Research Article

The Equivalence of Convergence Results between Mann and Multistep Iterations with Errors for Uniformly Continuous Generalized Weak $\Phi$-Pseudocontractive Mappings in Banach Spaces

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We prove the equivalence of the convergence of the Mann and multistep iterations with errors for uniformly continuous generalized weak $\Phi$-pseudocontractive mappings in Banach spaces. We also obtain the convergence results of Mann and multistep iterations with errors. Our results extend and improve the corresponding results.

1. Introduction

Let $E$ be a real Banach space, $E^*$ be its dual space, and $J : E \rightarrow 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\| = \|f\|^2 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The single-valued normalized duality mapping is denoted by $j$.

Definition 1.1. A mapping $T : E \rightarrow E$ is said to be

(1) strongly accretive if for all $x, y \in E$, there exist a constant $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq k\|x - y\|^2;$$

$$\langle Tx - Ty, j(x - y) \rangle \geq k\|x - y\|^2;$$

(1.2)
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(2) \( \Phi \)-strongly accretive if there exist \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \Phi : [0,+\infty) \rightarrow [0,+\infty) \) with \( \Phi(0) = 0 \) such that

\[
\langle Tx - Ty, j(x-y) \rangle \geq \Phi(\|x-y\|) \|x-y\|, \quad \forall x,y \in E;
\]  

(3) generalized \( \Phi \)-accretive if, for any \( x,y \in E \), there exist \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \Phi : [0,+\infty) \rightarrow [0,+\infty) \) with \( \Phi(0) = 0 \) such that

\[
\langle Tx - Ty, j(x-y) \rangle \geq \Phi(\|x-y\|).
\]

**Remark 1.2.** Let \( N(T) = \{ x \in E : Tx = 0 \} \neq \emptyset \). If \( x,y \in E \) in the formulas of Definition 1.1 is replaced by \( x \in E, q \in N(T) \), then \( T \) is called strongly quasi-accretive, \( \Phi \)-strongly quasi-accretive, generalized \( \Phi \)-quasi-accretive mapping, respectively.

Closely related to the class of accretive-type mappings are those of pseudocontractive type mappings.

**Definition 1.3.** A mapping \( T \) with domain \( D(T) \) and range \( R(T) \) is said to be

(1) strongly pseudocontractive if there exist a constant \( k \in (0,1) \) and \( j(x-y) \in J(x-y) \) such that for each \( x,y \in D(T) \),

\[
\langle Tx - Ty, j(x-y) \rangle \leq k \|x-y\|^2;
\]  

(2) \( \Phi \)-strongly pseudocontractive if there exist \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \Phi : [0,+\infty) \rightarrow [0,+\infty) \) with \( \Phi(0) = 0 \) such that

\[
\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \Phi(\|x-y\|) \|x-y\|, \quad \forall x,y \in D(T);
\]  

(3) generalized \( \Phi \)-pseudocontractive if, for any \( x,y \in D(T) \), there exist \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \Phi : [0,+\infty) \rightarrow [0,+\infty) \) with \( \Phi(0) = 0 \) such that

\[
\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \Phi(\|x-y\|).
\]

**Definition 1.4.** Let \( F(T) = \{ x \in E : Tx = x \} \neq \emptyset \). The mapping \( T \) is called \( \Phi \)-strongly pseudocontractive, generalized \( \Phi \)-pseudocontractive, if, for all \( x \in D(T), q \in F(T) \), the formula (2), (3) in the above Definition 1.3 hold.

**Definition 1.5.** A mapping \( T \) is said to be

(1) generalized weak \( \Phi \)-accretive if, for all \( x,y \in E \), there exist \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \Phi : [0,+\infty) \rightarrow [0,+\infty) \) with \( \Phi(0) = 0 \) such that

\[
\langle Tx - Ty, j(x-y) \rangle \geq \frac{\Phi(\|x-y\|)}{1 + \|x-y\|^2 + \Phi(\|x-y\|)};
\]  

\[1.2\ \Phi \text{-strongly accretive if there exist } j(x-y) \in J(x-y) \text{ and a strictly increasing function } \Phi : [0,+\infty) \rightarrow [0,+\infty) \text{ with } \Phi(0) = 0 \text{ such that} \]

\[
\langle Tx - Ty, j(x-y) \rangle \geq \Phi(\|x-y\|) \|x-y\|, \quad \forall x,y \in E; \]

\(1.3\) \( \Phi \)-accretive if, for any \( x,y \in E \), there exist \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \Phi : [0,+\infty) \rightarrow [0,+\infty) \) with \( \Phi(0) = 0 \) such that

\[
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\[
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(2) \( \Phi \)-strongly pseudocontractive if there exist \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \Phi : [0,+\infty) \rightarrow [0,+\infty) \) with \( \Phi(0) = 0 \) such that

\[
\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \Phi(\|x-y\|) \|x-y\|, \quad \forall x,y \in D(T);
\]  

(3) generalized \( \Phi \)-pseudocontractive if, for any \( x,y \in D(T) \), there exist \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \Phi : [0,+\infty) \rightarrow [0,+\infty) \) with \( \Phi(0) = 0 \) such that

\[
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\]

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**Definition 1.5.** A mapping \( T \) is said to be

(1) generalized weak \( \Phi \)-accretive if, for all \( x,y \in E \), there exist \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \Phi : [0,+\infty) \rightarrow [0,+\infty) \) with \( \Phi(0) = 0 \) such that

\[
\langle Tx - Ty, j(x-y) \rangle \geq \frac{\Phi(\|x-y\|)}{1 + \|x-y\|^2 + \Phi(\|x-y\|)}; \]

\[1.6\]
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(2) generalized weak $\Phi$-quasi-accretive if, for all $x \in E, q \in N(T)$, there exist $j(x - q) \in J(x - q)$ and a strictly increasing function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that

$$
\langle Tx - q, j(x - q) \rangle \geq \frac{\Phi(\|x - q\|)}{1 + \|x - q\|^2 + \Phi(\|x - q\|)};
$$

(1.9)

(3) generalized weak $\Phi$-pseudocontractive if, for any $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and a strictly increasing function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that

$$
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \frac{\Phi(\|x - y\|)}{1 + \|x - y\|^2 + \Phi(\|x - y\|)};
$$

(1.10)

(4) generalized weak $\Phi$-hemicontractive if, for any $x \in K, q \in F(T)$, there exist $j(x - q) \in J(x - q)$ and a strictly increasing function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that

$$
\langle Tx - q, j(x - q) \rangle \leq \|x - q\|^2 - \frac{\Phi(\|x - q\|)}{1 + \|x - q\|^2 + \Phi(\|x - q\|)}.
$$

(1.11)

It is very well known that a mapping $T$ is strongly pseudocontractive (hemicontractive), $\phi$-strongly pseudocontractive ($\phi$-strongly hemicontractive), generalized $\Phi$-pseudocontractive (generalized $\Phi$-hemicontractive), generalized weak $\Phi$-pseudocontractive (generalized weak $\Phi$-hemicontractive) if and only if $(I - T)$ are strongly accretive (quasi-accretive), $\phi$-strongly accretive ($\phi$-strongly quasi-accretive), $(I - T)$ is generalized $\Phi$-accretive (generalized $\Phi$-quasi-accretive), generalized weak $\Phi$-accretive (weak $\Phi$-quasi-accretive), respectively.

It is shown in [1] that the class of strongly pseudocontractive mappings is a proper subclass of $\phi$-strongly pseudocontractive mappings. Furthermore, an example in [2] shows that the class of $\phi$-strongly hemicontractive mappings with the nonempty fixed point set is a proper subclass of generalized $\Phi$-hemicontractive mappings. Obviously, generalized $\Phi$-hemicontractive mapping must be generalized weak $\Phi$-hemicontractive, but, on the contrary, it is not true. We have the following example.

Example 1.6. Let $E = (-\infty, +\infty)$ be real number space with usual norm and $K = [0, +\infty)$. $T : K \to E$ defined by

$$
Tx = \frac{x + x^3 + x^2\sqrt{x} - \sqrt{x}}{1 + x\sqrt{x} + x^2}, \quad \forall x \in K.
$$

(1.12)
Then $T$ has a fixed point $0 \in F(T)$. $\Phi : [0, +\infty) \to [0, +\infty)$ defined by $\Phi(t) = t^{3/2}$ is a strictly increasing function with $\Phi(0) = 0$. For all $x \in K$ and $0 \in F(T)$, we have

$$
\langle Tx - T0, j(x - 0) \rangle = \left( \frac{x^3 + x^2 \sqrt{x} - \sqrt{x}}{1 + x \sqrt{x} + x^2} - 0, x - 0 \right) = \frac{x^2 + x^4 + x^3 \sqrt{x} - x \sqrt{x}}{1 + x \sqrt{x} + x^2} = x^2 - \frac{x^{3/2}}{1 + x^{3/2} + x^2}
$$

$$
= |x - 0|^2 - \frac{\Phi(x)}{1 + \Phi(x) + x^2} = |x - 0|^2 - \sigma(x)
$$

$$
\geq |x - 0|^2 - \Phi(x).
$$

(1.13)

Then $T$ is a generalized weak $\Phi$-hemicontractive map, but it is not a generalized $\Phi$-hemicontractive map; that is, the class of generalized weak $\Phi$-hemicontractive maps properly contains the class of generalized $\Phi$-hemicontractive maps. Hence the class of generalized weak $\Phi$-hemicontractive mappings is the most general among those defined above.

**Definition 1.7.** The mapping $T : E \to E$ is called Lipschitz, if there exists a constant $L > 0$ such that

$$
\|Tx - Ty\| \leq L \|x - y\|, \quad \forall x, y \in E.
$$

(1.14)

It is clear that if $T$ is Lipschitz, then it must be uniformly continuous. Otherwise, it is not true. For example, the function $f(x) = \sqrt{x}, x \in [0, +\infty)$ is uniformly continuous but it is not Lipschitz.

Now let us consider the multi-step iteration with errors. Let $K$ be a nonempty convex subset of $E$, and let $\{T_i\}_{i=1}^M$ be a finite family of self-maps of $K$. For $x_0 \in K$, the sequence $\{x_n\}$ is generated as follows:

$$
x_{n+1} = (1 - \alpha_n - \delta_n)x_n + \alpha_n T_n y_n + \delta_n v_n,
$$

$$
y_i^j = (1 - \beta_n^{ij} - \eta_n^{ij}) x_n + \beta_n^{ij} T_n y_n^{j+1} + \eta_n^{ij} w_n^{ij}, \quad i = 1, \ldots, p - 2,
$$

$$
y_n^{p-1} = (1 - \beta_n^{p-1} - \eta_n^{p-1}) x_n + \beta_n^{p-1} T_n x_n + \eta_n^{p-1} w_n^{p-1}, \quad p \geq 2,
$$

(1.15)

where $\{v_n\}, \{w_n^{ij}\}$ are any bounded sequences in $K$ and $\{\alpha_n\}, \{\delta_n\}, \{\beta_n^{ij}\}, \{\eta_n^{ij}\}, (i = 1, 2, \ldots, p - 1)$ are sequences in $[0, 1]$ satisfying certain conditions.

If $p = 2$, (1.15) becomes the Ishikawa iteration sequence with errors $\{x_n\}_{n=0}^\infty$ defined iteratively by

$$
x_{n+1} = (1 - \alpha_n - \delta_n)x_n + \alpha_n T_n y_n + \delta_n v_n,
$$

$$
y_1 = (1 - \beta_n - \eta_n)x_n + \beta_n T_n x_n + \eta_n w_n, \quad \forall n \geq 0.
$$

(1.16)
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If \( \beta_n = \gamma_n = 0 \), for all \( n \geq 0 \), then from (1.16), we get the Mann iteration sequence with errors \( \{u_n\}_{n=0}^{\infty} \) defined by

\[
u_{n+1} = (1 - \alpha_n - \delta_n)u_n + \alpha_nT_nu_n + \delta_n\mu_n, \quad \forall n \geq 0, \tag{1.17}
\]

where \( \{\mu_n\} \subset K \) is bounded.

Recently, many authors have researched the iteration approximation of fixed points by Lipschitz pseudocontractive (accretive) type nonlinear mappings and have obtained some excellent results [3–12]. In this paper we prove the equivalence between the Mann and multi-step iterations with errors for uniformly continuous generalized weak \( \Phi \)-pseudocontractive mappings in Banach spaces. Our results extend and improve the corresponding results [3–12].

**Lemma 1.8** (see [13]). Let \( E \) be a real normed space. Then, for all \( x, y \in E \), the following inequality holds:

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle, \quad \forall j(x + y) \in J(x + y). \tag{1.18}
\]

**Lemma 1.9** (see [14]). Let \( \{\rho_n\} \) be a nonnegative sequence which satisfies the following inequality:

\[
\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n, \quad n \geq N, \tag{1.19}
\]

where \( \lambda_n \in (0, 1), \lim_{n \to \infty} \lambda_n = 0 \) and \( \sum_{n=0}^{\infty} \lambda_n = \infty, \sigma_n = o(\lambda_n) \). Then \( \rho_n \to 0 \) as \( n \to \infty \).

**Lemma 1.10.** Let \( \{\theta_n\}, \{c_n\}, \{e_n\} \) and \( \{t_n\} \) be four nonnegative real sequences satisfying the following conditions: (i) \( \lim_{n \to \infty} t_n = 0 \); (ii) \( \sum_{n=0}^{\infty} t_n = \infty \); (iii) \( c_n = o(t_n), e_n = o(t_n) \). Let \( \Phi : [0, +\infty) \to [0, +\infty) \) be a strictly increasing and continuous function with \( \Phi(0) = 0 \) such that

\[
\theta_{n+1}^2 \leq (1 + c_n)\theta_n^2 - t_n\frac{\Phi(\theta_{n+1})}{1 + \Phi(\theta_{n+1})} + \theta_{n+1}^2 + e_n, \quad n \geq 0. \tag{1.20}
\]

If \( \{\theta_n\} \) is bounded, then \( \theta_n \to 0 \) as \( n \to \infty \).

**Proof.** Since \( \lim_{n \to \infty} t_n = 0 \), \( \{\theta_n\} \) is bounded, we set \( R = \max\{\sup_{n \geq 0} t_n, \sup_{n \geq 0} \theta_n\} \), 
\( y = \lim \inf_{n \to \infty} (\Phi(\theta_{n+1}) / (1 + \theta_{n+1}^2)(1 + \Phi(R) + R^2)) \), then \( y = 0 \). Otherwise, we assume that \( y > 0 \), then there exists a constant \( \delta > 0 \) with \( \delta = \min\{1, y\} \) and a natural number \( N_1 \) such that

\[
\Phi(\theta_{n+1}) > \left( \delta + \delta \theta_{n+1}^2 \right) \left[ 1 + \Phi(R) + R^2 \right] > \delta \theta_{n+1}^2 \left[ 1 + \Phi(R) + R^2 \right], \tag{1.21}
\]

for \( n > N_1 \).

Then, from (1.20), we get

\[
\theta_{n+1}^2 \leq \frac{1 + c_n}{1 + \delta t_n} \theta_n^2 + e_n. \tag{1.22}
\]
Since $c_n = o(t_n)$, there exists a nature number $N_2 > N_1$, such that $c_n < (\delta/2)t_n$, $n > N_2$. Hence $(1 + c_n)/(1 + \delta t_n) < 1 - (\delta/2)t_n$ and (1.22) becomes

\[
\theta_{n+1}^2 \leq \left( 1 - \frac{\delta}{2}t_n \right) \theta_n^2 + e_n. \tag{1.23}
\]

By Lemma 1.9, we obtain that $\theta_n \to 0$ as $n \to \infty$. Since $\Phi$ is strictly increasing and continuous with $\Phi(0) = 0$. Hence $\gamma = 0$, which is contradicting with the assumption $\gamma > 0$. Then $\gamma = 0$, there exists a subsequence $\{\theta_{n_j}\}$ of $\{\theta_n\}$ such that $\theta_{n_j} \to 0$ as $j \to \infty$. Let $0 < \varepsilon < 1$ be any given. Since $c_n = o(t_n)$, $e_n = o(t_n)$, then there exists a natural number $N_3 > N_2$, such that

\[
\theta_{n_j} < \varepsilon, \quad c_{n_j} < \frac{\Phi(\varepsilon)}{2M^2(1 + R^2 + \Phi(R))} t_{n_j}, \quad e_{n_j} < \frac{\Phi(\varepsilon)}{2(1 + R^2 + \Phi(R))} t_{n_j} \tag{1.24}
\]

for all $j > N_3$. Next, we will show that $\theta_{n_j + m} < \varepsilon$ for all $m = 1, 2, 3, \ldots$. First, we want to prove that $\theta_{n+1} \to \varepsilon$. Suppose that it is not the case, then $\theta_{n+1} \geq \varepsilon$. Since $\Phi$ is strictly increasing,

\[
\Phi(\theta_{n+1}) \geq \Phi(\varepsilon). \tag{1.25}
\]

From (1.24) and (1.25), we obtain that

\[
\theta_{n+1}^2 \leq \left( 1 + c_{n_j} \right) \theta_{n_j}^2 - 2t_{n_j} \frac{\Phi(\varepsilon)}{1 + R^2 + \Phi(R)} + e_{n_j} \leq \theta_{n_j}^2 - \frac{\Phi(\varepsilon)}{(1 + R^2 + \Phi(R))} t_{n_j} < \theta_{n_j}^2 < \varepsilon^2. \tag{1.26}
\]

That is $\theta_{n+1} < \varepsilon$, which is a contradiction. Hence $\theta_{n+1} < \varepsilon$. Now we assume that $\theta_{n+1} < \varepsilon$ holds. Using the similar way, it follows that $\theta_{n+m+1} < \varepsilon$. Therefore, this shows that $\theta_n \to 0$ as $n \to \infty$. \hfill \Box

### 2. Main Results

**Theorem 2.1.** Let $K$ be a nonempty closed convex subset of a Banach space $E$. Suppose that $T_n = T_{n(\text{mod } M)}$, and $T_i : K \to K$, $i \in I = \{1, 2, \ldots, M\}$ are $M$ uniformly continuous generalized weak $\Phi$-hemicontractive mappings with $F = \bigcap_{i=1}^M F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a sequence in $K$ defined iteratively from some $u_0 \in K$ by (1.17), where $\{\alpha_n\}$ is an arbitrary bounded sequence in $K$ and $\{\alpha_n\}$, $\{\delta_n\}$ are two sequences in $[0, 1]$ satisfying the following conditions: (i) $\alpha_n + \delta_n \leq 1$, (ii) $\sum_{n=1}^\infty \alpha_n = \infty$, (iii) $\lim_{n \to \infty} \alpha_n = 0$, (iv) $\delta_n = o(\alpha_n)$. Then the iteration sequence $\{u_n\}$ converges strongly to the unique fixed point of $T$. 
Proof. Since $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, set $q \in F$. Since the mapping $T_n$ are generalized weak $\Phi$-hemicontractive mappings, there exist strictly increasing functions $\Phi_i : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ and $f(x - y) \in f(x - y)$ such that

$$
\langle T_i x - T_i y, (x - y) \rangle \leq \|x - y\|^2 - \frac{\Phi_i(\|x - y\|)}{1 + \|x - y\|^2 + \Phi_i(\|x - y\|)}, \quad \forall x, y \in K, \ i \in I. \quad (2.1)
$$

Firstly, we claim that there exists $u_0 \in K$ with $u_0 \neq Tu_0$ such that $t_0 = \|u_0 - Tu_0\| \cdot \|u_0 - q\| \cdot [1 + \|u_0 - q\|^2 + \Phi_i(\|u_0 - q\|)] \in R(\Phi_1)$. In fact, if $u_0 = Tu_0$, then we have done. Otherwise, there exists the smallest positive integer $n_0 \in N$ such that $u_{n_0} \neq Tu_{n_0}$. We denote $u_{n_0} = u_0$, then we will obtain that $t_0 \in R(\Phi_1)$. Indeed, if $R(\Phi_1) = [0, +\infty)$, then $t_0 \in R(\Phi)$. If $R(\Phi_1) = [0, A]$ with $0 < A < +\infty$, then for $q \in K$, there exists a sequence $\{w_n\} \subseteq K$ such that $w_n \to q$ as $n \to \infty$ with $w_n \neq q$, and we also obtain that the sequence $\{w_n - Tw_n\}$ is bounded. So there exists $n_0 \in N$ such that $\|w_n - Tw_n\| \cdot \|w_n - q\| \cdot [1 + \|w_n - q\|^2 + \Phi_i(\|w_n - q\|)] \in R(\Phi_1)$ for $n \geq n_0$, then we redefine $u_0 = w_{n_0}$, let $\omega_0 = \Phi^{-1}_i(t_0) > 0$.

Next we shall prove $\|u_n - q\| \leq \omega_0$ for $n \geq 0$. Clearly, $\|u_0 - q\| \leq \omega_0$ holds. Suppose that $\|u_n - q\| \leq \omega_0$, for some $n$, then we want to prove $\|u_{n+1} - q\| \leq \omega_0$. If it is not the case, then $\|u_{n+1} - q\| > \omega_0$. Since $T$ is a uniformly continuous mapping, setting $\varepsilon_0 = \Phi_i(\omega_0)/12\omega_0 [1 + \Phi_i((3/2)\omega_0) + ((3/2)\omega_0)^2]$, there exists $\delta_0 > 0$ such that $\|T_n x - T_n y\| < \varepsilon_0$, whenever $\|x - y\| < \delta$; and $T_n$ are bounded operators, set $M = \sup \{\|T_n x\| : \|x - q\| \leq \omega_0\} + \sup_n \|w_n\|$. Since $\lim_{n \to \infty} \alpha_n = 0$, $\delta_n = o(\alpha_n)$, without loss of generality, let

$$
\frac{\alpha_n}{\delta_n} < \min \left\{ \frac{1}{4}, \frac{\omega_0}{4\delta \Phi_i(\omega_0)}, \frac{\Phi_i(\omega_0)}{4\delta \Phi_i((3/2)\omega_0) + ((3/2)\omega_0)^2} \right\}, \quad n \geq 0. \quad (2.2)
$$

From (1.17), we have

$$
\|u_{n+1} - q\| = \|(1 - \alpha_n - \delta_n)(u_n - q) + \alpha_n (T_n u_n - q) + \delta_n (\omega_n - q)\|
$$

$$
\leq \|u_n - q\| + \alpha_n \|T_n u_n - q\| + \delta \|\omega_n - q\|
$$

$$
\leq \omega_0 + \alpha_n \|T_n u_n - q\| + \delta_n \|\omega_n - q\|
$$

$$
\leq \omega_0 + M(\alpha_n + \delta_n) \leq \omega_0 + 2M \alpha_n \leq \frac{3}{2} \omega_0,
$$

$$
\|u_{n+1} - u_n\| = \|\alpha_n T_n u_n + \delta_n \omega_n - (\alpha_n + \delta_n) u_n\|
$$

$$
\leq \alpha_n \|T_n u_n - q\| + \delta_n \|\omega_n - q\| + (\alpha_n + \delta_n) \|u_n - q\|
$$

$$
\leq (\alpha_n + \delta_n) (M + \omega_0) < \delta.
$$

Since $T_n$ are uniformly continuous mappings, so $\|T_n u_{n+1} - T_n u_n\| < \varepsilon_0.$
Applying Lemma 1.8, the recursion (1.17), and the above inequalities, we obtain

\[
\| u_{n+1} - q \|^2 = \| (1 - \alpha_n - \delta_n)(u_n - q) + \alpha_n (T_n u_n - q) + \delta_n (\omega_n - q) \|^2 \\
\leq (1 - \alpha_n - \delta_n)^2 \| u_n - q \|^2 + 2\alpha_n \langle T_n u_n - q, j(u_{n+1} - q) \rangle \\
+ 2\delta_n \| \omega_n - q \| \cdot \| u_{n+1} - q \| \\
\leq (1 - \alpha_n)^2 \| u_n - q \|^2 + 2\alpha_n \| T_n u_n - T_n u_{n+1} \| \cdot \| u_{n+1} - q \| + 2\alpha_n \| \omega_n - q \| \cdot \| u_{n+1} - q \| \\
\leq (1 - \alpha_n)^2 \| u_n - q \|^2 + 2\alpha_n \left[ \| u_{n+1} - q \|^2 - \frac{\Phi_1(\| u_{n+1} - q \|)}{1 + \Phi_1(\| u_{n+1} - q \|)} + \| u_{n+1} - q \|^2 \right] \\
+ 2\alpha_n \| T_n u_n - T_n u_{n+1} \| \cdot \| u_{n+1} - q \| + 2\alpha_n \| \omega_n - q \| \cdot \| u_{n+1} - q \|.
\]

(2.5)

Inequality (2.5) implies

\[
\| u_{n+1} - q \|^2 \leq \| u_n - q \|^2 - 2\alpha_n \frac{\Phi_1(\| u_{n+1} - q \|)}{1 + \Phi_1(\| u_{n+1} - q \|)} + \| u_{n+1} - q \|^2 + \frac{\alpha_n^2}{1 - 2\alpha_n} \| u_n - q \|^2 \\
+ \frac{2\alpha_n}{1 - 2\alpha_n} \| T_n u_n - T_n u_{n+1} \| \cdot \| u_{n+1} - q \| + \frac{2\alpha_n}{1 - 2\alpha_n} \| \omega_n - q \| \cdot \| u_{n+1} - q \| \\
\leq \omega_0^2 - 2\alpha_n \frac{\Phi_1(\omega_0)}{1 + \Phi_1((3/2)\omega_0) + ((3/2)\omega_0)^2} \\
+ \frac{2\alpha_n}{4} \frac{\Phi_1(\omega_0)}{1 + \Phi_1((3/2)\omega_0) + ((3/2)\omega_0)^2} \omega_0^2 \\
+ \frac{4\alpha_n}{12} \frac{\Phi_1(\omega_0)}{1 + \Phi_1((3/2)\omega_0) + ((3/2)\omega_0)^2} \cdot \frac{3\omega_0}{2} \\
+ \frac{4\alpha_n}{12} \frac{\Phi_1(\omega_0)}{1 + \Phi_1((3/2)\omega_0) + ((3/2)\omega_0)^2} \cdot \frac{3M\omega_0}{2} < \omega_0^2,
\]

(2.6)

which is a contradiction with the assumption \( \| u_{n+1} - q \| > \omega_0 \). Then \( \| u_{n+1} - q \| \leq \omega_0 \); that is, the sequence \( \{ u_n \} \) is bounded. Let \( N = \sup_n \| u_n - q \| \). From (2.4), we have

\[
\| u_{n+1} - u_n \| \leq (\alpha_n + \delta_n)(M + \omega_0) \to 0, \quad n \to \infty,
\]

(2.7)

that is, \( \lim_{n \to \infty} \| u_{n+1} - u_n \| = 0 \). Since \( T \) is on uniformly continuous, so

\[
\lim_{n \to \infty} \| T_n u_{n+1} - T_n u_n \| = 0.
\]

(2.8)
Let $E$ be a Banach space and $K$ be a nonempty closed convex subset of $E$, $T_n$ are as in Theorem 2.1. For $x_0, u_0 \in K$, the sequence iterations $\{x_n\}, \{u_n\}$ are defined by (1.15) and (1.17), respectively. $\{\alpha_n\}, \{\delta_n\}, \{\beta^i_n\}, \{\eta^i_n\}, i = 1, 2, \ldots, p - 1$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $0 \leq \alpha_n + \delta_n \leq 1, 0 \leq \beta^i_n + \eta^i_n \leq 1, 1 \leq i \leq p - 1$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\lim_{n \to \infty} \alpha_n = 0$;
(iv) $\lim_{n \to \infty} \beta^i_n = \lim_{n \to \infty} \eta^i_n = 0, i = 1, \ldots, p - 1$;
(v) $\delta_n = o(\alpha_n)$.

Then the following two assertions are equivalent:

(I) the iteration sequence $\{x_n\}$ strongly converges to the common point of $F(T_i), i \in I$;

(II) the sequence iteration $\{u_n\}$ strongly converges to the common point of $F(T_i), i \in I$.

Proof. Since $F = \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$, set $q \in F$. If the iteration sequence $\{x_n\}$ strongly converges to $q$, then setting $p = 2, \beta_n = \delta_n = 0$, we obtain the convergence of the iteration sequence $\{u_n\}$. Conversely, we only prove that (II)$\Rightarrow$(I). The proof is divided into two parts.

Step I. We show that $\{x_n - u_n\}$ is bounded.

By the proof method of Theorem 2.1, there exists $x_0 \in K$ with $x_0 \neq T_1 x_0$ such that $r_0 = \|x_0 - T_1 x_0\| \cdot \|x_0 - q\| \cdot [1 + \|x_0 - q\|^2 + \Phi_1(\|x_0 - q\|)] \in R(\Phi)$. Setting $a_0 = \Phi_1^{-1}(r_0)$, we have $\|x_0 - q\| \leq a_0$. Set $B_1 = \{\|x - q\| \leq a_0 : x \in K\}, B_2 = \{\|x - q\| \leq 2a_0 : x \in K\}$. Since $T_i$ are bounded mappings and $\{\omega^i_n\} (i = 1, \ldots, p - 1), \{\nu_n\}$ are some bounded sequences in $K$, we can set $M = \max \{\sup_{x \in B_1} \|T_n x - q\|; \sup_{n \in N} \|\omega^i_n - q\|; \sup_{n \in N} \|\nu_n - q\|\}$. Since $T_i$ are uniformly continuous mappings, given $e_0 = \Phi_1(a_0)/4a_0[1 + (5a_0/4)^2 + \Phi_1(5a_0/4)], \exists \delta > 0$, such that $\|T x - T y\| < e_0$ whenever $\|x - y\| < \delta$, for all $x, y \in B_2$. Now, we define $\tau_0 = \min\{1/2, a_0/8M, a_0/8(M + a_0), \delta/8(M + a_0), \Phi_1(a_0)/5a_0^2[1 + (5a_0/4)^2 + \Phi_1(5a_0/4)], \Phi_1(a_0)/5a_0M[1 + (5a_0/4)^2 + \Phi_1(5a_0/4)]\}$. Since the control conditions (iii)-(iv), without loss of generality, we let $0 < \alpha_n, \delta_n/\alpha_n, \beta^i_n, \eta^i_n < \tau_0$, $n \geq 0$.

Now we claim that if $x_n \in B_1$, then $y^i_n \in B_2, 1 \leq i \leq p - 1$. 

\begin{equation}
\|u_{n+1} - q\|^2 \leq \|u_n - q\|^2 - 2\alpha_n \frac{\Phi_1(\|u_{n+1} - q\|)}{1 + \Phi_1(\|u_{n+1} - q\|) + \|u_{n+1} - q\|^2} + A_n,
\end{equation}

where

\[ A_n = \alpha^2_n N^2 + 2\alpha_n N\|T_n u_n - T_n u_{n+1}\| + 2\delta_n M N. \]

By (2.8), the conditions (iii) and (iv), we get $A_n = o(\alpha_n)$. So applying Lemma 1.10 on (2.9), we obtain $\lim_{n \to \infty} \|u_n - q\| = 0$. □
From (1.15), we obtain that
\[
\left\| y_n^{p-1} - q \right\| \leq \left(1 - \beta_n^{p-1} - \eta_n^{p-1}\right) \left\| x_n - q \right\| + \beta_n^{p-1} \left\| T_n x_n - q \right\| + \eta_n^{p-1} \left\| \omega_n^{p-1} - q \right\|
\]
\[
\leq \left\| x_n - q \right\| + \left(\beta_n^{p-1} + \eta_n^{p-1}\right) M
\]
\[
\leq \left\| x_n - q \right\| + 2\tau_0 M \leq 2a_0,
\]
\[
\left\| y_n^{p-2} - q \right\| \leq \left(1 - \beta_n^{p-2} - \eta_n^{p-2}\right) \left\| x_n - q \right\| + \beta_n^{p-2} \left\| T_n y_n^{p-1} - q \right\| + \eta_n^{p-2} \left\| \omega_n^{p-2} - q \right\|
\]
\[
\leq \left\| x_n - q \right\| + \left(\beta_n^{p-2} + \eta_n^{p-2}\right) M
\]
\[
\leq \left\| x_n - q \right\| + 2\tau_0 M \leq 2a_0,
\]
(2.11)
we also obtain that
\[
\left\| y_n^1 - q \right\| \leq 2a_0.
\]
(2.12)

Now we suppose that \( \left\| x_n - q \right\| \leq a_0 \) holds. We will prove that \( \left\| x_{n+1} - q \right\| \leq a_0 \). If it is not the case, we assume that \( \left\| x_{n+1} - q \right\| > a_0 \). From (1.15), we obtain that
\[
\left\| x_{n+1} - q \right\| = \left\| \left(1 - \alpha_n - \delta_n\right) (x_n - q) + \alpha_n \left( T_n y_n^1 - q \right) + \delta_n (v_n - q) \right\|
\]
\[
\leq \left\| x_n - q \right\| + \alpha_n \left\| T_n y_n^1 - q \right\| + \delta_n \left\| v_n - q \right\|
\]
\[
\leq \left\| x_n - q \right\| + \left(\alpha_n + \delta_n\right) M
\]
\[
\leq \left\| x_n - q \right\| + 2\tau_0 M \leq \left\| x_n - q \right\| + \frac{1}{4}a_0 \leq \frac{5}{4}a_0.
\]
(2.13)

Consequently, by (2.11) and (2.12), we obtain
\[
\left\| x_{n+1} - y_n^1 \right\| = \left\| \left(\beta_n^1 - \alpha_n\right) \left( x_n - q \right) + \alpha_n \left( T_n y_n^1 - q \right) - \beta_n^1 \left( T_n^a y_n^2 - q \right) + \delta_n (v_n - q) - \eta_n^1 \left( \omega_n^1 - q \right) \right\|
\]
\[
\leq \left(\beta_n^1 + \alpha_n + \eta_n^1 + \delta_n\right) \left\| x_n - q \right\| + \alpha_n \left\| T_n y_n^1 - q \right\| + \beta_n^1 \left\| T_n y_n^2 - q \right\| + \delta_n \left\| v_n - q \right\| + \eta_n^1 \left\| \omega_n^1 - q \right\|
\]
\[
\leq 4\tau_0 (\mu_0 + M) \leq \delta.
\]
(2.14)

Since \( T_n \) are uniformly continuous mappings, we get
\[
\left\| T_n x_{n+1} - T_n y_n^1 \right\| \leq \epsilon_0.
\]
(2.15)
Using (2.1), Lemma 1.8, and the recursion formula (1.15), we have

\[
\|x_{n+1} - q\|^2 = \left\| (1 - \alpha_n - \delta_n) (x_n - q) + \alpha_n (T_n y_n^1 - q) + \delta_n (v_n - q) \right\|^2 \\
\leq \left(1 - \alpha_n\right)^2 \|x_n - q\|^2 + 2\alpha_n \left( T_n y_n^1 - q, j(x_{n+1} - q) \right) + 2\delta_n \|v_n - q\| \cdot \|x_{n+1} - q\| \\
\leq \left(1 - \alpha_n\right)^2 \|x_n - q\|^2 + 2\alpha_n \left( T_n x_n, j(x_{n+1} - q) \right) + 2\delta_n \|v_n - q\| \cdot \|x_{n+1} - q\| \\
+ 2\alpha_n \left( T_n y_n^1 - T_n x_n, j(x_{n+1} - q) \right) + 2\delta_n \|v_n - q\| \cdot \|x_{n+1} - q\| \\
\leq \left(1 - \alpha_n\right)^2 \|x_n - q\|^2 + 2\alpha_n \left[ \|x_{n+1} - q\|^2 - \frac{\Phi_1(\|x_{n+1} - q\|)}{1 + \Phi_1(\|x_{n+1} - q\|) + \|x_{n+1} - q\|^2} \right] \\
+ 2\alpha_n \left\| T_n y_n^1 - T_n x_n \right\| : \|x_{n+1} - q\| + 2\delta_n M : \|x_{n+1} - q\|. 
\]

(2.16)

Which implies

\[
\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \frac{2\alpha_n}{1 - 2\alpha_n} \frac{\Phi_1(a_0)}{1 + \Phi_1(\|x_{n+1} - q\|) + \|x_{n+1} - q\|^2} + \frac{\alpha_n^2}{1 - 2\alpha_n} \|x_n - q\|^2 \\
+ \frac{2\alpha_n}{1 - 2\alpha_n} \left\| T_n y_n^1 - T_n x_n \right\| : \|x_{n+1} - q\| + \frac{2\delta_n}{1 - 2\alpha_n} M : \|x_{n+1} - q\| \\
\leq a_0^2 \frac{2\alpha_n}{1 - 2\alpha_n} \frac{\Phi_1(a_0)}{1 + (5a_0/4)^2 + \Phi_1(5a_0/4)} \\
+ \frac{2\alpha_n}{1 - 2\alpha_n} \frac{\Phi_1(a_0)}{1 + (5a_0/4)^2 + \Phi_1(5a_0/4)} \cdot \frac{5a_0}{4} \\
+ \frac{2\alpha_n}{1 - 2\alpha_n} \frac{\Phi_1(a_0)}{1 + (5a_0/4)^2 + \Phi_1(5a_0/4)} \cdot \frac{5a_0 M}{4} < a_0^2 
\]

(2.17)

which is a contradiction with the assumption \(\|x_{n+1} - q\| > \mu_0, \) then \(\|x_{n+1} - q\| \leq \mu_0; \) that is, the sequence \(\{x_n - q\} \) is bounded. Since \(u_n \rightarrow q; \) as \(n \rightarrow \infty, \) so the sequence \(\{x_n - u_n\} \) is bounded.

**Step 2.** We prove \(\lim_{n \rightarrow \infty} \|x_n - q\| = 0.\)

Since \(\{x_n - u_n\} \) is bounded, again applying (2.11) and (2.12), we get the boundedness of \(\{y_n^i - u_n\}, i = 1, 2, \ldots, p - 1. \) Since \(T_n = T_n(\mod M) \) are bounded mappings, set \(L = \)
\[
\begin{align*}
\max \{ & \sup_{n \geq 0} \| x_n - u_n \|, \sup_{n \geq 0} \| T_n x_n - u_n \|, \sup_{n \geq 0} \| T_n y_n^1 - u_n \|, \sup_{n \geq 0} \| \mu_n - u_n \|, \sup_{n \geq 0} \| \nu_n - u_n \|, \sup_{n \geq 0} \| \omega_n^1 - u_n \| \}, \quad (i = 1, 2, \ldots, p - 1). \end{align*}
\]

From (1.15) and (1.17), we obtain

\[
\begin{align*}
\| x_{n+1} - u_{n+1} \|^2 &= \left\| (1 - \alpha_n - \delta_n) (x_n - u_n) + \alpha_n \left( T_n y_n^1 - T_n u_n \right) + \delta_n (\nu_n - \mu_n) \right\|^2 \\
&\leq (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2 \alpha_n \left( T_n y_n^1 - T_n u_n, j (x_{n+1} - u_{n+1}) \right) \\
&\quad + 2 \delta_n \| \nu_n - \mu_n \| \cdot \| x_{n+1} - u_{n+1} \| \\
&\leq (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2 \alpha_n \left( T_n x_{n+1} - T_n u_{n+1}, j (x_{n+1} - u_{n+1}) \right) \\
&\quad + 2 \alpha_n \left( T_n y_n^1 - T_n x_{n+1} + T_n u_{n+1} - T_n u_n, j (x_{n+1} - u_{n+1}) \right) \\
&\quad + 2 \delta_n \| \nu_n - \mu_n \| \cdot \| x_{n+1} - u_{n+1} \| \\
&\leq \left( 1 + \alpha_n^2 \right) \| x_n - q \|^2 - 2 \alpha_n \frac{\Phi(t(\| x_{n+1} - u_{n+1} \|))}{1 + \Phi(t(\| x_{n+1} - u_{n+1} \|) + \| x_{n+1} - u_{n+1} \|^2)} \\
&\quad + 2 \alpha_n \left| T_n y_n^1 - T_n x_{n+1} \right| \cdot \| x_{n+1} - u_{n+1} \| + 2 \alpha_n \left| T_n u_{n+1} - T_n u_n \right| \cdot \| x_{n+1} - u_{n+1} \| \\
&\quad + 2 \delta_n M \cdot \| x_{n+1} - u_{n+1} \| \\
&\quad (2.18)
\end{align*}
\]

\[
\begin{align*}
\| x_{n+1} - y_n^1 \| &\leq \left( \beta_n^1 + \alpha_n + \eta_n^1 + \delta_n \right) \| x_n - u_n \| + \alpha_n \left| T_n y_n^1 - u_n \right| \\
&\quad + \beta_n^1 \left| T_n y_n^2 - u_n \right| + \delta_n \| \nu_n - u_n \| + \eta_n^1 \left| \omega_n^1 - u_n \right| \\
&\leq 2 \left( \beta_n^1 + \alpha_n + \eta_n^1 + \delta_n \right) L \cdot \| x_{n+1} - u_{n+1} \| \quad (2.19)
\end{align*}
\]

By the conditions (iii)-(v), we have

\[
\lim_{n \to \infty} \| x_{n+1} - y_n^1 \| = 0. \quad (2.20)
\]

Since \( \lim_{n \to \infty} \| u_n - q \| = 0 \), so

\[
\| u_{n+1} - u_n \| \leq \| u_n - q \| + \| u_n - q \|. \quad (2.21)
\]

That is:

\[
\lim_{n \to \infty} \| u_{n+1} - u_n \| = 0. \quad (2.22)
\]
By the uniform continuity of $T$, we obtain

$$
\lim_{n \to \infty} \|T_n x_{n+1} - T_n y_n^1\| = 0, \quad \lim_{n \to \infty} \|T_n u_{n+1} - T_n u_n\| = 0. \tag{2.23}
$$

From (2.23) and the conditions (iii) and (v), (2.18) becomes

$$
\|x_{n+1} - u_{n+1}\|^2 \leq \left(1 + \alpha_n^2\right)\|x_n - u_n\|^2 - 2\alpha_n \frac{\Phi_i(\|x_n - u_{n+1}\|)}{1 + \Phi_i(\|x_{n+1} - u_{n+1}\|) + \|x_{n+1} - u_{n+1}\|^2} + o(\alpha_n).
$$

By Lemma 1.10, we get $\lim_{n \to \infty} \|x_n - u_n\| = 0$. Since $\lim_{n \to \infty} \|u_n - q\| = 0$, and the inequality $0 \leq \|x_n - q\| \leq \|x_n - u_n\| + \|u_n - q\|$, so $\lim_{n \to \infty} \|x_n - q\| = 0$. \hfill \Box

From Theorems 2.1 and 2.2, we can obtain the following corollary.

**Corollary 2.3.** Let $E$ be a Banach space and $K$ be a nonempty closed convex subset of $E$, $T_n$ are as in Theorem 2.1. For $x_0 \in K$, the sequence iterations $\{x_n\}$ is defined by (1.15), $\{\alpha_n\}$, $\{\delta_n\}$, $\{\beta_n^i\}$, $\{\eta_n^i\}$, $(i = 1, 2, \ldots, p-1)$ are sequences in $[0, 1]$ satisfying the following conditions:

(i) $0 \leq \alpha_n + \delta_n \leq 1$, $0 \leq \beta_n^i + \eta_n^i \leq 1$, $1 \leq i \leq p - 1$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\lim_{n \to \infty} \alpha_n = 0$;
(iv) $\lim_{n \to \infty} \beta_n^i = \lim_{n \to \infty} \eta_n^i = 0$, $i = 1, \ldots, p - 1$;
(v) $\delta_n = o(\alpha_n)$.

Then the iteration sequence $\{x_n\}$ strongly converges to the common point of $F(T_i)$, $i \in I$.

**Corollary 2.4.** Let $T_n = S_n (\mathrm{mod} \ M)$, $T_l : E \to E$, $l \in I = \{1, 2, \ldots, M\}$ are $M$ uniformly continuous generalized weak $\Phi$-quasi-accretive mappings. Suppose $N(F) = \cap_{i=1}^{M} N(T_i) \neq \emptyset$, that is, there exists $x^* \in N(F)$. Let $\{\alpha_n\}$, $\{\delta_n\}$, $\{\beta_n^i\}$, $\{\eta_n^i\}$, $(i = 1, 2, \ldots, p - 1)$ be sequences in $[0, 1]$ satisfying the following conditions:

(i) $0 \leq \alpha_n + \delta_n \leq 1$, $0 \leq \beta_n^i + \eta_n^i \leq 1$, $1 \leq i \leq p - 1$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\lim_{n \to \infty} \alpha_n = 0$;
(iv) $\lim_{n \to \infty} \beta_n^i = \lim_{n \to \infty} \eta_n^i = 0$, $i = 1, \ldots, p - 1$;
(v) $\delta_n = o(\alpha_n)$.

Let the sequence $\{x_n\}$ in $E$ be generated iteratively from some $x_0 \in E$ by

$$
x_{n+1} = (1 - \alpha_n - \delta_n)x_n + \alpha_n y_n^1 + \delta_n y_n,
$$

$$
y_n^i = \left(1 - \beta_n^i - \eta_n^i\right)x_n + \beta_n^i S_n y_n^{i+1} + \eta_n^i \omega_n^i, \quad i = 1, \ldots, p - 2,
$$

$$
y_n^{p-1} = \left(1 - \beta_n^{p-1} - \eta_n^{p-1}\right)x_n + \beta_n^{p-1} S_n x_n + \eta_n^{p-1} \omega_n^{p-1}, \quad p \geq 2,
$$

where $S_l x := x - T_l x$ for all $x \in E$ and $\{\nu_n\}$, $\{\omega_n^i\}$ are any bounded sequences in $K$. 

Then \( \{x_n\} \) defined by (2.25) converges strongly to \( x^* \).

**Proof.** We simply observe that \( S_l := I - T_l \), \( l \in I \) are \( M \)-uniformly continuous generalized weak \( \Phi \)-hemicontractive mappings. The result follows from Corollary 2.3. \( \square \)

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**References**


