} \langle v, x_j \rangle_i \right] \geq \|v\|^p - \sum_{j=1}^n p_j \|x_j\|^p \geq 0. \quad (2.23)$$

Rearranging the first inequality in (2.23), we also have that

$$(p-1)\|v\|^p + \sum_{j=1}^n p_j \|x_j\|^p \geq p \sum_{j=1}^n p_j \|x_j\|^{p-2} \langle v, x_j \rangle_i. \quad (2.24)$$

If the space is smooth, then the condition (2.22) becomes

$$\sum_{j=1}^n p_j \|x_j\|^{p-2} [v, x_j] \geq \sum_{j=1}^n p_j \|x_j\|^p, \quad p \geq 1, \quad (2.25)$$

implying the inequality

$$p \left[\|v\|^p - \sum_{j=1}^n p_j \|x_j\|^{p-2} [v, x_j] \right] \geq \|v\|^p - \sum_{j=1}^n p_j \|x_j\|^p \geq 0. \quad (2.26)$$

Notice also that the first inequality in (2.26) is equivalent with

$$(p-1)\|v\|^p + \sum_{j=1}^n p_j \|x_j\|^p \geq p \sum_{j=1}^n p_j \|x_j\|^{p-2} [v, x_j] \left(\geq p \sum_{j=1}^n p_j \|x_j\|^p \geq 0 \right). \quad (2.27)$$

Corollary 2.4. *Let $f : X \rightarrow \mathbb{R}$ be a convex function on the real linear space X , $x = (x_1, \dots, x_n) \in X^n$ an n -tuple of vectors, and $p = (p_1, \dots, p_n) \in \mathbb{P}^n$ a probability distribution. If*

$$\sum_{i=1}^n p_i \nabla_+ f(x_i)(x_i) \geq (<) 0, \quad (2.28)$$

and there exists a vector $s \in X$ with

$$\sum_{i=1}^n p_i \nabla_{+(-)} f(x_i)(s) \geq (\leq) 1, \quad (2.29)$$

then

$$\begin{aligned} & \nabla_- f \left(\sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) s \right) \left(\sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) s \right) - \sum_{i=1}^n p_i \nabla_- f \left(\sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) s \right) (x_i) \\ & \geq f \left(\sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) s \right) - \sum_{i=1}^n p_i f(x_i) \geq 0. \end{aligned} \quad (2.30)$$

Proof. Assume that $\sum_{i=1}^n p_i \nabla_+ f(x_i)(x_i) \geq 0$ and $\sum_{i=1}^n p_i \nabla_{+(-)} f(x_i)(s) \geq 1$ and define $v := \sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) s$. We claim that $v \in \text{Sla}_+(f, \mathbf{x}, \mathbf{p})$.

By the subadditivity and positive homogeneity of the mapping $\nabla_+ f(\cdot)(\cdot)$ in the second variable, we have

$$\begin{aligned}
\sum_{i=1}^n p_i \nabla_+ f(x_i)(v - x_i) &\geq \sum_{i=1}^n p_i \nabla_+ f(x_i)(v) - \sum_{i=1}^n p_i \nabla_+ f(x_i)(x_i) \\
&= \sum_{i=1}^n p_i \nabla_+ f(x_i) \left(\sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) s \right) - \sum_{i=1}^n p_i \nabla_+ f(x_i)(x_i) \\
&= \sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) \sum_{i=1}^n p_i \nabla_+ f(x_i)(s) - \sum_{i=1}^n p_i \nabla_+ f(x_i)(x_i) \\
&= \sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) \left[\sum_{i=1}^n p_i \nabla_+ f(x_i)(s) - 1 \right] \geq 0,
\end{aligned} \tag{2.31}$$

as claimed. Applying Theorem 2.2 for this v , we get the desired result.

If $\sum_{i=1}^n p_i \nabla_+ f(x_i)(x_i) < 0$ and $\sum_{i=1}^n p_i \nabla_- f(x_i)(s) \leq 1$, then for

$$w := \sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) s, \tag{2.32}$$

we also have that

$$\begin{aligned}
\sum_{i=1}^n p_i \nabla_+ f(x_i)(w - x_i) &\geq \sum_{i=1}^n p_i \nabla_+ f(x_i) \left(\sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) s \right) - \sum_{i=1}^n p_i \nabla_+ f(x_i)(x_i) \\
&= \sum_{i=1}^n p_i \nabla_+ f(x_i) \left(\left(- \sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) \right) (-s) \right) - \sum_{i=1}^n p_i \nabla_+ f(x_i)(x_i) \\
&= \left(- \sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) \right) \sum_{i=1}^n p_i \nabla_+ f(x_i)(-s) - \sum_{i=1}^n p_i \nabla_+ f(x_i)(x_i) \\
&= \left(- \sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) \right) \left(1 + \sum_{i=1}^n p_i \nabla_+ f(x_i)(-s) \right) \\
&= \left(- \sum_{j=1}^n p_j \nabla_+ f(x_j)(x_j) \right) \left(1 - \sum_{i=1}^n p_i \nabla_- f(x_i)(s) \right) \geq 0,
\end{aligned} \tag{2.33}$$

where, for the last equality, we have used the property (2.4). Therefore, $w \in \text{Sla}_+(f, \mathbf{x}, \mathbf{p})$ and by Theorem 2.2 we get the desired result. $\square$

It is natural to consider the case of normed spaces.

Remark 2.5. Let $(X, \|\cdot\|)$ be a normed linear space, $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ an n -tuple of vectors from X , and $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{P}^n$ a probability distribution. Then for any vector $s \in X$ with the property that

$$p \sum_{i=1}^n p_i \|x_i\|^{p-2} \langle s, x_i \rangle_s \geq 1, \quad (2.34)$$

we have the inequalities

$$\begin{aligned} & p^p \|s\|^{p-1} \left(\sum_{j=1}^n p_j \|x_j\|^p \right)^{p-1} \left(p \|s\| \sum_{j=1}^n p_j \|x_j\|^p - \sum_{j=1}^n p_j \langle x_j, s \rangle_i \right) \\ & \geq p^p \|s\|^p \left(\sum_{j=1}^n p_j \|x_j\|^p \right)^p - \sum_{j=1}^n p_j \|x_j\|^p \geq 0. \end{aligned} \quad (2.35)$$

The case of smooth spaces can be easily derived from the above; however, the details are left to the interested reader.

3. The Case of Finite Dimensional Linear Spaces

Consider now the finite dimensional linear space $X = \mathbb{R}^m$ and assume that C is an open convex subset of $\mathbb{R}^m$. Assume also that the function $f : C \rightarrow \mathbb{R}$ is differentiable and convex on C . Obviously, if $x = (x^1, \dots, x^m) \in C$, then for any $y = (y^1, \dots, y^m) \in \mathbb{R}^m$ we have

$$\nabla f(x)(y) = \sum_{k=1}^m \frac{\partial f(x^1, \dots, x^m)}{\partial x^k} \cdot y^k. \quad (3.1)$$

For the convex function $f : C \rightarrow \mathbb{R}$ and a given n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in C^n$ with $x_i = (x_i^1, \dots, x_i^m)$ with $i \in \{1, \dots, n\}$, we consider the sets

$$\begin{aligned} \text{Sla}(f, \mathbf{x}, C) &:= \left\{ v \in C \mid \sum_{k=1}^m \frac{\partial f(x_i^1, \dots, x_i^m)}{\partial x_i^k} \cdot v^k \right. \\ &\quad \left. \geq \sum_{k=1}^m \frac{\partial f(x_i^1, \dots, x_i^m)}{\partial x_i^k} \cdot x_i^k \quad \forall i \in \{1, \dots, n\} \right\}, \\ \text{Sla}(f, \mathbf{x}, \mathbf{p}, C) &:= \left\{ v \in C \mid \sum_{i=1}^n \sum_{k=1}^m p_i \frac{\partial f(x_i^1, \dots, x_i^m)}{\partial x_i^k} \cdot v^k \right. \\ &\quad \left. \geq \sum_{i=1}^n \sum_{k=1}^m p_i \frac{\partial f(x_i^1, \dots, x_i^m)}{\partial x_i^k} \cdot x_i^k \right\}, \end{aligned} \quad (3.2)$$

where $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{P}^n$ is a given probability distribution.

As in the previous section the sets, $\text{Sla}(f, \mathbf{x}, C)$ and $\text{Sla}(f, \mathbf{x}, \mathbf{p}, C)$ are convex and closed subsets of $\text{clo}(C)$, the closure of C . Also $\{x_1, \dots, x_n\} \subset \text{Sla}(f, \mathbf{x}, C) \subset \text{Sla}(f, \mathbf{x}, \mathbf{p}, C)$ for any $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{P}^n$ is a probability distribution.

Proposition 3.1. *Let $f : C \rightarrow \mathbb{R}$ be a convex function on the open convex set C in the finite dimensional linear space $\mathbb{R}^m$, $(x_1, \dots, x_n) \in C^n$ an n -tuple of vectors and $(p_1, \dots, p_n) \in \mathbb{P}^n$ a probability distribution. Then for any $v = (v^1, \dots, v^m) \in \text{Sla}(f, \mathbf{x}, \mathbf{p}, C)$, one has the inequalities*

$$\begin{aligned} & \sum_{k=1}^m \frac{\partial f(v^1, \dots, v^m)}{\partial x^k} \cdot v^k - \sum_{i=1}^n \sum_{k=1}^m p_i \frac{\partial f(x_i^1, \dots, x_i^m)}{\partial x_i^k} \cdot v^k \\ & \geq f(v^1, \dots, v^m) - \sum_{i=1}^n p_i f(x_i^1, \dots, x_i^m) \geq 0. \end{aligned} \quad (3.3)$$

The unidimensional case, that is, $m = 1$ is of interest for applications. We will state this case with the general assumption that $f : I \rightarrow \mathbb{R}$ is a convex function on an *open* interval I . For a given n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in I^n$, we have

$$\begin{aligned} \text{Sla}_{+(-)}(f, \mathbf{x}, I) &:= \left\{ v \in I \mid f'_{+(-)}(x_i) \cdot (v - x_i) \geq 0 \ \forall i \in \{1, \dots, n\} \right\}, \\ \text{Sla}_{+(-)}(f, \mathbf{x}, \mathbf{p}, I) &:= \left\{ v \in I \mid \sum_{i=1}^n p_i f'_{+(-)}(x_i) \cdot (v - x_i) \geq 0 \right\}, \end{aligned} \quad (3.4)$$

where $(p_1, \dots, p_n) \in \mathbb{P}^n$ is a probability distribution. These sets inherit the general properties pointed out in Lemma 2.1. Moreover, if we make the assumption that $\sum_{i=1}^n p_i f'_+(x_i) \neq 0$, then for $\sum_{i=1}^n p_i f'_+(x_i) > 0$ we have

$$\text{Sla}_+(f, \mathbf{x}, \mathbf{p}, I) = \left\{ v \in I \mid v \geq \frac{\sum_{i=1}^n p_i f'_+(x_i) x_i}{\sum_{i=1}^n p_i f'_+(x_i)} \right\}, \quad (3.5)$$

while for $\sum_{i=1}^n p_i f'_+(x_i) < 0$ we have

$$\text{Sla}_+(f, \mathbf{x}, \mathbf{p}, I) = \left\{ v \in I \mid v \leq \frac{\sum_{i=1}^n p_i f'_+(x_i) x_i}{\sum_{i=1}^n p_i f'_+(x_i)} \right\}. \quad (3.6)$$

Also, if we assume that $f'_+(x_i) \geq 0$ for all $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n p_i f'_+(x_i) > 0$, then

$$v_s := \frac{\sum_{i=1}^n p_i f'_+(x_i) x_i}{\sum_{i=1}^n p_i f'_+(x_i)} \in I, \quad (3.7)$$

due to the fact that $x_i \in I$ and I is a convex set.

Proposition 3.2. Let $f : I \rightarrow \mathbb{R}$ be a convex function on an open interval I . For a given n -tuple of vectors $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ and $(p_1, \dots, p_n) \in \mathbb{P}^n$ a probability distribution, one has

$$f'_-(v) \left(v - \sum_{i=1}^n p_i x_i \right) \geq f(v) - \sum_{i=1}^n p_i f(x_i) \geq 0, \quad (3.8)$$

for any $v \in \text{Sla}_+(f, \mathbf{x}, \mathbf{p}, \mathbf{I})$.

In particular, if one assumes that $\sum_{i=1}^n p_i f'_+(x_i) \neq 0$ and

$$\frac{\sum_{i=1}^n p_i f'_+(x_i) x_i}{\sum_{i=1}^n p_i f'_+(x_i)} \in I, \quad (3.9)$$

then

$$\begin{aligned} f'_- \left(\frac{\sum_{i=1}^n p_i f'_+(x_i) x_i}{\sum_{i=1}^n p_i f'_+(x_i)} \right) \left[\frac{\sum_{i=1}^n p_i f'_+(x_i) x_i}{\sum_{i=1}^n p_i f'_+(x_i)} - \sum_{i=1}^n p_i x_i \right] \\ \geq f \left(\frac{\sum_{i=1}^n p_i f'_+(x_i) x_i}{\sum_{i=1}^n p_i f'_+(x_i)} \right) - \sum_{i=1}^n p_i f(x_i) \geq 0. \end{aligned} \quad (3.10)$$

Moreover, if $f'_+(x_i) \geq 0$ for all $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n p_i f'_+(x_i) > 0$, then (3.10) holds true as well.

Remark 3.3. We remark that the first inequality in (3.10) provides a reverse inequality for the classical result due to Slater.

4. Some Applications for f -Divergences

Given a convex function $f : [0, \infty) \rightarrow \mathbb{R}$, the f -divergence functional

$$I_f(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^n q_i f \left(\frac{p_i}{q_i} \right), \quad (4.1)$$

where $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{q} = (q_1, \dots, q_n)$ are positive sequences, was introduced by Csiszár in [9], as a generalized measure of information, a “distance function” on the set of probability distributions $\mathbb{P}^n$. As in [9], we interpret undefined expressions by

$$\begin{aligned} f(0) &= \lim_{t \rightarrow 0^+} f(t), & 0f\left(\frac{0}{0}\right) &= 0, \\ 0f\left(\frac{a}{0}\right) &= \lim_{q \rightarrow 0^+} qf\left(\frac{a}{q}\right) = a \lim_{t \rightarrow \infty} \frac{f(t)}{t}, & a &> 0. \end{aligned} \quad (4.2)$$

The following results were essentially given by Csiszár and Körner [10]:

- (i) if f is convex, then $I_f(\mathbf{p}, \mathbf{q})$ is jointly convex in $\mathbf{p}$ and $\mathbf{q}$;
- (ii) for every $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^n$, we have

$$I_f(\mathbf{p}, \mathbf{q}) \geq \sum_{j=1}^n q_j f\left(\frac{\sum_{j=1}^n p_j}{\sum_{j=1}^n q_j}\right). \quad (4.3)$$

If f is strictly convex, equality holds in (4.3) if and only if

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \dots = \frac{p_n}{q_n}. \quad (4.4)$$

If f is normalized, that is, $f(1) = 0$, then for every $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i$, we have the inequality

$$I_f(\mathbf{p}, \mathbf{q}) \geq 0. \quad (4.5)$$

In particular, if $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$, then (4.5) holds. This is the well-known positivity property of the f -divergence.

It is obvious that the above definition of $I_f(\mathbf{p}, \mathbf{q})$ can be extended to any function $f : [0, \infty) \rightarrow \mathbb{R}$; however, the positivity condition will not generally hold for normalized functions and $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^n$ with $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i$.

For a normalized convex function $f : [0, \infty) \rightarrow \mathbb{R}$ and two probability distributions $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$, we define the set

$$\text{Sla}_+(f, \mathbf{p}, \mathbf{q}) := \left\{ v \in [0, \infty) \mid \sum_{i=1}^n q_i f'_+\left(\frac{p_i}{q_i}\right) \cdot \left(v - \frac{p_i}{q_i}\right) \geq 0 \right\}. \quad (4.6)$$

Now, observe that

$$\sum_{i=1}^n q_i f'_+\left(\frac{p_i}{q_i}\right) \cdot \left(v - \frac{p_i}{q_i}\right) \geq 0, \quad (4.7)$$

is equivalent with

$$v \sum_{i=1}^n q_i f'_+\left(\frac{p_i}{q_i}\right) \geq \sum_{i=1}^n p_i f'_+\left(\frac{p_i}{q_i}\right). \quad (4.8)$$

If $\sum_{i=1}^n q_i f'_+(p_i/q_i) > 0$, then (4.8) is equivalent with

$$v \geq \frac{\sum_{i=1}^n p_i f'_+(p_i/q_i)}{\sum_{i=1}^n q_i f'_+(p_i/q_i)}, \quad (4.9)$$

therefore in this case

$$\text{Sla}_+(f, \mathbf{p}, \mathbf{q}) = \begin{cases} [0, \infty) & \text{if } \sum_{i=1}^n p_i f'_+(p_i/q_i) < 0, \\ \left[\frac{\sum_{i=1}^n p_i f'_+(p_i/q_i)}{\sum_{i=1}^n q_i f'_+(p_i/q_i)}, \infty \right) & \text{if } \sum_{i=1}^n p_i f'_+(p_i/q_i) \geq 0. \end{cases} \quad (4.10)$$

If $\sum_{i=1}^n q_i f'_+(p_i/q_i) < 0$, then (4.8) is equivalent with

$$v \leq \frac{\sum_{i=1}^n p_i f'_+(p_i/q_i)}{\sum_{i=1}^n q_i f'_+(p_i/q_i)}, \quad (4.11)$$

therefore

$$\text{Sla}_+(f, \mathbf{p}, \mathbf{q}) = \begin{cases} \left[0, \frac{\sum_{i=1}^n p_i f'_+(p_i/q_i)}{\sum_{i=1}^n q_i f'_+(p_i/q_i)} \right] & \text{if } \sum_{i=1}^n p_i f'_+\left(\frac{p_i}{q_i}\right) \leq 0, \\ \emptyset & \text{if } \sum_{i=1}^n p_i f'_+\left(\frac{p_i}{q_i}\right) > 0. \end{cases} \quad (4.12)$$

Utilising the extended f -divergences notation, we can state the following result.

Theorem 4.1. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a normalized convex function and $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$ two probability distributions. If $v \in \text{Sla}_+(f, \mathbf{p}, \mathbf{q})$, then one has*

$$f'_-(v)(v - 1) \geq f(v) - I_f(\mathbf{p}, \mathbf{q}) \geq 0. \quad (4.13)$$

In particular, if one assumes that $I_{f'_+}(\mathbf{p}, \mathbf{q}) \neq 0$ and

$$\frac{I_{f'_+(\cdot)(\cdot)}(\mathbf{p}, \mathbf{q})}{I_{f'_+}(\mathbf{p}, \mathbf{q})} \in [0, \infty), \quad (4.14)$$

then

$$f'_-\left(\frac{I_{f'_+(\cdot)(\cdot)}(\mathbf{p}, \mathbf{q})}{I_{f'_+}(\mathbf{p}, \mathbf{q})}\right) \left[\frac{I_{f'_+(\cdot)(\cdot)}(\mathbf{p}, \mathbf{q})}{I_{f'_+}(\mathbf{p}, \mathbf{q})} - 1 \right] \geq f\left(\frac{I_{f'_+(\cdot)(\cdot)}(\mathbf{p}, \mathbf{q})}{I_{f'_+}(\mathbf{p}, \mathbf{q})}\right) - I_f(\mathbf{p}, \mathbf{q}) \geq 0. \quad (4.15)$$

Moreover, if $f'_+(p_i/q_i) \geq 0$ for all $i \in \{1, \dots, n\}$ and $I_{f'_+}(\mathbf{p}, \mathbf{q}) > 0$, then (4.15) holds true as well.

The proof follows immediately from Proposition 3.2 and the details are omitted.

The K. Pearson χ^2 -divergence is obtained for the convex function $f(t) = (1 - t)^2$, $t \in \mathbb{R}$ and given by

$$\chi^2(\mathbf{p}, \mathbf{q}) := \sum_{j=1}^n q_j \left(\frac{p_j}{q_j} - 1 \right)^2 = \sum_{j=1}^n \frac{(p_j - q_j)^2}{q_j} = \sum_{j=1}^n \frac{p_j^2}{q_j} - 1. \quad (4.16)$$

The *Kullback-Leibler divergence* can be obtained for the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = t \ln t$ and is defined by

$$\text{KL}(\mathbf{p}, \mathbf{q}) := \sum_{j=1}^n q_j \cdot \frac{p_j}{q_j} \ln \left(\frac{p_j}{q_j} \right) = \sum_{j=1}^n p_j \ln \left(\frac{p_j}{q_j} \right). \quad (4.17)$$

If we consider the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = -\ln t$, then we observe that

$$I_f(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^n q_i f \left(\frac{p_i}{q_i} \right) = - \sum_{i=1}^n q_i \ln \left(\frac{p_i}{q_i} \right) = \sum_{i=1}^n q_i \ln \left(\frac{q_i}{p_i} \right) = \text{KL}(\mathbf{q}, \mathbf{p}). \quad (4.18)$$

For the function $f(t) = -\ln t$, we will obviously have that

$$\begin{aligned} \text{Sla}(-\ln, \mathbf{p}, \mathbf{q}) &:= \left\{ v \in [0, \infty) \mid - \sum_{i=1}^n q_i \left(\frac{p_i}{q_i} \right)^{-1} \cdot \left(v - \frac{p_i}{q_i} \right) \geq 0 \right\} \\ &= \left\{ v \in [0, \infty) \mid v \sum_{i=1}^n \frac{q_i^2}{p_i} - 1 \leq 0 \right\} \\ &= \left[0, \frac{1}{\chi^2(\mathbf{q}, \mathbf{p}) + 1} \right]. \end{aligned} \quad (4.19)$$

Utilising the first part of Theorem 4.1, we can state the following.

Proposition 4.2. *Let $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$ be two probability distributions. If $v \in [0, (1/(\chi^2(\mathbf{q}, \mathbf{p}) + 1))]$, then one has*

$$\frac{1-v}{v} \geq -\ln(v) - \text{KL}(\mathbf{q}, \mathbf{p}) \geq 0. \quad (4.20)$$

In particular, for $v = 1/(\chi^2(\mathbf{q}, \mathbf{p}) + 1)$, one gets

$$\chi^2(\mathbf{q}, \mathbf{p}) \geq \ln[\chi^2(\mathbf{q}, \mathbf{p}) + 1] - \text{KL}(\mathbf{q}, \mathbf{p}) \geq 0. \quad (4.21)$$

If we consider now the function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = t \ln t$, then $f'(t) = \ln t + 1$ and

$$\begin{aligned} \text{Sla}((\cdot) \ln(\cdot), \mathbf{p}, \mathbf{q}) &:= \left\{ v \in [0, \infty) \mid \sum_{i=1}^n q_i \left(\ln \left(\frac{p_i}{q_i} \right) + 1 \right) \cdot \left(v - \frac{p_i}{q_i} \right) \geq 0 \right\} \\ &= \left\{ v \in [0, \infty) \mid v \sum_{i=1}^n q_i \left(\ln \left(\frac{p_i}{q_i} \right) + 1 \right) - \sum_{i=1}^n p_i \cdot \left(\ln \left(\frac{p_i}{q_i} \right) + 1 \right) \geq 0 \right\} \quad (4.22) \\ &= \{ v \in [0, \infty) \mid v(1 - \text{KL}(\mathbf{q}, \mathbf{p})) \geq 1 + \text{KL}(\mathbf{p}, \mathbf{q}) \}. \end{aligned}$$

We observe that if $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$ are two probability distributions such that $0 < \text{KL}(\mathbf{q}, \mathbf{p}) < 1$, then

$$\text{Sla}((\cdot) \ln(\cdot), \mathbf{p}, \mathbf{q}) = \left[\frac{1 + \text{KL}(\mathbf{p}, \mathbf{q})}{1 - \text{KL}(\mathbf{q}, \mathbf{p})}, \infty \right). \quad (4.23)$$

If $\text{KL}(\mathbf{q}, \mathbf{p}) \geq 1$, then $\text{Sla}((\cdot) \ln(\cdot), \mathbf{p}, \mathbf{q}) = \emptyset$.

By the use of Theorem 4.1, we can state now the following.

Proposition 4.3. *Let $\mathbf{p}, \mathbf{q} \in \mathbb{P}^n$ be two probability distributions such that $0 < \text{KL}(\mathbf{q}, \mathbf{p}) < 1$. If $v \in [(1 + \text{KL}(\mathbf{p}, \mathbf{q})) / (1 - \text{KL}(\mathbf{q}, \mathbf{p})), \infty)$, then one has*

$$(\ln v + 1)(v - 1) \geq v \ln(v) - \text{KL}(\mathbf{p}, \mathbf{q}) \geq 0. \quad (4.24)$$

In particular, for $v = (1 + \text{KL}(\mathbf{p}, \mathbf{q})) / (1 - \text{KL}(\mathbf{q}, \mathbf{p}))$, one gets

$$\begin{aligned} &\left(\ln \left[\frac{1 + \text{KL}(\mathbf{p}, \mathbf{q})}{1 - \text{KL}(\mathbf{q}, \mathbf{p})} \right] + 1 \right) \left(\frac{1 + \text{KL}(\mathbf{p}, \mathbf{q})}{1 - \text{KL}(\mathbf{q}, \mathbf{p})} - 1 \right) \\ &\geq \frac{1 + \text{KL}(\mathbf{p}, \mathbf{q})}{1 - \text{KL}(\mathbf{q}, \mathbf{p})} \ln \left[\frac{1 + \text{KL}(\mathbf{p}, \mathbf{q})}{1 - \text{KL}(\mathbf{q}, \mathbf{p})} \right] - \text{KL}(\mathbf{p}, \mathbf{q}) \geq 0. \end{aligned} \quad (4.25)$$

Similar results can be obtained for other divergence measures of interest such as the *Jeffreys divergence* and *Hellinger discrimination*. However, the details are left to the interested reader.

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