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*Research Article*

# Almost Periodic (Type) Solutions to Parabolic Cauchy Inverse Problems

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We first show the existence and uniqueness of (pseudo) almost periodic solutions of some types of parabolic equations. Then, we apply the results to a type of Cauchy parabolic inverse problems and show the existence, uniqueness, and stability.

## 1. Introduction

Zhang in [1, 2] defined pseudo almost periodic functions. As almost periodic functions, pseudo almost periodic functions are applied to many mathematical areas, particular to the theory of ordinary differential equations. (e.g., see [3–26] and references therein). However, the study of the related topic on partial differential equations has only a few important developments. On the other hand, almost periodic functions to various problems have been investigated (e.g., see [27–32] and references therein), but little has been done about the inverse problems except for our work in [33–36]. In [36], we study pseudo almost periodic solutions to parabolic boundary value inverse problems. In this paper, we devote such solutions to cauchy problems.

To this end, we need first to define the spaces in a more general setting. Let $J \in \{\mathbf{R}, \mathbf{R}^n\}$. Let $\mathcal{C}(J)$ (resp., $\mathcal{C}(J \times \Omega)$, where $\Omega \subset \mathbf{R}^m$) denote the $C^*$-algebra of bounded continuous complex-valued functions on $J$ (resp. $J \times \Omega$) with the supremum norm. For $f \in \mathcal{C}(J)$ (resp., $\mathcal{C}(J \times \Omega)$) and $s \in J$, the translation of $f$ by $s$ is the function $R_s f(t) = f(t+s)$ (resp., $R_s f(t, Z) = f(t+s, Z)$, $(t, Z) \in J \times \Omega$).

*Definition 1.1.* (1) A function $f \in \mathcal{C}(J)$ is called almost periodic if for every $\epsilon > 0$ the set

$$T(f, \epsilon) = \{\tau \in J : \|R_\tau f - f\| < \epsilon\} \tag{1.1}$$

is relatively dense in $J$. Denote by $\mathcal{AP}(J)$ the set of all such functions. The number (vector) $\tau$ is called $\epsilon$-translation number (vector) of $f$.

(2) A function $f \in \mathcal{C}(J \times \Omega)$ is said to be almost periodic in $t \in J$ and uniform on compact subsets of $\Omega$ if $f(\cdot, Z) \in \mathcal{AP}(J)$ for each $Z \in \Omega$ and is uniformly continuous on $J \times K$ for any compact subset $K \subset \Omega$. Denote by $\mathcal{AP}(J \times \Omega)$ the set of all such functions. For convenience, such functions are also called uniformly almost periodic.

(3) A function $f \in \mathcal{C}(J)(\mathcal{C}(J \times \Omega))$ is called pseudo almost periodic if

$$f = g + \varphi, \tag{1.2}$$

where $g \in \mathcal{AP}(J)(\mathcal{AP}(J \times \Omega))$ and $\varphi \in \mathcal{PAP}_0(J)(\mathcal{PAP}_0(J \times \Omega))$,

$$\begin{aligned} \mathcal{PAP}_0(J) &= \left\{ \varphi \in \mathcal{C}(J) : \lim_{r \to \infty} \frac{1}{(2r)^n} \int_{[-r,r]^n} |\varphi(x)| dx = 0 \right\}, \\ \mathcal{PAP}_0(J \times \Omega) &= \left\{ \varphi \in \mathcal{C}(J \times \Omega) : \lim_{r \to \infty} \frac{1}{(2r)^n} \int_{[-r,r]^n} |\varphi(x, Z)| dx = 0 \right\}, \end{aligned} \tag{1.3}$$

uniformly with respect to $Z \in K$, where $K$ is any compact subset of $\Omega$. Denote by $\mathcal{PAP}(J)(\mathcal{PAP}(J \times \Omega))$ the set of all such functions.

Set

$$\begin{aligned} \mathcal{APT}(J) &\in \{\mathcal{AP}(J),\ \mathcal{PAP}(J)\}, \\ \mathcal{APT}(J \times \Omega) &\in \{\mathcal{AP}(J \times \Omega),\ \mathcal{PAP}(J \times \Omega)\}. \end{aligned} \tag{1.4}$$

Members of $\mathcal{APT}(J)(\mathcal{APT}(J \times \Omega))$ are called almost periodic type.

We will use the notations throughout the paper: $\mathbf{R}_T^m = \mathbf{R}^m \times (0, T)$, $\|F\|_T = \sup\{|F(x,t)| : x \in \mathbf{R}^n,\ 0 \le t \le T\}$. $F \in \mathcal{APT}(\mathbf{R}^n \times \mathbf{R}_T^m)$ means that $F(x^{(1)}, x^{(2)}, t)$ is almost periodic type in $x^{(1)} \in \mathbf{R}^n$ and uniformly for $(x^{(2)}, t) \in \mathbf{R}_T^m$; $F \in \mathcal{APT}(\mathbf{R}^n \times \mathbf{R}^m)$ means that $F(x^{(1)}, x^{(2)})$ is almost periodic type in $x^{(1)} \in \mathbf{R}^n$ and uniformly for $x^{(2)} \in \mathbf{R}^m$.

Let

$$Z(x, t; \xi, s) = \frac{1}{\left(2\sqrt{\pi(t-s)}\right)^{n+m}} \exp\left\{ -\frac{\sum (x_i - \xi_i)^2}{4(t-s)} \right\}, \quad (x,\ \xi \in \mathbf{R}^{n+m}), \tag{1.5}$$

be the fundamental solution of the heat equation [37].

In the next section, we will show the existence and uniqueness of some type of parabolic equations. Sections 3 is devoted to a type of Cauchy Problem respectively.

## 2. Solutions of Parabolic Equations

**Lemma 2.1.** *Let $T > 0$. If $\varphi \in \mathcal{AP}\mathcal{T}(\mathbf{R}^n \times \mathbf{R}^m)$ and*

$$u(x,t;s) = \int_{\mathbf{R}^{n+m}} \varphi(\xi) Z(x,t;\xi,s)\,d\xi, \tag{2.1}$$

*then for each fixed $s \in [0,T)$ $u \in \mathcal{AP}\mathcal{T}(\mathbf{R}^n \times \mathbf{R}^m \times [s,T])$.*

*Proof.* First consider the case that $\varphi \in \mathcal{AP}(\mathbf{R}^n \times \mathbf{R}^m)$. Let $\tau \in \mathbf{R}^n$ be an $\epsilon$-translation vector of $\varphi$:

$$\begin{aligned}
&u\left(x^{(1)}+\tau, x^{(2)}, t; s\right) - u\left(x^{(1)}, x^{(2)}, t; s\right) \\
&\quad = \int_{\mathbf{R}^{n+m}} \varphi\left(\xi^{(1)}, \xi^{(2)}\right)\left[Z\left(x^{(1)}+\tau, x^{(2)}, t; \xi^{(1)}, \xi^{(2)}, s\right) - Z\left(x^{(1)}, x^{(2)}, t; \xi^{(1)}, \xi^{(2)}, s\right)\right] d\xi^{(1)} d\xi^{(2)} \\
&\quad = \int_{\mathbf{R}^{n+m}} \left[\varphi\left(x^{(1)}+\tau+\xi^{(1)}, x^{(2)}+\xi^{(2)}\right) - \varphi\left(x^{(1)}+\xi^{(1)}, x^{(2)}+\xi^{(2)}\right)\right] Z(0,t;\xi s)\,d\xi,
\end{aligned} \tag{2.2}$$

where $0 \in \mathbf{R}^{n+m}$ is the zero vector. Note that $\int_{\mathbf{R}^{n+m}} Z(0,t;\xi,s)\,d\xi = 1$, we get

$$\|R_\tau u - u\| \le \left\|R_\tau \varphi - \varphi\right\| \int_{\mathbf{R}^{n+m}} Z(0,t;\xi,s)\,d\xi < \epsilon, \tag{2.3}$$

where $t \in [s,T]$ and $x^{(2)} \in B$ with $B$ a bounded subset of $\mathbf{R}^m$. This shows that $u \in \mathcal{AP}(\mathbf{R}^n \times \mathbf{R}^m \times [s,T])$.

To show that $u \in \mathcal{PAP}(\mathbf{R}^n \times \mathbf{R}^m \times [s,T])$ if $\varphi \in \mathcal{PAP}(\mathbf{R}^n \times \mathbf{R}^m)$, we only need to show that $u \in \mathcal{PAP}_0(\mathbf{R}^n \times \mathbf{R}^m \times [s,T])$ if $\varphi \in \mathcal{PAP}_0(\mathbf{R}^n \times \mathbf{R}^m)$. That is,

$$\lim_{r\to\infty} \frac{1}{(2r)^n} \int_{[-r,r]^n} \left|u\left(x^{(1)}, x^{(2)}, t; s\right)\right| ds^{(1)} = 0, \tag{2.4}$$

uniformly with respect to $(x^{(2)}, t) \in \Omega$, here $\Omega$ is any compact subset of $\mathbf{R}^m \times [s,T]$.

Since $\varphi \in \mathcal{PAP}_0(\mathbf{R}^n \times \mathbf{R}^m)$, for $\epsilon > 0$ there exist positive numbers $A$ and $r_0$ such that, when $r \ge r_0$ for all $\xi^{(1)} \in [-A,A]^n$ and $\xi^{(2)} \in \Omega \cap \mathbf{R}^m$, one has

$$\begin{gathered}
\frac{1}{(2r)^n} \int_{[-r,r]^n} \left|\varphi\left(x^{(1)}+\xi^{(1)}, \xi^{(2)}\right)\right| dx^{(1)} < \frac{\epsilon}{3}, \\
\int_{-\infty}^{-A} + \int_{A}^{\infty} \left|\varphi\left(x^{(1)}+\xi^{(1)}, \xi^{(2)}\right)\right| Z\left(0, x^{(2)}, t; \xi, s\right) d\xi < \frac{2\epsilon}{3}.
\end{gathered} \tag{2.5}$$

Therefore,

$$\begin{aligned}
&\frac{1}{(2r)^n}\int_{[-r,r]^n}\left|u\left(x^{(1)},x^{(2)},t;s\right)\right|dx^{(1)}\\
&\quad\le \frac{1}{(2r)^n}\int_{[-r,r]^n}\int_{\mathbf{R}^{n+m}}\left|\varphi\left(x^{(1)}+\xi^{(1)},\xi^{(2)}\right)\right|Z\left(0,x^{(2)},t;\xi,s\right)d\xi dx^{(1)}\\
&\quad= \int_{-\infty}^{-A}+\int_{-A}^{A}+\int_{A}^{\infty}Z\left(0,x^{(2)},t;\xi,s\right)d\xi\frac{1}{(2r)^n}\int_{[-r,r]^n}\left|\varphi\left(x^{(1)}+\xi^{(1)},\xi^{(2)}\right)\right|dx^{(1)}<\epsilon
\end{aligned}\tag{2.6}$$

uniformly with respect to $(x^{(2)},t)\in\Omega$, where by $\int_a^b F(\xi)d\xi$ we mean that

$$\int_a^b F(\xi)\xi=\int_{[a,b]^{n+m}}F(\xi)d\xi=\int_a^b\cdots\int_a^b F(\xi)d\xi_1 d\xi_2\cdots d\xi_{n+m}.\tag{2.7}$$

This shows that $u\in\mathcal{PAP}_0(\mathbf{R}^n\times\mathbf{R}^m)$. The proof is complete. □

**Corollary 2.2.** *Let $\varphi,\partial\varphi/\partial x_i\in\mathcal{APT}(\mathbf{R}^n\times\mathbf{R}^m)$, and let $u$ be as in Lemma 2.1. Then, $\partial u/\partial x_i\in\mathcal{APT}(\mathbf{R}^n\times\mathbf{R}^m\times[s,T])$.*

*Proof.* Note that

$$\begin{aligned}
\frac{\partial u(x,t;s)}{\partial x_i}&=\int_{\mathbf{R}^{n+m}}\varphi(\xi)\frac{\partial Z(x,t;\xi,s)}{\partial x_i}d\xi\\
&=-\int_{\mathbf{R}^{n+m}}\varphi(\xi)\frac{\partial Z(x,t;\xi,s)}{\partial \xi_i}d\xi\\
&=\int_{\mathbf{R}^{n+m}}\frac{\partial\varphi(\xi)}{\partial\xi_i}Z(x,t;\xi,s)d\xi.
\end{aligned}\tag{2.8}$$

□

By Lemma 2.1 we get the conclusion.

**Lemma 2.3.** *If $f(x,t)\in\mathcal{APT}(\mathbf{R}^n\times\overline{\mathbf{R}_T^m})$ and*

$$u(x,t)=\int_0^t ds\int_{\mathbf{R}^{n+m}}f(\xi,s)Z(x,t;\xi,s)d\xi,\tag{2.9}$$

*then $u$ and $\partial u(x,t)/\partial x_i (i=1,2,\ldots,n+m)$ are all in $\mathcal{APT}(\mathbf{R}^n\times\overline{\mathbf{R}_T^m})$.*

The proof is similar to that of Lemma 2.1, so we omit it.

**Theorem 2.4.** *Consider the heat problem*

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n+m}\left[\frac{\partial^2 u}{\partial x_i^2} + b_i(x,t)\frac{\partial u}{\partial x_i}\right] - c(x,t)u = f(x,t), \quad (x,t) \in \mathbf{R}_T^{n+m},$$
$$u(x,0) = \varphi(x), \quad x \in \mathbf{R}^{n+m}. \tag{2.10}$$

*If $f(x,t), b_i(x,t), \partial b_i/\partial x_j$ $(i,j = 1,2,\ldots,n+m)$, $c(x,t)$ are in $\mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$ and $\varphi, \partial\varphi/\partial x_i$ $(i = 1,2,\ldots,n+m)$ are in $\mathcal{APT}(\mathbf{R}^n \times \mathbf{R}^m)$, then* (2.10) *has a unique solution $u \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$.*

*Proof.* Problem (2.10) has the standard solution (see [37]):

$$u(x,t) = \int_{\mathbf{R}^{n+m}} \varphi(\xi)\Gamma(x,t;\xi,0)d\xi + \int_0^t ds \int_{\mathbf{R}^{n+m}} f(\xi,s)\Gamma(x,t;\xi,s)d\xi = I_1 + I_2, \tag{2.11}$$

where

$$\Gamma(x,t;\xi,s) = Z(x,t;\xi,s) + \int_s^t \int_{\mathbf{R}^{n+m}} Z(x,t;y,\eta)\cdot\Phi(y,\eta;\xi,s)dyd\eta,$$
$$\Phi(y,\eta;\xi,s) = \sum_{l=1}^{\infty}(LZ)_l(y,\eta;\xi,s), \quad (LZ)_1 = LZ, \tag{2.12}$$
$$(LZ)_{l+1}(y,\eta;\xi,s) = \int_s^t \int_{\mathbf{R}^{n+m}} [LZ(y,\eta;v,\sigma)](LZ)_l(v,\sigma;\xi,s)dvd\sigma,$$

and $L$ is the parabolic operator

$$L = \sum_{i=1}^{n+m}\left[\frac{\partial^2}{\partial x_i^2} + b_i(x,t)\frac{\partial}{\partial x_i}\right] + c(x,t) - \frac{\partial}{\partial t}. \tag{2.13}$$

Now, we show that $u \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$ :

$$\begin{aligned} I_1 &= \int_{\mathbf{R}^{n+m}} \varphi(\xi)Z(x,t;\xi,0)d\xi \\ &\quad + \int_{\mathbf{R}^{n+m}} \varphi(\xi)d\xi \int_0^t d\eta \int_{\mathbf{R}^{n+m}} Z(x,t;y,\eta)\sum_{l=1}^{\infty}(LZ)_l(y,\eta;\xi,0)dy \\ &= u_1(x,t) + \sum_{l=1}^{\infty}\int_0^t d\eta \int_{\mathbf{R}^{n+m}} Z(x,t;y,\eta)dy \int_{\mathbf{R}^{n+m}} \varphi(\xi)(LZ)_l(y,\eta;\xi,0)d\xi, \end{aligned}$$

$$
\begin{aligned}
I_2 &= \int_0^t ds \int_{\mathbf{R}^{n+m}} f(\xi,s) Z(x,t;\xi,s)\, d\xi \\
&\quad + \int_0^t ds \int_{\mathbf{R}^{n+m}} f(\xi,s)\, d\xi \int_s^t d\eta \int_{\mathbf{R}^{n+m}} Z(x,t;y,\eta) \sum_{l=1}^{\infty} (LZ)_l Z(y,\eta;\xi,s)\, dy \\
&= u_2(x,t) + \sum_{l=1}^{\infty} \int_0^t d\eta \int_{\mathbf{R}^{n+m}} Z(x,t;y,\eta)\, dy \int_0^{\eta} ds \int_{\mathbf{R}^{n+m}} f(\xi,s)(LZ)_l(y,\eta;\xi,s)\, d\xi.
\end{aligned}
\tag{2.14}
$$

By Lemmas 2.1 and 2.3, $u_i \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$ $(i = 1, 2, )$.

Obviously,

$$
LZ(x,t;\xi,s) = \sum_{i=1}^{n+m} b_i(x,t) \frac{\partial Z(x,t;\xi,s)}{\partial x_i} + c(x,t) Z(x,t;\xi,s). \tag{2.15}
$$

By Lemmas 2.1 and 2.3 we only need to show that the functions

$$
\begin{aligned}
w_l(x,t) &= \int_{\mathbf{R}^{n+m}} \varphi(\xi)(LZ)_l(x,t;\xi,0)\, d\xi, \\
v_l(x,t) &= \int_0^t ds \int_{\mathbf{R}^{n+m}} f(\xi,s)(LZ)_l(x,t;\xi,s)\, d\xi,
\end{aligned}
\tag{2.16}
$$

are in $\mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$. We do this by induction. By Lemmas 2.1 and 2.3 and Corollary 2.2, it is true for the case $l = 1$. Suppose that $w_l(x,t),\ v_l(x,t) \in \mathcal{APT}(\mathbf{R}^n \times \mathbf{R}_T^m)$. Then,

$$
\begin{aligned}
w_{l+1}(x,t) &= \int_{\mathbf{R}^{n+m}} \varphi(\xi)\, d\xi \int_0^t d\eta \int_{\mathbf{R}^{n+m}} LZ(x,t;y,\eta)(LZ)_l(y,\eta;\xi,0)\, dy \\
&= \int_0^t d\eta \int_{\mathbf{R}^{n+m}} LZ(x,t;y,\eta)\, dy \int_{\mathbf{R}^{n+m}} \varphi(\xi)(LZ)_l(y,\eta;\xi,0)\, d\xi \\
&= \int_0^t d\eta \int_{\mathbf{R}^{n+m}} w_l(y,\eta) LZ(x,t;y,\eta)\, dy.
\end{aligned}
\tag{2.17}
$$

By the induction assumption and Lemma 2.3, we have $w_{l+1}(x,t) \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$. Similarly one shows that $v_{l+1}(x,t) \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T^m})$. The proof is complete. □

## 3. Cauchy Problem

Starting this section we will apply the results of the last section to inverse problems of partial differential equations. We will investigate two types of initial value problems in this and

the next sections, respectively. We will keep the notation in Section 2 and, at the same time, introduce the following new notation:

$$x = (x_1, x_2, \ldots, x_{n-1}), \qquad \xi = (\xi_1, \xi_2, \ldots, \xi_{n-1}),$$
$$X = (x, x_n), \qquad \zeta = (\xi, \xi_n). \tag{3.1}$$

The following estimates are easily obtained:

$$\left\| \int_0^t ds \int_{\mathbf{R}^n} Z(X, t; \zeta, s) d\zeta \right\|_T \leq m(T), \tag{3.2}$$

where $m(T)$ are positive and increasing for $T \geq 0$ and $m(T) \to 0$ as $T \to 0$.

To show the main results of this and the next sections, the following lemmas are needed. The first lemma is the Gronwall-Bellman lemma; the convenient reference should be an ODE text, for instance, it is proved on page 15 of [38].

**Lemma 3.1.** *Let $\varphi$, $\phi$, and $\chi$ be real, continuous functions on $[0, T]$ with $\chi \geq 0$. If*

$$\varphi(t) \leq \phi(t) + \int_0^t \chi(s)\varphi(s) ds \quad (t \in [0, T]), \tag{3.3}$$

*then*

$$\varphi(t) \leq \phi(t) + \int_0^t \chi(s)\phi(s) \exp\left\{ \int_s^t \chi(\rho) d\rho \right\} ds \quad (t \in [0, T]). \tag{3.4}$$

**Lemma 3.2.** *Let $\varphi$ be a continuous function on $[0, T]$. If $\phi$, $\chi_1$, and $\chi_2$ are nondecreasing and nonnegative on $[0, T]$ and*

$$\varphi(t) \leq \phi(t) + \chi_1(t) \int_0^t \varphi(s) ds + \chi_2(t) \int_0^t \frac{\varphi(s)}{\sqrt{t-s}} ds \quad (t \in [0, T]), \tag{$*$}$$

*then*

$$\varphi(t) \leq \phi(t) \left[ 1 + t\chi_1(t) + 2\sqrt{t}\chi_2(t) \right] e^{t\chi(t)}, \tag{3.5}$$

*where*

$$\chi(t) = t\chi_1^2(t) + 4\sqrt{t}\chi_1(t)\chi_2(t) + \pi\chi_2^2(t). \tag{3.6}$$

*Proof.* Replacing $\varphi(s)$ in the two integrals of $(*)$ by the expression on the right-hand side in (26), changing the integral order of the resulting inequality, and making use of the monotonicity of $\phi$, $\chi_1$, and $\chi_2$, one gets

$$\varphi(t) \le \phi(t)\left[1 + t\chi_1(t) + 2\sqrt{t}\chi_2(t)\right] + \left[t\chi_1^2(t) + 4\sqrt{t}\chi_1(t)\chi_2(t) + \pi\chi_2^2(t)\right]\int_0^t \varphi(s)Ds. \tag{3.7}$$

Using Lemma 3.1 leads to the conclusion. □

**Lemma 3.3.** *Let $F(X,t) \in \mathcal{C}(\overline{\mathbf{R}_T^n})$ and $\varphi \in \mathcal{C}(\mathbf{R}^n)$. If $u(X,t)$ is a solution of the problem*

$$\begin{gathered} u_t - \Delta u + qu = F(X,t), \quad (X,t) \in \mathbf{R}_T^n, \\ u(X,0) = \varphi(X), \quad X \in \mathbf{R}^n, \end{gathered} \tag{3.8}$$

*then*

$$\|u\|_T \le K(T)\left(T\|F\|_T + \|\varphi\|\right), \tag{3.9}$$

*where $K(T) = 1 + T\|q\|_T e^{T\|q\|_T}$.*

One sees that $K(T)$ depends on $\|q\|_T$ only and is bounded near zero.

*Proof.* The solution $u$ can been written as

$$\begin{aligned} u(X,t) = &\int_{\mathbf{R}^n} \varphi(\zeta)Z(X,t;\zeta,0)d\zeta + \int_0^t ds \int_{\mathbf{R}^n} F(\zeta,s)Z(X,t;\zeta,s)d\zeta \\ &- \int_0^t ds \int_{\mathbf{R}^n} q(\xi,s)u(\zeta,s)Z(X,t;\zeta,s)d\zeta, \end{aligned} \tag{3.10}$$

so,

$$\|u\|_t \le \|\varphi\| + \int_0^t \|F\|_s ds + \int_0^t \|q\|_s \|u\|_s ds. \tag{3.11}$$

By Lemma 3.1 one gets the desired result. The proof is complete. □

*Problem 1.* Find functions $u(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$ such that

$$u_t - \Delta u + qu = F(X,t), \quad (X,t) \in \mathbf{R}_T^n, \tag{3.12}$$

$$u(X,0) = \varphi(X), \quad X \in \mathbf{R}^n, \tag{3.13}$$

$$u(x,0,t) = h(x,t), \quad (x,t) \in \overline{\mathbf{R}_T^{n-1}}, \tag{3.14}$$

where $\varphi(x,x_n)$, $\varphi_{x_nx_n}(x,x_n) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \mathbf{R})$, $h(x,t) \ge \text{const} > 0$, $h(x,t)$, $(\Delta h - h_t) \in \mathcal{APT}(\overline{\mathbf{R}_{T_0}^{n-1}})$, and $F(x,x_n,t)$, $F_{x_nx_n}(x,x_n,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_{T_0}})$. $T_0 > T > 0$ are constants.

By (3.13) and (3.14), one sees that $h(x,0) = \varphi(x,0)$.

We have the following additional problem.

*Problem 2.* Find functions $W(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$ such that

$$W_t - \Delta W + qW = F_{x_n x_n}(X,t), \quad (X,t) \in \mathbf{R}_T^n, \tag{3.15}$$

$$W(X,0) = \varphi_{x_n x_n}(X), \quad X \in \mathbf{R}^n, \tag{3.16}$$

$$W(x,0,t) = h_t - \Delta h + qh - F(x,0,t), \quad (x,t) \in \overline{\mathbf{R}_T^{n-1}}. \tag{3.17}$$

The Cauchy problems with unknown coefficient belong to inverse problems [39]. "In the last two decades, the field of inverse problems has certainly been one of the fastest growing areas in applied mathematics. This growth has largely been driven by the needs of applications both in other sciences and in industry." [40]. For the two problems above, we have the following.

**Lemma 3.4.** *Problems 1 and 2 are equivalent to each other.*

*Proof.* Let $V(X,t) = u_{x_n}(X,t)$. Then, $V(X,t)$ satisfies

$$V_t - \Delta V + qV = F_{x_n}(X,t), \quad (X,t) \in \mathbf{R}_T^n, \tag{3.18}$$

$$V(X,0) = \varphi_{x_n}(X), \quad X \in \mathbf{R}^n, \tag{3.19}$$

$$V_{x_n}(x,0,t) = h_t - \Delta h + qh - F(x,0,t), \qquad \varphi(x,0) = h(x,0), \quad (x,t) \in \overline{\mathbf{R}_T^{n-1}}. \tag{3.20}$$

So, if Problem 1 has a solution $(u,q)$, then Problems (3.18)–(3.20) have the solution $(V,q)$ with $V(X,t) = u_{x_n}(X,t)$. Obviously $V(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ if $u(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

On the other hand, if $V(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$ satisfy (3.18)–(3.20), then we will show that Problem 1 has a unique solution $(u,q)$ and $u(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

The uniqueness comes from the uniqueness of Cauchy Problem (1)-(2). For the existence, note the fact that if $(u,q)$ is a solution of (3.12)–(3.14), then $V = u_{x_n}$. Thus, we define

$$u(X,t) = \int_0^{x_n} V(x,y,t)\,dy + \Phi(x,t). \tag{3.21}$$

It follows from (3.14) that $\Phi = h$. Now, $u$ satisfies (3.13) because

$$u(X,0) = \int_0^{x_n} V(x,y,0)\,dy + h(x,0) = \varphi(X) - \varphi(x,0) + h(x,0) = \varphi(X). \tag{3.22}$$

It follows from (3.18), (3.20), and (3.21) that

$$\begin{aligned} u_t - \Delta u + qu &= \int_0^{x_n} (V_t - \Delta V + qV)\,dy + \int_0^{x_n} \frac{\partial^2 V(x,y,t)}{\partial y^2}\,dy \\ &\quad - \frac{\partial^2}{\partial x_n^2}\int_0^{x_n} V(x,y,t)\,dy + h_t - \nabla h + qh \\ &= F(x,x_n,t) - F(x,0,t) + V_{x_n}(x,x_n,t) - V_{x_n}(x,0,t) - V_{x_n}(x,x_n,t) \\ &\quad + h_t - \Delta h + qh = F(X,t). \end{aligned} \tag{3.23}$$

Thus, $u$ satisfies (3.12) and $(u,q)$ is a unique solution of Problem 1.

It follows from $V(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T}$, $h(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$, and (3.21) that $u(X,t)$ in (3.21) is in $\mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

Since we have shown that Problem 1 is equivalent to (3.18)–(3.20), to show the lemma we only need to show that Problem 2, equivalent to (3.18)–(3.20) too.

If $(V,q)$ is a solution of (3.18)–(3.20), let $W(X,t) = V_{x_n}(X,t)$. Then one can directly calculate that $(W,q)$ is a solution of (3.15)–(3.17) and $W(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

On the other hand, if $(W,q)$ is a solution of (3.15)–(3.17), let

$$V(X,t) = \int_0^{x_n} W(x,y,t)\,dy + \Phi(x,t), \tag{3.24}$$

where $\Phi$ is the solution of the Cauchy problem

$$\begin{aligned} \Phi_t - \Delta\Phi + q\Phi &= W_{x_n}(x,0,t) + F_{x_n}(x,0,t), \quad (x,t) \in \mathbf{R}_T^{n-1} \\ \Phi(x,0) &= \varphi_{x_n}(x,0), \quad x \in \mathbf{R}^{n-1}. \end{aligned} \tag{3.25}$$

Since $W(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$, $W_{x_n}(x,0,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$. By Theorem 2.4, $\Phi(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$ and so $V(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$.

Obviously, $V_{x_n}(x,0,t) = W(x,0,t) = h_t - \Delta h + qh - F(x,0,t)$, and this shows that $V$ satisfies (3.20). $V$ satisfies (3.19) because

$$\begin{aligned} V(X,0) &= \int_0^{x_n} W(x,y,0)\,dy + \Phi(x,0) = \int_0^{x_n} \varphi_{x_n x_n}(x,y)\,dy + \Phi(x,0) \\ &= \varphi_{x_n}(x,x_n) - \varphi_{x_n}(x,0) + \Phi(x,0) = \varphi_{x_n}(X). \end{aligned} \tag{3.26}$$

Finally,

$$\begin{aligned}
V_t - \Delta V + qV &= \int_0^{x_n} (W_t - \Delta W + qW)\,dy + \int_0^{x_n} \frac{\partial^2 W(x,y,t)}{\partial y^2}\,dy \\
&\quad - \frac{\partial^2}{\partial x_n^2} \int_0^{x_n} W(x,\eta,t)\,d\eta + \Phi_t - \Delta\Phi + q\Phi \\
&= F_{x_n}(x,x_n,t) - F_{x_n}(x,0,t) + W_{x_n}(x,x_n,t) - W_{x_n}(x,0,t) - W_{x_n}(x,x_n,t) \\
&\quad + \Phi_t - \Delta\Phi + q\Phi = F_{x_n}(X,t).
\end{aligned} \tag{3.27}$$

This shows that $V$ satisfies (3.18). The proof is complete. □

By (3.15)-(3.16) we have the integral equation about $W(X,t)$:

$$\begin{aligned}
W(X,t) &= \int_{\mathbf{R}^n} \varphi_{x_n x_n}(\zeta) Z(X,t;\zeta,0)\,d\zeta + \int_0^t ds \int_{\mathbf{R}^n} F_{x_n x_n}(\zeta,s) Z(X,t;\zeta,s)\,d\zeta \\
&\quad - \int_0^t ds \int_{\mathbf{R}^n} q(\xi,s) W(\zeta,s) Z(X,t;\zeta,s)\,d\zeta,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
q &= Lq \\
&= [\Delta h - h_t + F(x,0,t)]h^{-1} + h^{-1} \int_{\mathbf{R}^n} \varphi_{x_n x_n}(\zeta) Z(x,0,t;\zeta,0)\,d\zeta \\
&\quad + h^{-1} \int_0^t ds \int_{\mathbf{R}^n} F_{x_n x_n}(\zeta,s) Z(x,0,t;\zeta,s)\,d\zeta \\
&\quad - h^{-1} \int_0^t ds \int_{\mathbf{R}^n} q(\xi,s) W(\zeta,s) Z(x,0,t;\zeta,s)\,d\zeta \quad (x,t) \in \mathbf{R}_T^{n-1},
\end{aligned} \tag{3.29}$$

where $W$ is determined by (3.28).

It is readily to show that (3.15)–(3.17) are equivalent to (3.28)-(3.29).

Note that, for a given $q(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$, Theorem 2.4 shows the Cauchy problem (3.15) and (3.16) (or equivalently (3.28)) has a unique solution $W \in \mathcal{APT}(\mathbf{R}^n \times \overline{\mathbf{R}_T})$. Thus, (3.29) does define an operator $L$. To show that Problem 2 (and so Problem 1) has a unique solution, we only need to show that (3.29) has a solution $q(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$. That is, $L$ has a fixed point in $\mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$. To this end, let

$$\begin{aligned}
&\Bigg\| [\Delta h - h_t + F(x,0,t)]h^{-1} + h^{-1} \int_{\mathbf{R}^n} \varphi_{x_n x_n}(\zeta) Z(x,0,t;\zeta,0)\,d\zeta \\
&\quad + h^{-1} \int_0^T ds \int_{\mathbf{R}^n} F_{x_n x_n}(\zeta,s) Z(x,0,t;\zeta,s)\,d\zeta \Bigg\|_{T_0} = \frac{M}{2}.
\end{aligned} \tag{3.30}$$

Set $B(M,T) = \{q(x,t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}}) : \|q\|_T \le M\}$. Now, we show that for small $T$ the operator $L$ in (3.29) is a contraction from $B(M,T)$ into itself.

If $q \in B(M,T)$, then, according to Theorem 2.4, the function $W$ determined by (3.15)-(3.16) and therefore by (3.28) belongs to $\mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$. Note that $\varphi_{x_n x_n} \in \mathcal{APT}(\mathbf{R}^{n-1} \times \mathbf{R})$, $(\Delta h - h_t) \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$, and $F_{x_n x_n} \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$. It follows from Lemma 2.3, Theorem 2.4, and (3.29) that $Lq \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$ and

$$\begin{aligned}
\|Lq\|_T \le{}& \Big\| [\Delta h - t_t + F(x,0,t)]h^{-1} + h^{-1} \int_{\mathbf{R}^n} \varphi_{x_n x_n}(\zeta) Z(x,0,t;\zeta,0)\, d\zeta \\
&+ h^{-1} \int_0^t ds \int_{\mathbf{R}^n} F_{x_n x_n}(\zeta,s) Z(x,0,t;\zeta,s)\, d\zeta \Big\|_T \\
&+ \left\| h^{-1} \right\|_T \|q\|_T \|W\|_T m(T) \\
\le{}& \frac{1}{2} M + \left\| h^{-1} \right\|_T M K_0(T) \left( \|\varphi_{x_n x_n}\| + T \|F_{x_n x_n}\|_T \right) m(T),
\end{aligned} \tag{3.31}$$

where $K_0$ comes from Lemma 3.3. Noting that $m(T) \to 0$ ad $T \to 0$, we choose $T_1 \le T_0$ such that when $T < T_1$ one has

$$\left\| h^{-1} \right\|_T K_0(T) \left( \|\varphi_{x_n x_n}\| + T \|F_{x_n x_n}\|_T \right) m(T) \le \frac{1}{2}. \tag{3.32}$$

So, $Lq \in B(M,T)$. For $q_1, q_2 \in B(M,T)$, by (3.29)

$$\begin{aligned}
\|Lq_1 - Lq_2\| &\le \left\| h^{-1} \right\|_T \|W_1 q_1 - W_2 q_2\|_T m(T) \\
&\le m(T) \left\| h^{-1} \right\|_T \left[ \|W_1\|_T \|q_1 - q_2\|_T + \|q_2\|_T \|W_1 - W_2\|_T \right].
\end{aligned} \tag{3.33}$$

The function $V = W_1 - W_2$ is a solution of the Cauchy problem

$$\begin{gathered}
V_t - \Delta V + q_1 V = W_2 (q_2 - q_1), \quad (X,t) \in \mathbf{R}_T^n, \\
V(X,0) = 0, \quad X \in \mathbf{R}^n.
\end{gathered} \tag{3.34}$$

Thus, by Lemma 3.3

$$\|V\|_T \le K_1(T) \|W_2\|_T \|q_2 - q_1\|. \tag{3.35}$$

Applying Lemma 3.3 to $(W_1, q_1)$ and $(W_2, q_2)$, respectively, one gets

$$\begin{aligned} \|W_1\| &\le K_1(T)\left(\left\|\varphi_{x_n x_n}\right\| + T\|F_{x_n x_n}\|\right), \\ \|W_2\| &\le K_2(T)\left(\left\|\varphi_{x_n x_n}\right\| + T\|F_{x_n x_n}\|\right). \end{aligned} \tag{3.36}$$

If we choose $T_2 \le T_1$ so that when $T \le T_2$

$$2m(T)\left\|h^{-1}\right\|_T\left(\left\|\varphi_{x_n x_n}\right\| + T\|F_{x_n x_n}\|\right)K_1(T)[1 + MK_2(T)] \le 1, \tag{3.37}$$

then

$$\begin{aligned} \|Lq_1 - Lq_2\|_T &\le m(T)\|h^{-1}\|_T[\|W_1\| + MK_1(T)\|W_2\|]\|q_2 - q_1\|_T \\ &\le m(T)\|h^{-1}\|_T K_1(T)(1 + MK_2(T))(\|\varphi_{x_n x_n}\| + T\|F_{x_n x_n}\|_T)\|q_2 - q_1\|_T \\ &\le \frac{1}{2}\|q_2 - q_1\|_T. \end{aligned}$$

One sees that for such $T$, the operator $L$ is a contraction from $B(M,T)$ into itself and, therefore, has a unique fixed point in $B(M,T)$. Thus, we have shown.

**Theorem 3.5.** *If functions $F, \varphi$, and $h$ satisfy the conditions of Problem 1, $M$ and $T$ are determined by* (3.30) *and* (3.32), (3.37) *respectively, then in $\mathbf{R}^n_T$, Problem 1 has a unique solution $(u, q)$ with $u \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q \in \mathcal{APT}(\overline{\mathbf{R}^{n-1}_T})$.*

Furthermore, we have the following.

**Theorem 3.6.** *Let $F, \varphi$, and $h$ be as in Problem 1. Then, there exists an almost periodic type solution for Problem 1 in $\mathbf{R}^n_{T_0}$.*

*Proof.* We show that the conclusion of Theorem 3.5 can be extended to $\mathbf{R}^n_{T_0}$. Let $T = \sup\{s :$ Problem 1 has solution in $\mathbf{R}^n_s\}$. By Theorem 3.5, $T > 0$. Suppose that $T < T_0$. Consider the problem

$$\begin{aligned} &u_t - \Delta u + qu = F(X,t), \quad X \in \mathbf{R}^n,\ T \le t \le T_0, \\ &u(X,T) = f(X), \quad X \in \mathbf{R}^n, \\ &u(x,0,t) = h(x,t), \quad (x,t) \in \mathbf{R}^{n-1} \times [T, T_0]. \end{aligned} \tag{3.38}$$

□

For $q$ we can write the integral equation similar to (3.29), but this time its domain is $x \in \mathbf{R}^{n-1},\ t \in [T, T_0]$. As the proof above, define the ball $B_1(M,S)$ in $\mathcal{APT}(\mathbf{R}^{n-1} \times [T, T_0])$; then there exists a $t_0 > 0$ such that the operator $L_{t_0}$ is a contraction from $B_1(M,S)$ into itself. So, (3.29) has a solution for the domain $x \in \mathbf{R}^{n-1},\ t \in [T, T + t_0]$. This contradicts the definition of $T$. We must have $T = T_0$.

For the stability, we have the following.

**Theorem 3.7.** *Let functions $h_i$, $\varphi_i$, and $F_i (i = 1, 2)$ be as in Problem 1. If $W_i(X,t) \in \mathcal{APT}(\mathbf{R}^{n-1} \times \overline{\mathbf{R}_T})$ and $q_i \in \mathcal{APT}(\overline{\mathbf{R}_T^{n-1}})$ $(i = 1, 2)$ are solutions to* (3.15)–(3.17)*, then*

$$\begin{aligned} \|q_2 - q_1\|_t \le{}& c_1\|h_2 - h_1\|_t + c_2\left\|\left(\frac{\partial}{\partial t} - \Delta\right)(h_2 - h_1)\right\|_t \\ &+ c_3\left\|\frac{\partial^2}{\partial x_n^2}(\varphi_2 - \varphi_1)\right\| + c_4\|F_2 - F_1\|_t + c_5\left\|\frac{\partial^2}{\partial x_n^2}(F_2 - F_1)\right\|_t, \end{aligned} \tag{3.39}$$

*where $c_j$ $(1 \le j \le 5)$ depends on $T$, $\|h_1\|_t$, $\|(\partial^2/\partial x_n^2)\varphi_1\|$, $\|F_1\|_t$, $\|(\partial^2/\partial x_n^2)F_1\|_t$, and $\|q_i\|_t$ $(i = 1, 2)$ only.*

*Proof.* By (3.29),

$$\begin{aligned} h_1(q_2 - q_1) ={}& -q_2(h_2 - h_1) - \left(\frac{\partial}{\partial t} - \Delta\right)(h_2 - h_1) + F_2(x,0,t) - F_1(x,0,t) \\ &+ \int_{\mathbf{R}^n} \frac{\partial^2}{\partial \xi_n^2}(\varphi_2(\zeta) - \varphi_1(\zeta))Z(x,0,t;\zeta,0)d\zeta \\ &+ \int_0^t ds \int_{\mathbf{R}^n} \frac{\partial^2}{\partial \xi_n^2}(F_2 - F_1)Z(x,0,t;\zeta,s)d\zeta \\ &- \int_0^t ds \int_{\mathbf{R}^n} [(W_2 - W_1)q_2 + W_1(q_2 - q_1)]Z(x,0,t;\zeta,s)d\zeta. \end{aligned} \tag{3.40}$$

By Lemma 3.3,

$$\|W_1\|_t \le K_1(t)\left[\left\|\frac{\partial^2\varphi_1}{\partial x_n^2}\right\| + t\left\|\frac{\partial^2 F_1}{\partial x_n^2}\right\|_t\right]. \tag{3.41}$$

Since the function $V = W_2 - W_1$ is the solution of the Cauchy problem

$$\begin{aligned} V_t - \Delta V + q_2 V &= \frac{\partial^2}{\partial x_n^2}(F_2 - F_1) - W_1(q_2 - q_1), \quad (X,t) \in \mathbf{R}_T^n \\ V(X,0) &= \frac{\partial^2}{\partial x_n^2}[\varphi_2(X) - \varphi_1(X)], \quad X \in \mathbf{R}^n, \end{aligned} \tag{3.42}$$

one has

$$\begin{aligned} \|V\|_t \le K_2(T)\Bigg[&\left\|\frac{\partial^2}{\partial x_n^2}(\varphi_2 - \varphi_1)\right\| + t\left\|\frac{\partial^2}{\partial x_n^2}(F_2 - F_1)\right\|_t \\ &+ K_1(t)\left(\left\|\frac{\partial^2}{\partial x_n^2}\varphi_1\right\| + t\left\|\frac{\partial^2}{\partial x_n^2}F_1\right\|\right)\|q_2 - q_1\|_t\Bigg]. \end{aligned} \tag{3.43}$$

Therefore,

$$\begin{aligned}\|q_2-q_1\|_t \le \left\|h_1^{-1}\right\|_t \Bigg\{ & \|q_2\|_t\|h_2-h_1\|_t + \left\|\left(\frac{\partial}{\partial t}-\Delta\right)(h_2-h_1)\right\|_t + \|F_2(x,0,t)-F_1(x,0,t)\|_t \\ & + \left\|\frac{\partial^2}{\partial x_n^2}(\varphi_2-\varphi_1)\right\| + T\left\|\frac{\partial^2}{\partial x_n^2}(F_2-F_1)\right\|_t \\ & + \int_0^t K_2(s)\left[\left\|\frac{\partial^2}{\partial x_n^2}(\varphi_2-\varphi_1)\right\| + s\left\|\frac{\partial^2}{\partial x_n^2}(F_2-F_1)\right\|_s\right]\|q_2\|_s ds \\ & + \int_0^t K_1(s)\left[\left\|\frac{\partial^2}{\partial x_n^2}\varphi_1\right\| + s\left\|\frac{\partial^2}{\partial x_n^2}F_1\right\|_s\right]\|q_2-q_1\|_s ds \\ & + \int_0^t K_1(s)K_2(s)\left[\left\|\frac{\partial^2}{\partial x_n^2}\varphi_1\right\| + s\left\|\frac{\partial^2}{\partial x_n^2}F_1\right\|_s\right]\|q_2-q_1\|_s\|q_2\|_s ds\Bigg\}.\end{aligned} \tag{3.44}$$

Using Lemma 3.1, we get the estimates desired if we let

$$\begin{aligned} c_1 &= \left\|h_1^{-1}\right\|_t\|q_2\|_t\left[1+\left\|h_1^{-1}\right\|_t\int_0^t \chi(s)\exp\left\{\left\|h_1^{-1}\right\|_t\int_s^t\chi(\rho)d\rho\right\}ds\right], \\ c_2 &= \left\|h_1^{-1}\right\|_t\left[1+\left\|h_1^{-1}\right\|_t\int_0^t \chi(s)\exp\left\{\left\|h_1^{-1}\right\|_t\int_s^t\chi(\rho)d\rho\right\}ds\right], \\ c_3 &= \left\|h_1^{-1}\right\|_t\left[1+\int_0^t\|q_2\|_s K_2(s)ds\right]\left[1+\left\|h_1^{-1}\right\|_t\int_0^t \chi(s)\exp\left\{\left\|h_1^{-1}\right\|_t\int_s^t\chi(\rho)d\rho\right\}ds\right], \\ c_4 &= c_2 \qquad c_5 = c_3, \end{aligned} \tag{3.45}$$

where

$$\chi(t) = \left[\left\|\frac{\partial^2}{\partial x_n^2}\varphi_1\right\| + t\left\|\frac{\partial^2}{\partial x_n^2}F_1\right\|_t\right]K_1(t)\left(1+K_2(t)\|q_2\|_t\right). \tag{3.46}$$

The proof is complete. □

**Corollary 3.8.** *If Problem 1 has a solution in* $\mathbf{R}^n_{T_0}$, *then it has a unique one.*

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