Research Article

Inequalities for the Polar Derivative of a Polynomial

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For a polynomial $p(z)$ of degree $n$, we consider an operator $D_a$ which map a polynomial $p(z)$ into $D_a p(z) := (a - z)p'(z) + np(z)$ with respect to $a$. It was proved by Liman et al. (2010) that if $p(z)$ has no zeros in $|z| < 1$, then for all $a, \beta \in \mathbb{C}$ with $|a| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$, $|zD_a p(z) + \beta ((|a| - 1)/2)p(z)| \leq (n/2) [|a + \beta ((|a| - 1)/2)| + |z + \beta ((|a| - 1)/2)| \max |p(z)| - [|a + \beta ((|a| - 1)/2)| - |z + \beta ((|a| - 1)/2)| \min |p(z)|].$ In this paper we extend the above inequality for the polynomials having no zeros in $|z| < k$, where $k \leq 1$. Our result generalizes certain well-known polynomial inequalities.

1. Introduction and Statement of Results

According to a result well known as Bernstein’s inequality on the derivative of a polynomial $p(z)$ of degree $n$, we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \quad (1.1)$$

The result is best possible, and equality holds for a polynomial having all its zeros at the origin (see [1, 2]).

The inequality (1.1) can be sharpened, by considering the class of polynomials having no zeros in $|z| < 1$.

In fact, P. Erdős conjectured, and later Lax [3] proved that if $p(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (1.2)$$
As a refinement of (1.2), Aziz and Dawood [4] proved that if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}.
\]  

(1.3)

As an improvement of (1.3), Dewan and Hans [5] proved that if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then for any \( \beta \) with \( |\beta| \leq 1 \) and \( |z| = 1 \),

\[
\left| zp'(z) + \frac{n\beta}{2} p(z) \right| \leq \frac{n}{2} \left\{ \left( 1 + \frac{\beta}{2} \right) \max_{|z|=1} |p(z)| - \left( 1 + \frac{\beta}{2} \right) \min_{|z|=1} |p(z)| \right\}.
\]  

(1.4)

Let \( \alpha \) be a complex number. For a polynomial \( p(z) \) of degree \( n \), \( D_{\alpha}p(z) \), the polar derivative of \( p(z) \) is defined as

\[
D_{\alpha}p(z) = np(z) + (\alpha - z)p'(z).
\]  

(1.5)

It is easy to see that \( D_{\alpha}p(z) \) is a polynomial of degree at most \( n - 1 \) and that \( D_{\alpha}p(z) \) generalizes the ordinary derivative in the sense that

\[
\lim_{\alpha \to \infty} \left[ \frac{D_{\alpha}p(z)}{\alpha} \right] = p'(z).
\]  

(1.6)

As an extension to (1.1) for the polynomial derivative \( D_{\alpha}p(z) \), Aziz and Shah [6] proved that if \( p(z) \) is a polynomial of degree \( n \), then for every \( \alpha \) with \( |\alpha| \geq 1 \),

\[
\max_{|z|=1} |D_{\alpha}p(z)| \leq n|\alpha| \max_{|z|=1} |p(z)|.
\]  

(1.7)

As a refinement and extension of (1.7), Aziz and Mohammad Shah [7] proved that if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then, for every \( \alpha \) with \( |\alpha| \geq 1 \),

\[
\max_{|z|=1} |D_{\alpha}p(z)| \leq \frac{n}{2} \left\{ (|\alpha| + 1) \max_{|z|=1} |p(z)| - (|\alpha| - 1) \min_{|z|=1} |p(z)| \right\}.
\]  

(1.8)

Recently Dewan et al. [8] generalized (1.8) to the polynomial of the form \( p(z) = a_0 + \sum_{t=1}^{n} a_{t}z^{t} \), \( 1 \leq t \leq n \) and proved that if \( p(z) = a_0 + \sum_{t=1}^{n} a_{t}z^{t} \), \( 1 \leq t \leq n \) is a polynomial of degree \( n \) having no zeros in \( |z| < k \), \( k \geq 1 \), then for \( |\alpha| \geq 1 \)

\[
\max_{|z|=1} |D_{\alpha}p(z)| \leq \frac{n}{1 + s_0} \left\{ (|\alpha| + s_0) \max_{|z|=1} |p(z)| - (|\alpha| - 1) \min_{|z|=1} |p(z)| \right\},
\]  

(1.9)

where \( s_0 = k^{t+1} \left\{ ( ((t/n)|a_1|/(|a_0| - m)) k^{t+1} + 1 \right\} / ((((t/n)|a_1|/(|a_0| - m))) k^{t+1} + 1) \), and \( m = \min_{|z|=k} |p(z)| \).
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As a generalization of (1.9), Bidkham et al. [9] proved that if \( p(z) = a_0 + \sum_{\nu=1}^{n} a_{\nu} z^{\nu} \), \( 1 \leq \mu \leq n \) is a polynomial of degree \( n \) having no zeros in \( |z| < k, \ k \geq 1 \), then for \( 0 < r \leq R \leq k \) and \( |\alpha| \geq R \)

\[
\max_{|z|=R} |D_{a} p(z)| \leq \frac{n}{1 + s'_{0}} \left\{ \left( \frac{|\alpha|}{R} + s'_{0} \right) \exp \left\{ n \int_{r}^{R} A_{t} dt \right\} \max_{|z|=r} |p(z)| \right. \\
+ s'_{0} + 1 - \left( \frac{|\alpha|}{R} + s'_{0} \right) \exp \left\{ n \int_{r}^{R} A_{t} dt \right\} \min_{|z|=k} |p(z)| \right\},
\]

(1.10)

where

\[
A_{t} = \frac{(\mu/n)(|a_{\mu}|/(|a_{0}| - m))k^{\mu+1} + \mu}{\mu+1 + k^{\mu+1} + (\mu/n)(|a_{\mu}|/(|a_{0}| - m))(k^{\mu+1} + k^{2\mu})} \quad \quad m = \min_{|z|=k} |p(z)|.
\]

(1.11)

As an improvement and generalization to (1.8) and (1.4), Liman et al. [10] proved that if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then, for all \( \alpha, \beta \) with \( |\alpha| \geq 1, \ |\beta| \leq 1 \) and \( |z| = 1 \)

\[
\left| zD_{a} p(z) + n\beta \frac{|\alpha| - 1}{2} p(z) \right| \leq \frac{n}{2} \left\{ \left( \left| \alpha + \beta \frac{|\alpha| - 1}{2} \right| + \left| z + \beta \frac{|\alpha| - 1}{2} \right| \right) \max_{|z|=1} |p(z)| \right. \\
- \left( \left| \alpha + \beta \frac{|\alpha| - 1}{2} \right| - \left| z + \beta \frac{|\alpha| - 1}{2} \right| \right) \min_{|z|=1} |p(z)| \right\}.
\]

(1.12)

In this paper, we obtain the following extension of (1.12).

**Theorem 1.1.** Let \( p(z) \) be a polynomial of degree \( n \) that does not vanish in \( |z| < k, \ k \leq 1 \), then, for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \geq k, \ |\beta| \leq 1 \) and \( |z| = 1 \), we have

\[
\left| zD_{a} p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| \leq \frac{n}{2} \left\{ \left( k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| + \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right) \max_{|z|=1} |p(z)| \right. \\
- \left( k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| - \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right) \min_{|z|=1} |p(z)| \right\}.
\]

(1.13)

If we take \( k = 1 \) in Theorem 1.1, then (1.13) reduces to (1.12).

Theorem 1.1 simplifies to the following result by taking \( \beta = 0 \).
Corollary 1.2. Let \( p(z) \) be a polynomial of degree \( n \) does not vanish in \( |z| < k, k \leq 1 \), then for any \( \alpha \in \mathbb{C} \) with \( |\alpha| \geq k \), we have

\[
\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{2} \left\{ (k^{-n}|\alpha| + 1)\max_{|z|=1} |p(z)| - (k^{-n}|\alpha| - 1)\min_{|z|=k} |p(z)| \right\}. \tag{1.14}
\]

If we take \( k = 1 \) in Corollary 1.2, then (1.14) reduce to (1.8). Dividing two sides of inequality (1.13) by \( |\alpha| \) and letting \( |\alpha| \to \infty \), we have the following generalization of the inequality (1.4).

Corollary 1.3. Let \( p(z) \) be a polynomial of degree \( n \), having no zeros in \( |z| < k, k \leq 1 \), then, for any \( \beta \in \mathbb{C} \) with \( |\beta| \leq 1 \) and \( |z| = 1 \), we have

\[
\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| \leq \frac{n}{2} \left\{ \left( k^{-n}|\beta| + \frac{\beta}{1+k} \right)\max_{|z|=1} |p(z)| \right. \\
- \left. \left( k^{-n}|\beta| - \frac{\beta}{1+k} \right)\min_{|z|=k} |p(z)| \right\}. \tag{1.15}
\]

Taking \( \beta = 0 \) and \( k = 1 \) in Corollary 1.3, (1.15) reduces to (1.3).

2. Lemmas

For proof of the theorem, we need the following lemmas. The first lemma is due to Laguerre [11, 12].

Lemma 2.1. If all the zeros of an \( n \)th degree polynomial \( p(z) \) lie in a circular region \( C \), and \( \omega \) is any zero of \( D_\alpha p(z) \), then at most one of the points \( \omega \) and \( \alpha \) may lie outside \( C \).

Lemma 2.2. If \( p(z) \) is a polynomial of degree \( n \), having all its zeros in the closed disk \( |z| \leq k, k \leq 1 \), then on \( |z| = 1 \)

\[
|p'(z)| \geq \frac{n}{1+k} |p(z)|. \tag{2.1}
\]

This lemma is due to Malik [13].

Lemma 2.3. Let \( p(z) \) be a polynomial of degree \( n \) and have no zero in \( |z| < k, k \geq 1 \), then on \( |z| = 1 \)

\[
k|p'(z)| \leq |q(z)|, \tag{2.2}
\]

where \( q(z) = z^n p(1/z) \).

The above lemma is due to Chan and Malik [14].
Lemma 2.4. If \( p(z) \) is a polynomial of degree \( n \), having all its zeros in the closed disk \( |z| \leq k \), then on \( |z| = 1 \)

\[
|q'(z)| \leq k|p'(z)|,  \tag{2.3}
\]

where \( q(z) = z^n p(1/z) \).

Proof. Since \( p(z) \) has all its zeros in \( |z| \leq k \), \( k \leq 1 \); therefore, \( q(z) \) has no zero in \( |z| < 1/k \), \( 1/k \geq 1 \). Now applying Lemma 2.3 to the polynomial \( q(z) \) and the result follows.

Lemma 2.5. If \( p(z) \) is a polynomial of degree \( n \), having all its zeros in the closed disk \( |z| \leq k \), \( k \leq 1 \), then for all real or complex number \( \alpha \) with \( |\alpha| \geq k \) and \( |z| = 1 \), we have

\[
|D_\alpha p(z)| \geq n \frac{|\alpha| - k}{1 + k} |p(z)|.  \tag{2.4}
\]

Proof. Let \( q(z) = z^n p(1/z) \), then \( |q'(z)| = |np(z) - zp'(z)| \) on \( |z| = 1 \). Thus on \( |z| = 1 \)

\[
|D_\alpha p(z)| = |np(z) + (\alpha - z)p'(z)|
= |ap'(z) + np(z) - zp'(z)| \tag{2.5}
\geq |ap'(z)| - |np(z) - zp'(z)|,
\]

which implies that

\[
|D_\alpha p(z)| \geq |\alpha||p'(z)| - |q'(z)|.  \tag{2.6}
\]

Combining (2.3) and (2.6), we get the following:

\[
|D_\alpha p(z)| \geq (|\alpha| - k)|p'(z)|.  \tag{2.7}
\]

along with Lemma 2.2, which gives the following:

\[
|D_\alpha p(z)| \geq n \frac{|\alpha| - k}{1 + k} |p(z)|.  \tag{2.8}
\]

Lemma 2.6. Let \( p(z) \) be a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), \( k \leq 1 \). Then for every \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \geq k \), \( |\beta| \leq 1 \) and \( |z| = 1 \), we have

\[
\left|zD_\alpha p(z) + np\left|\frac{\alpha - k}{1 + k} p(z)\right| \geq nk^{-n} \left|\alpha + \beta \frac{|\alpha| - k}{1 + k} \min_{|z| = k} |p(z)|\right| \tag{2.9}
\]

Proof. If \( p(z) \) has a zero on \( |z| = k \), then (2.9) is trivial. Therefore, we assume that \( p(z) \) has all its zeros in \( |z| < k \). Let \( m = \min_{|z| = k} |p(z)| \), then \( m > 0 \) and \( |p(z)| \geq m \) where \( |z| = k \). Therefore,
for $|\lambda| < 1$, it follows by Rouche’s Theorem that the polynomial $G(z) = p(z) - \lambda m(z/k)^n$ has all its zeros in $|z| < k$. By using Lemma 2.1, $D_\alpha G(z) = D_\alpha p(z) - \alpha \lambda m n (z^{n-1}/k^n)$ has all its zeros in $|z| < k$, where $|\alpha| \geq k$. Applying Lemma 2.5 to the polynomial $G(z)$ yields

$$|zD_\alpha G(z)| \geq n|\alpha| - k \over 1+k |G(z)|, \quad |z| = 1. \quad (2.10)$$

Since $zD_\alpha G(z)$ has all its zeros in $|z| < k \leq 1$, by using Rouche’s Theorem, it can be easily verified from (2.10) that the polynomial

$$zD_\alpha G(z) + \beta n|\alpha| - k \over 1+k G(z) \quad (2.11)$$

has all its zeros in $|z| < 1$, where $|\beta| < 1$.

Substituting for $G(z)$, we conclude that the polynomial

$$T(z) = \left(zD_\alpha p(z) + n\beta |\alpha| - k \over 1+k p(z)\right) - \lambda mn \left(\frac{z}{k}\right)^n \left(\alpha + \beta |\alpha| - k \over 1+k \right) \quad (2.12)$$

will have no zeros in $|z| \geq 1$. This implies for every $\beta$ with $|\beta| < 1$ and $|z| \geq 1$,

$$\left|zD_\alpha p(z) + n\beta |\alpha| - k \over 1+k p(z)\right| \geq nm \left|\frac{z}{k}\right|^n \left|\alpha + \beta |\alpha| - k \over 1+k \right|. \quad (2.13)$$

If (2.13) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\left|z_0D_\alpha p(z_0) + n\beta |\alpha| - k \over 1+k p(z_0)\right| < nm \left|\frac{z_0}{k}\right|^n \left|\alpha + \beta |\alpha| - k \over 1+k \right|. \quad (2.14)$$

Take

$$\lambda = \frac{z_0D_\alpha p(z_0) + n\beta (|\alpha| - k)/(1+k)) p(z_0)}{nm(z_0/k)^n (\alpha + \beta (|\alpha| - k)/(1+k))}, \quad (2.15)$$

then $|\lambda| < 1$ and with this choice of $\lambda$, we have $T(z_0) = 0$ for $|z_0| \geq 1$, from (2.12). But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. For $\beta$ with $|\beta| = 1$, (2.13) follows by continuity. This completes the proof of Lemma 2.6.

**Lemma 2.7.** If $p(z)$ is a polynomial of degree $n$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|\alpha| \geq k$, where $k \leq 1$, we have

$$\left|zD_\alpha p(z) + n\beta |\alpha| - k \over 1+k p(z)\right| \leq nk^{-n} \left|\alpha + \beta |\alpha| - k \over 1+k \right| \max_{|z| = k}|p(z)|, \quad |z| = 1. \quad (2.16)$$

**Proof.** Let $M = \max_{|z| = k}|p(z)|$, if $|\lambda| < 1$, then $|\lambda p(z)| < |M(z/k)^n|$ for $|z| = k$. Therefore, it follows by Rouche’s Theorem that the polynomial $G(z) = M(z/k)^n - \lambda p(z)$ has all its zeros
in $|z| < k$. By using Lemma 2.1, $D_a G(z) = \alpha Mn(z^n / k^n) - \lambda D_a p(z)$ has all its zeros in $|z| < k$ for $|\alpha| \geq k$.

On applying Lemma 2.5 to the polynomial $G(z)$, we have

$$|zD_a G(z)| \geq n \frac{|\alpha| - k}{1 + k} |G(z)|, \quad |z| = 1.$$  \hspace{1cm} (2.17)

Now, using a similar argument as used in the proof of Lemma 2.6, the result follows.

\Box

**Lemma 2.8.** If $p(z)$ is a polynomial of degree $n$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|\alpha| \geq k$, where $k \leq 1$, we have

$$|zD_a p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z)| + |zD_a Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z)| \leq n \left( k^{-n} |\alpha + \beta \frac{|\alpha| - k}{1 + k} | + |z + \beta \frac{|\alpha| - k}{1 + k}| \right) \max_{|z|=1} |p(z)|, \quad |z| = 1,$$

where $Q(z) = (z/k)^n p(k^2/z)$.

**Proof.** Let $M = \max_{|z|=k} |p(z)|$. For $\lambda$ with $|\lambda| > 1$, it follows by Rouche’s Theorem that the polynomial $G(z) = p(z) - \lambda M$ has no zeros in $|z| < k$. Consequently the polynomial

$$H(z) = \left( \frac{z}{k} \right)^n G \left( \frac{k^2}{z} \right)$$

has all its zeros in $|z| \leq k$, also $|G(z)| = |H(z)|$ for $|z| = k$. Since all the zeros of $H(z)$ lie in $|z| \leq k$; therefore, for $\delta$ with $|\delta| > 1$, by Rouche’s Theorem all the zeros of $G(z) + \delta H(z)$ lie in $|z| \leq k$. Hence by Lemma 2.5 for every $\alpha$ with $|\alpha| \geq k$, and $|z| = 1$, we have

$$n \frac{|\alpha| - k}{1 + k} |G(z) + \delta H(z)| \leq |zD_a (G(z) + \delta H(z))|.$$  \hspace{1cm} (2.20)

On the other hand by Lemma 2.1, all the zeros of $D_a (G(z) + \delta H(z))$ lie in $|z| < k \leq 1$, where $|\alpha| \geq k$. Therefore, for any $\beta$ with $|\beta| \leq 1$, Rouche’s Theorem implies that all the zeros of $zD_a (G(z) + \delta H(z)) + \beta n(\lambda - k)/(1 + k))(G(z) + \delta H(z))$ lie in $|z| < 1$. This means that the polynomial

$$T(z) = zD_a G(z) + n\beta \frac{|\alpha| - k}{1 + k} G(z) + \delta \left( zD_a H(z) + n\beta \frac{|\alpha| - k}{1 + k} H(z) \right)$$

will have no zeros in $|z| \geq 1$. Now using a similar argument as used in the proof of Lemma 2.6, we get for $|z| \geq 1$,

$$|zD_a G(z) + n\beta \frac{|\alpha| - k}{1 + k} G(z)| \leq |zD_a H(z) + n\beta \frac{|\alpha| - k}{1 + k} H(z)|.$$  \hspace{1cm} (2.22)
Therefore by the equalities

\[ H(z) = \left( \frac{z}{k} \right)^n G \left( \frac{k^2}{\bar{z}} \right) = \left( \frac{z}{k} \right)^n p \left( \frac{k^2}{\bar{z}} \right) - \lambda n M \left( \frac{z}{k} \right)^n = Q(z) - \lambda n M \left( \frac{z}{k} \right)^n, \quad (2.23) \]

or

\[ H(z) = Q(z) - \lambda n M \left( \frac{z}{k} \right)^n, \quad (2.24) \]

and substitute for \( G(z) \) and \( H(z) \) in (2.22), we get the following:

\[
\left| \left( zD_a p(z) + n \beta \frac{|\alpha| - k}{1 + k} p(z) \right) - \lambda n M \left( z + \beta \frac{|\alpha| - k}{1 + k} \right) \right|
\leq \left| \left( zD_a Q(z) + n \beta \frac{|\alpha| - k}{1 + k} Q(z) \right) - \lambda n M \left( \frac{z}{k} \right)^n \left( \alpha + \beta \frac{|\alpha| - k}{1 + k} \right) \right|. \quad (2.25)
\]

This implies that

\[
\left| zD_a p(z) + n \beta \frac{|\alpha| - k}{1 + k} p(z) \right| - \left| \lambda n M \left( z + \beta \frac{|\alpha| - k}{1 + k} \right) \right|
\leq \left| \left( zD_a Q(z) + n \beta \frac{|\alpha| - k}{1 + k} Q(z) \right) - \lambda n M \left( \frac{z}{k} \right)^n \left( \alpha + \beta \frac{|\alpha| - k}{1 + k} \right) \right|. \quad (2.26)
\]

As \( |p(z)| = |Q(z)| \) for \( |z| = k \), that is, \( \max_{|z|=k}|p(z)| = \max_{|z|=k}|Q(z)| = M \), by Lemma 2.7 for \( Q(z) \), we obtain the following:

\[
\left| zD_a Q(z) + n \beta \frac{|\alpha| - k}{1 + k} Q(z) \right| < |\lambda| n M k^n |\alpha + \beta \frac{|\alpha| - k}{1 + k}|. \quad (2.27)
\]

Thus, taking suitable choice of argument of \( \lambda \), result is

\[
\left| zD_a Q(z) + n \beta \frac{|\alpha| - k}{1 + k} Q(z) \right| - \lambda n M \left( \frac{z}{k} \right)^n \left( \alpha + \beta \frac{|\alpha| - k}{1 + k} \right)
= |\lambda| n M k^n |\alpha + \beta \frac{|\alpha| - k}{1 + k}| - \left| zD_a Q(z) + n \beta \frac{|\alpha| - k}{1 + k} Q(z) \right|. \quad (2.28)
\]

By combining right hand side of (2.26) and (2.28) for \( |z| = 1 \) and \( |\beta| \leq 1 \), we get that

\[
\left| zD_a p(z) + n \beta \frac{|\alpha| - k}{1 + k} p(z) \right| - \left| \lambda n M \left( z + \beta \frac{|\alpha| - k}{1 + k} \right) \right|
\leq |\lambda| |\alpha + \beta \frac{|\alpha| - k}{1 + k}| n k^{-n} M - \left| zD_a Q(z) + n \beta \frac{|\alpha| - k}{1 + k} Q(z) \right|, \quad (2.29)
\]
That is,
\[
|zD_\alpha p(z) + n\beta |\alpha - k| 1 + k p(z)| + |zD_\alpha Q(z) + n\beta |\alpha - k| 1 + k Q(z)| \\
\leq |\lambda| \left( |\alpha + \beta |\alpha - k| 1 + k| k^n + |z + \beta |\alpha - k| 1 + k| \right) nM. \tag{2.30}
\]

Taking $|\lambda| \to 1$, we have
\[
|zD_\alpha p(z) + n\beta |\alpha - k| 1 + k p(z)| + |zD_\alpha Q(z) + n\beta |\alpha - k| 1 + k Q(z)| \\
\leq \left( |\alpha + \beta |\alpha - k| 1 + k| k^n + |z + \beta |\alpha - k| 1 + k| \right) nM. \tag{2.31}
\]

Then, by applying the Principal Maximum Modulus for polynomial $p(z)$ when $k \leq 1$, we get
\[
\max_{|z|=1} |p(z)| \leq \max_{|z|=1} |p(z)|. \tag{2.32}
\]

This in conjunction with (2.31) gives the following result. \hfill \Box

**Lemma 2.9.** Let $H(z)$ be a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$, and $G(z)$ be a polynomial of degree not exceeding that of $H(z)$. If $|G(z)| \leq |H(z)|$ for $|z| = k$, $k \leq 1$, then for all $\alpha$, $\beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| = 1$, we have
\[
|zD_\alpha G(z) + n\beta \left( |\alpha - k| 1 + k \right) G(z)| \leq |zD_\alpha H(z) + n\beta \left( |\alpha - k| 1 + k \right) H(z)|. \tag{2.33}
\]

**Proof.** Since $|\lambda G(z)| \leq |G(z)| \leq |H(z)|$, for $|\lambda| < 1$, and $|z| = k$, then by Rouche’s Theorem $H(z) - \lambda G(z)$ and $H(z)$ have the same number of zeros in $|z| < k$. On the other hand by inequality $|G(z)| \leq |H(z)|$ for $|z| = k$, any zero of $H(z)$, that lies on $|z| = k$, is the zero of $G(z)$. Therefore, $H(z) - \lambda G(z)$ has all its zeros in the closed disk $|z| \leq k$. Hence by Lemma 2.5, for all real or complex numbers $\alpha$ with $|\alpha| \geq k$ and $|z| = 1$, we have
\[
|zD_\alpha (H(z) - \lambda G(z))| \geq n |\alpha - k| 1 + k |H(z) - \lambda G(z)|. \tag{2.34}
\]

Now, consider a similar argument as used in the proof of Lemma 2.6, that for any value $\beta$ with $|\beta| < 1$, we have
\[
|zD_\alpha (H(z) - \lambda G(z))| \geq n |\alpha - k| 1 + k |H(z) - \lambda G(z)| \\
> n |\beta| |\alpha - k| 1 + k |H(z) - \lambda G(z)|, \tag{2.35}
\]

where \(|z| = 1\), resulting in
\[
T(z) = [zD_αH(z) - \lambda zD_αG(z)] + n \beta |\alpha - k| \frac{k}{1 + k}[H(z) - \lambda G(z)] \neq 0,
\] (2.36)
where \(|z| = 1\).
That is,
\[
T(z) = \left[ zD_αH(z) + n \beta |\alpha - k| H(z) \right] - \lambda \left[ zD_αG(z) + n \beta |\alpha - k| G(z) \right] \neq 0,
\] (2.37)
for \(|z| = 1\).
We also conclude that
\[
\left| zD_αH(z) + n \beta |\alpha - k| H(z) \right| \geq \left| zD_αG(z) + n \beta |\alpha - k| G(z) \right|
\] (2.38)
for \(|z| = 1\).
If (2.38) is not true, then there is a point \(z = z_0\) with \(|z_0| = 1\) such that
\[
\left| z_0D_αH(z_0) + n \beta |\alpha - k| H(z_0) \right| < \left| z_0D_αG(z_0) + n \beta |\alpha - k| G(z_0) \right|. \tag{2.39}
\]
Take
\[
\lambda = \frac{z_0D_αH(z_0) + n \beta ((|\alpha - k|)/(1 + k))H(z_0)}{z_0D_αG(z_0) + n \beta ((|\alpha - k|)/(1 + k))G(z_0)},
\] (2.40)
than \(|\lambda| < 1\) and with this choice of \(\lambda\), we have from (2.37), \(T(z_0) = 0\) for \(|z_0| = 1\). But this contradicts the fact that \(T(z) \neq 0\) for \(|z| = 1\). For \(\beta\) with \(|\beta| = 1\), (2.38) follows by continuity. This completes the proof. \(\Box\)

3. Proof of the Theorem

Proof of Theorem 1.1. Under the assumption of Theorem 1.1, the polynomial \(p(z) \neq 0\) in \(|z| < k\), and thus if \(m = \min_{|z|=k}|p(z)|\), then \(m \leq |p(z)|\) for \(|z| \leq k\). Now, for \(\lambda\) with \(|\lambda| < 1\), we have
\[
|\lambda m| < m \leq |p(z)|, \tag{3.1}
\]
where \(|z| = k\).
It follows by Rouche’s Theorem that the polynomial \(G(z) = p(z) - \lambda m\) has no zero in \(|z| < k\). Therefore, the polynomial
\[
H(z) = \left( \frac{z}{k} \right)^n G\left( \frac{k^2}{z} \right) = Q(z) - \lambda m \left( \frac{z}{k} \right)^n, \tag{3.2}
\]
Lemma 2.6 to conclude that for every \(\alpha, \beta\)
Applying Lemma 2.9 for the polynomials \(H(z)\) and \(G(z)\), we have

\[
|zD_\alpha G(z) + n\beta |\alpha - k|\frac{1}{1 + k}G(z)| \leq |zD_\alpha H(z) + n\beta |\alpha - k|\frac{1}{1 + k}H(z)|,
\]

where \(|\alpha| \geq k, |\beta| \leq 1\) and \(|z| = 1\). Substituting for \(G(z)\) and \(H(z)\) in the above inequality, we conclude that for every \(\alpha, \beta\), with \(|\alpha| \geq k, |\beta| \leq 1\), and \(|z| = 1\)

\[
|zD_\alpha p(z) - \lambda nmz + n\beta |\alpha - k|\frac{1}{1 + k}(p(z) - \lambda m)|
\]

\[
\leq |zD_\alpha Q(z) - \lambda nm\left(\frac{z}{k}\right)^n + n\beta |\alpha - k|\frac{1}{1 + k}(Q(z) - \lambda m\left(\frac{z}{k}\right)^n)|,
\]

that is,

\[
|zD_\alpha p(z) + n\beta |\alpha - k|\frac{1}{1 + k}p(z) - \lambda nm\left(z + \beta |\alpha - k|\frac{1}{1 + k}\right)|
\]

\[
\leq |zD_\alpha Q(z) + n\beta |\alpha - k|\frac{1}{1 + k}Q(z) - \lambda nm\left(z + \beta |\alpha - k|\frac{1}{1 + k}\right)^n
\]

Since all the zeros of \(Q(z)\) lie in \(|z| \leq k\) and \(|p(z)| = |Q(z)|\) for \(|z| = k\); therefore, by applying Lemma 2.6 to \(Q(z)\), we have

\[
|zD_\alpha Q(z) + n\beta |\alpha - k|\frac{1}{1 + k}Q(z)| \geq nk^{-n}|\alpha + \beta |\alpha - k|\frac{1}{1 + k}|\min_{|z|=k}|Q(z)|
\]

\[
= nk^{-n}|\alpha + \beta |\alpha - k|\frac{1}{1 + k}|m.
\]

Then, for an appropriate choice of the argument of \(\lambda\), we have

\[
|zD_\alpha Q(z) + n\beta |\alpha - k|\frac{1}{1 + k}Q(z) - \lambda nm\left(z + \beta |\alpha - k|\frac{1}{1 + k}\right)^n
\]

\[
\leq |zD_\alpha Q(z) + n\beta |\alpha - k|\frac{1}{1 + k}Q(z)|
\]

\[
- |\lambda|nmk^{-n}|\alpha + \beta |\alpha - k|\frac{1}{1 + k}|,
\]

where \(|z| = 1\).
Then combining the right hand sides of (3.5) and (3.7), we can rewrite (3.5) as

\[
\left| zD_\alpha p(z) + n\beta \frac{\alpha - k}{1 + k} p(z) \right| - |\lambda| n m \left| z + \beta \frac{\alpha - k}{1 + k} \right|
\leq \left| zD_\alpha Q(z) + n\beta \frac{\alpha - k}{1 + k} Q(z) \right| - |\lambda| n m \left| \alpha + \beta \frac{\alpha - k}{1 + k} \right|
\]

(3.8)

where \( |z| = 1 \).

Equivalently,

\[
\left| zD_\alpha p(z) + n\beta \frac{\alpha - k}{1 + k} p(z) \right| \leq \left| zD_\alpha Q(z) + n\beta \frac{\alpha - k}{1 + k} Q(z) \right|
\]

\[
- |\lambda| n m \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - k}{1 + k} \right| - z + \beta \frac{\alpha - k}{1 + k} \right\}
\]

(3.9)

As \( |\lambda| \to 1 \) we have

\[
\left| zD_\alpha p(z) + n\beta \frac{\alpha - k}{1 + k} p(z) \right| \leq \left| zD_\alpha Q(z) + n\beta \frac{\alpha - k}{1 + k} Q(z) \right|
\]

\[
- n m \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - k}{1 + k} \right| - z + \beta \frac{\alpha - k}{1 + k} \right\}
\]

(3.10)

It implies for every real or complex number \( \beta \) with \( |\beta| \leq 1 \) and \( |z| = 1 \),

\[
2 \left| zD_\alpha p(z) + n\beta \frac{\alpha - k}{1 + k} p(z) \right| \leq \left| zD_\alpha p(z) + n\beta \frac{\alpha - k}{1 + k} p(z) \right|
\]

\[
+ \left| zD_\alpha Q(z) + n\beta \frac{\alpha - k}{1 + k} Q(z) \right|
\]

\[
- n m \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - k}{1 + k} \right| - z + \beta \frac{\alpha - k}{1 + k} \right\}
\]

(3.11)

This in conjunction with Lemma 2.8 gives for \( |\beta| \leq 1 \) and \( |z| = 1 \),

\[
2 \left| zD_\alpha p(z) + n\beta \frac{\alpha - k}{1 + k} p(z) \right| \leq n \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - k}{1 + k} \right| + \left| z + \beta \frac{\alpha - k}{1 + k} \right| \right\} \max_{|z|=1} |p(z)|
\]

\[
- n \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - k}{1 + k} \right| - \left| z + \beta \frac{\alpha - k}{1 + k} \right| \right\} \min_{|z|=k} |p(z)|.
\]

(3.12)

The proof is complete. \( \square \)
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References


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