Research Article

Inequalities for the Polar Derivative of a Polynomial

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For a polynomial $p(z)$ of degree n , we consider an operator D_α which map a polynomial $p(z)$ into $D_\alpha p(z) := (\alpha - z)p'(z) + np(z)$ with respect to α . It was proved by Liman et al. (2010) that if $p(z)$ has no zeros in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$, $|zD_\alpha p(z) + n\beta((|\alpha| - 1)/2)p(z)| \leq (n/2)\{[|\alpha + \beta((|\alpha| - 1)/2)| + |z + \beta((|\alpha| - 1)/2)|]\max_{|z|=1}|p(z)| - [|\alpha + \beta((|\alpha| - 1)/2)| - |z + \beta((|\alpha| - 1)/2)|]\min_{|z|=1}|p(z)|\}$. In this paper we extend the above inequality for the polynomials having no zeros in $|z| < k$, where $k \leq 1$. Our result generalizes certain well-known polynomial inequalities.

1. Introduction and Statement of Results

According to a result well known as Bernstein's inequality on the derivative of a polynomial $p(z)$ of degree n , we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \quad (1.1)$$

The result is best possible, and equality holds for a polynomial having all its zeros at the origin (see [1, 2]).

The inequality (1.1) can be sharpened, by considering the class of polynomials having no zeros in $|z| < 1$.

In fact, P. Erdős conjectured, and later Lax [3] proved that if $p(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (1.2)$$

As a refinement of (1.2), Aziz and Dawood [4] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}. \quad (1.3)$$

As an improvement of (1.3), Dewan and Hans [5] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for any β with $|\beta| \leq 1$ and $|z| = 1$,

$$\left| zp'(z) + \frac{n\beta}{2} p(z) \right| \leq \frac{n}{2} \left\{ \left(\left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right) \max_{|z|=1} |p(z)| - \left(\left| 1 + \frac{\beta}{2} \right| - \left| \frac{\beta}{2} \right| \right) \min_{|z|=1} |p(z)| \right\}. \quad (1.4)$$

Let α be a complex number. For a polynomial $p(z)$ of degree n , $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z). \quad (1.5)$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and that $D_\alpha p(z)$ generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \rightarrow \infty} \left[\frac{D_\alpha p(z)}{\alpha} \right] = p'(z). \quad (1.6)$$

As an extension to (1.1) for the polar derivative $D_\alpha p(z)$, Aziz and Shah [6] proved that if $p(z)$ is a polynomial of degree n , then for every α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq n|\alpha| \max_{|z|=1} |p(z)|. \quad (1.7)$$

As a refinement and extension of (1.7), Aziz and Mohammad Shah [7] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then, for every α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{2} \left\{ (|\alpha| + 1) \max_{|z|=1} |p(z)| - (|\alpha| - 1) \min_{|z|=1} |p(z)| \right\}. \quad (1.8)$$

Recently Dewan et al. [8] generalized (1.8) to the polynomial of the form $p(z) = a_0 + \sum_{v=t}^n a_v z^v$, $1 \leq t \leq n$ and proved that if $p(z) = a_0 + \sum_{v=t}^n a_v z^v$, $1 \leq t \leq n$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $|\alpha| \geq 1$

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{1 + s_0} \left\{ (|\alpha| + s_0) \max_{|z|=1} |p(z)| - (|\alpha| - 1) \min_{|z|=k} |p(z)| \right\}, \quad (1.9)$$

where $s_0 = k^{t+1} \{ (((t/n)(|a_t|/(|a_0| - m)))k^{t-1} + 1) / (((t/n)(|a_t|/(|a_0| - m)))k^{t+1} + 1) \}$, and $m = \min_{|z|=k} |p(z)|$.

As a generalization of (1.9), Bidkham et al. [9] proved that if $p(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $0 < r \leq R \leq k$ and $|\alpha| \geq R$

$$\begin{aligned} \max_{|z|=R} |D_\alpha p(z)| &\leq \frac{n}{1+s'_0} \left\{ \left(\frac{|\alpha|}{R} + s'_0 \right) \exp \left\{ n \int_r^R A_t dt \right\} \max_{|z|=r} |p(z)| \right. \\ &\quad \left. + s'_0 + 1 - \left(\frac{|\alpha|}{R} + s'_0 \right) \exp \left\{ n \int_r^R A_t dt \right\} \min_{|z|=k} |p(z)| \right\}, \end{aligned} \quad (1.10)$$

where

$$\begin{aligned} A_t &= \frac{(\mu/n)(|a_\mu|/(|a_0| - m))k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + (\mu/n)(|a_\mu|/(|a_0| - m))(k^{\mu+1}t^\mu + k^2\mu t)}, \\ s'_0 &= \left(\frac{k}{R} \right)^{\mu+1} \left\{ \frac{(\mu/n)(|a_\mu|/(|a_0| - m))Rk^{\mu-1} + 1}{(\mu/n)(|a_\mu|/(R(|a_0| - m)))k^{\mu+1} + 1} \right\}, \\ m &= \min_{|z|=k} |p(z)|. \end{aligned} \quad (1.11)$$

As an improvement and generalization to (1.8) and (1.4), Liman et al. [10] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then, for all α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} \left| z D_\alpha p(z) + n\beta \frac{|\alpha| - 1}{2} p(z) \right| &\leq \frac{n}{2} \left\{ \left(\left| \alpha + \beta \frac{|\alpha| - 1}{2} \right| + \left| z + \beta \frac{|\alpha| - 1}{2} \right| \right) \max_{|z|=1} |p(z)| \right. \\ &\quad \left. - \left(\left| \alpha + \beta \frac{|\alpha| - 1}{2} \right| - \left| z + \beta \frac{|\alpha| - 1}{2} \right| \right) \min_{|z|=1} |p(z)| \right\}. \end{aligned} \quad (1.12)$$

In this paper, we obtain the following extension of (1.12).

Theorem 1.1. *Let $p(z)$ be a polynomial of degree n that does not vanish in $|z| < k$, $k \leq 1$, then, for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\begin{aligned} \left| z D_\alpha p(z) + n\beta \frac{|\alpha| - k}{1+k} p(z) \right| &\leq \frac{n}{2} \left\{ \left(k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1+k} \right| + \left| z + \beta \frac{|\alpha| - k}{1+k} \right| \right) \max_{|z|=1} |p(z)| \right. \\ &\quad \left. - \left(k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1+k} \right| - \left| z + \beta \frac{|\alpha| - k}{1+k} \right| \right) \min_{|z|=k} |p(z)| \right\}. \end{aligned} \quad (1.13)$$

If we take $k = 1$ in Theorem 1.1, then (1.13) reduces to (1.12).

Theorem 1.1 simplifies to the following result by taking $\beta = 0$.

Corollary 1.2. Let $p(z)$ be a polynomial of degree n does not vanish in $|z| < k$, $k \leq 1$, then for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, we have

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{2} \left\{ (k^{-n}|\alpha| + 1) \max_{|z|=1} |p(z)| - (k^{-n}|\alpha| - 1) \min_{|z|=k} |p(z)| \right\}. \quad (1.14)$$

If we take $k = 1$ in Corollary 1.2, then (1.14) reduce to (1.8).

Dividing two sides of inequality (1.13) by $|\alpha|$ and letting $|\alpha| \rightarrow \infty$, we have the following generalization of the inequality (1.4).

Corollary 1.3. Let $p(z)$ be a polynomial of degree n , having no zeros in $|z| < k$, $k \leq 1$, then, for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|z| = 1$, we have

$$\left| zp'(z) + \frac{n\beta}{1+k} p(z) \right| \leq \frac{n}{2} \left\{ \left(k^{-n} \left| 1 + \frac{\beta}{1+k} \right| + \left| \frac{\beta}{1+k} \right| \right) \max_{|z|=1} |p(z)| - \left(k^{-n} \left| 1 + \frac{\beta}{1+k} \right| - \left| \frac{\beta}{1+k} \right| \right) \min_{|z|=k} |p(z)| \right\}. \quad (1.15)$$

Taking $\beta = 0$ and $k = 1$ in Corollary 1.3, (1.15) reduces to (1.3).

2. Lemmas

For proof of the theorem, we need the following lemmas. The first lemma is due to Laguerre [11, 12].

Lemma 2.1. If all the zeros of an n th degree polynomial $p(z)$ lie in a circular region C , and w is any zero of $D_\alpha p(z)$, then at most one of the points w and α may lie outside C .

Lemma 2.2. If $p(z)$ is a polynomial of degree n , having all its zeros in the closed disk $|z| \leq k$, $k \leq 1$, then on $|z| = 1$

$$|p'(z)| \geq \frac{n}{1+k} |p(z)|. \quad (2.1)$$

This lemma is due to Malik [13].

Lemma 2.3. Let $p(z)$ be a polynomial of degree n and have no zero in $|z| < k$, $k \geq 1$, then on $|z| = 1$

$$k|p'(z)| \leq |q'(z)|, \quad (2.2)$$

where $q(z) = z^n \overline{p(1/\bar{z})}$.

The above lemma is due to Chan and Malik [14].

Lemma 2.4. *If $p(z)$ is a polynomial of degree n , having all its zeros in the closed disk $|z| \leq k$, $k \leq 1$, then on $|z| = 1$*

$$|q'(z)| \leq k|p'(z)|, \quad (2.3)$$

where $q(z) = z^n \overline{p(1/\bar{z})}$.

Proof. Since $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$; therefore, $q(z)$ has no zero in $|z| < 1/k$, $1/k \geq 1$. Now applying Lemma 2.3 to the polynomial $q(z)$ and the result follows. $\square$

Lemma 2.5. *If $p(z)$ is a polynomial of degree n , having all its zeros in the closed disk $|z| \leq k$, $k \leq 1$, then for all real or complex number α with $|\alpha| \geq k$ and $|z| = 1$, we have*

$$|D_\alpha p(z)| \geq n \frac{|\alpha| - k}{1 + k} |p(z)|. \quad (2.4)$$

Proof. Let $q(z) = z^n \overline{p(1/\bar{z})}$, then $|q'(z)| = |np(z) - zp'(z)|$ on $|z| = 1$. Thus on $|z| = 1$

$$\begin{aligned} |D_\alpha p(z)| &= |np(z) + (\alpha - z)p'(z)| \\ &= |\alpha p'(z) + np(z) - zp'(z)| \\ &\geq |\alpha p'(z)| - |np(z) - zp'(z)|, \end{aligned} \quad (2.5)$$

which implies that

$$|D_\alpha p(z)| \geq |\alpha||p'(z)| - |q'(z)|. \quad (2.6)$$

Combining (2.3) and (2.6), we get the following:

$$|D_\alpha p(z)| \geq (|\alpha| - k)|p'(z)|, \quad (2.7)$$

along with Lemma 2.2, which gives the following:

$$|D_\alpha p(z)| \geq n \frac{|\alpha| - k}{1 + k} |p(z)|. \quad (2.8)$$

$\square$

Lemma 2.6. *Let $p(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$. Then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\left| zD_\alpha p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| \geq nk^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| \min_{|z|=k} |p(z)|. \quad (2.9)$$

Proof. If $p(z)$ has a zero on $|z| = k$, then (2.9) is trivial. Therefore, we assume that $p(z)$ has all its zeros in $|z| < k$. Let $m = \min_{|z|=k} |p(z)|$, then $m > 0$ and $|p(z)| \geq m$ where $|z| = k$. Therefore,

for $|\lambda| < 1$, it follows by Rouché's Theorem that the polynomial $G(z) = p(z) - \lambda m(z/k)^n$ has all its zeros in $|z| < k$. By using Lemma 2.1, $D_\alpha G(z) = D_\alpha p(z) - \alpha \lambda m n (z^{n-1}/k^n)$ has all its zeros in $|z| < k$, where $|\alpha| \geq k$. Applying Lemma 2.5 to the polynomial $G(z)$ yields

$$|z D_\alpha G(z)| \geq n \frac{|\alpha| - k}{1 + k} |G(z)|, \quad |z| = 1. \quad (2.10)$$

Since $z D_\alpha G(z)$ has all its zeros in $|z| < k \leq 1$, by using Rouché's Theorem, it can be easily verified from (2.10) that the polynomial

$$z D_\alpha G(z) + \beta n \frac{|\alpha| - k}{1 + k} G(z) \quad (2.11)$$

has all its zeros in $|z| < 1$, where $|\beta| < 1$.

Substituting for $G(z)$, we conclude that the polynomial

$$T(z) = \left(z D_\alpha p(z) + n \beta \frac{|\alpha| - k}{1 + k} p(z) \right) - \lambda m n \left(\frac{z}{k} \right)^n \left(\alpha + \beta \frac{|\alpha| - k}{1 + k} \right) \quad (2.12)$$

will have no zeros in $|z| \geq 1$. This implies for every β with $|\beta| < 1$ and $|z| \geq 1$,

$$\left| z D_\alpha p(z) + n \beta \frac{|\alpha| - k}{1 + k} p(z) \right| \geq n m \left| \frac{z}{k} \right|^n \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right|. \quad (2.13)$$

If (2.13) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\left| z_0 D_\alpha p(z_0) + n \beta \frac{|\alpha| - k}{1 + k} p(z_0) \right| < n m \left| \frac{z_0}{k} \right|^n \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right|. \quad (2.14)$$

Take

$$\lambda = \frac{z_0 D_\alpha p(z_0) + n \beta ((|\alpha| - k)/(1 + k)) p(z_0)}{n m (z_0/k)^n (\alpha + \beta ((|\alpha| - k)/(1 + k)))}, \quad (2.15)$$

then $|\lambda| < 1$ and with this choice of λ , we have $T(z_0) = 0$ for $|z_0| \geq 1$, from (2.12). But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. For β with $|\beta| = 1$, (2.13) follows by continuity. This completes the proof of Lemma 2.6. $\square$

Lemma 2.7. *If $p(z)$ is a polynomial of degree n , then for all $\alpha, \beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|\alpha| \geq k$, where $k \leq 1$, we have*

$$\left| z D_\alpha p(z) + n \beta \frac{|\alpha| - k}{1 + k} p(z) \right| \leq n k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| \max_{|z|=k} |p(z)|, \quad |z| = 1. \quad (2.16)$$

Proof. Let $M = \max_{|z|=k} |p(z)|$, if $|\lambda| < 1$, then $|\lambda p(z)| < |M(z/k)^n|$ for $|z| = k$. Therefore, it follows by Rouché's Theorem that the polynomial $G(z) = M(z/k)^n - \lambda p(z)$ has all its zeros

in $|z| < k$. By using Lemma 2.1, $D_\alpha G(z) = \alpha M n(z^{n-1}/k^n) - \lambda D_\alpha p(z)$ has all its zeros in $|z| < k$ for $|\alpha| \geq k$.

On applying Lemma 2.5 to the polynomial $G(z)$, we have

$$|zD_\alpha G(z)| \geq n \frac{|\alpha| - k}{1 + k} |G(z)|, \quad |z| = 1. \quad (2.17)$$

Now, using a similar argument as used in the proof of Lemma 2.6, the result follows. $\square$

Lemma 2.8. *If $p(z)$ is a polynomial of degree n , then for all $\alpha, \beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|\alpha| \geq k$, where $k \leq 1$, we have*

$$\begin{aligned} & \left| zD_\alpha p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| + \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| \\ & \leq n \left\{ k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| + \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right\} \max_{|z|=1} |p(z)|, \quad |z| = 1, \end{aligned} \quad (2.18)$$

where $Q(z) = (z/k)^n \overline{p(k^2/\bar{z})}$.

Proof. Let $M = \max_{|z|=k} |p(z)|$. For λ with $|\lambda| > 1$, it follows by Rouché's Theorem that the polynomial $G(z) = p(z) - \lambda M$ has no zeros in $|z| < k$. Consequently the polynomial

$$H(z) = \left(\frac{z}{k} \right)^n \overline{G\left(\frac{k^2}{\bar{z}} \right)} \quad (2.19)$$

has all its zeros in $|z| \leq k$, also $|G(z)| = |H(z)|$ for $|z| = k$. Since all the zeros of $H(z)$ lie in $|z| \leq k$; therefore, for δ with $|\delta| > 1$, by Rouché's Theorem all the zeros of $G(z) + \delta H(z)$ lie in $|z| \leq k$. Hence by Lemma 2.5 for every α with $|\alpha| \geq k$, and $|z| = 1$, we have

$$n \frac{|\alpha| - k}{1 + k} |G(z) + \delta H(z)| \leq |zD_\alpha (G(z) + \delta H(z))|. \quad (2.20)$$

On the other hand by Lemma 2.1, all the zeros of $D_\alpha (G(z) + \delta H(z))$ lie in $|z| < k \leq 1$, where $|\alpha| \geq k$. Therefore, for any β with $|\beta| \leq 1$, Rouché's Theorem implies that all the zeros of $zD_\alpha (G(z) + \delta H(z)) + \beta n((|\alpha| - k)/(1 + k))(G(z) + \delta H(z))$ lie in $|z| < 1$. This means that the polynomial

$$T(z) = zD_\alpha G(z) + n\beta \frac{|\alpha| - k}{1 + k} G(z) + \delta \left(zD_\alpha H(z) + n\beta \frac{|\alpha| - k}{1 + k} H(z) \right) \quad (2.21)$$

will have no zeros in $|z| \geq 1$. Now using a similar argument as used in the proof of Lemma 2.6, we get for $|z| \geq 1$,

$$\left| zD_\alpha G(z) + n\beta \frac{|\alpha| - k}{1 + k} G(z) \right| \leq \left| zD_\alpha H(z) + n\beta \frac{|\alpha| - k}{1 + k} H(z) \right|. \quad (2.22)$$

Therefore by the equalities

$$H(z) = \left(\frac{z}{k}\right)^n \overline{G\left(\frac{k^2}{z}\right)} = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{z}\right)} - \bar{\lambda} M \left(\frac{z}{k}\right)^n = Q(z) - \bar{\lambda} M \left(\frac{z}{k}\right)^n, \quad (2.23)$$

or

$$H(z) = Q(z) - \bar{\lambda} M \left(\frac{z}{k}\right)^n, \quad (2.24)$$

and substitute for $G(z)$ and $H(z)$ in (2.22), we get the following:

$$\begin{aligned} & \left| \left(z D_\alpha p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right) - \lambda n M \left(z + \beta \frac{|\alpha| - k}{1 + k} \right) \right| \\ & \leq \left| \left(z D_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right) - \bar{\lambda} n M \left(\frac{z}{k} \right)^n \left(\alpha + \beta \frac{|\alpha| - k}{1 + k} \right) \right|. \end{aligned} \quad (2.25)$$

This implies that

$$\begin{aligned} & \left| z D_\alpha p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| - \left| \lambda n M \left(z + \beta \frac{|\alpha| - k}{1 + k} \right) \right| \\ & \leq \left| \left(z D_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right) - \bar{\lambda} n M \left(\frac{z}{k} \right)^n \left(\alpha + \beta \frac{|\alpha| - k}{1 + k} \right) \right|. \end{aligned} \quad (2.26)$$

As $|p(z)| = |Q(z)|$ for $|z| = k$, that is, $\max_{|z|=k} |p(z)| = \max_{|z|=k} |Q(z)| = M$, by Lemma 2.7 for $Q(z)$, we obtain the following:

$$\left| z D_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| < |\lambda| n M k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right|. \quad (2.27)$$

Thus, taking suitable choice of argument of λ , result is

$$\begin{aligned} & \left| \left(z D_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right) - \bar{\lambda} n M \left(\frac{z}{k} \right)^n \left(\alpha + \beta \frac{|\alpha| - k}{1 + k} \right) \right| \\ & = |\lambda| n M k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| - \left| z D_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right|. \end{aligned} \quad (2.28)$$

By combining right hand side of (2.26) and (2.28) for $|z| = 1$ and $|\beta| \leq 1$, we get that

$$\begin{aligned} & \left| z D_\alpha p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| - \left| \lambda n M \left(z + \beta \frac{|\alpha| - k}{1 + k} \right) \right| \\ & \leq |\lambda| \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| n k^{-n} M - \left| z D_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right|, \end{aligned} \quad (2.29)$$

That is,

$$\begin{aligned} & \left| zD_\alpha p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| + \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| \\ & \leq |\lambda| \left\{ \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| k^{-n} + \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right\} nM. \end{aligned} \quad (2.30)$$

Taking $|\lambda| \rightarrow 1$, we have

$$\begin{aligned} & \left| zD_\alpha p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| + \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| \\ & \leq \left\{ \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| k^{-n} + \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right\} nM. \end{aligned} \quad (2.31)$$

Then, by applying the Principal Maximum Modulus for polynomial $p(z)$ when $k \leq 1$, we get

$$\max_{|z|=k} |p(z)| \leq \max_{|z|=1} |p(z)|. \quad (2.32)$$

This in conjunction with (2.31) gives the following result. □

Lemma 2.9. *Let $H(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, and $G(z)$ be a polynomial of degree not exceeding that of $H(z)$. If $|G(z)| \leq |H(z)|$ for $|z| = k$, $k \leq 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\left| zD_\alpha G(z) + n\beta \left(\frac{|\alpha| - k}{1 + k} \right) G(z) \right| \leq \left| zD_\alpha H(z) + n\beta \left(\frac{|\alpha| - k}{1 + k} \right) H(z) \right|. \quad (2.33)$$

Proof. Since $|\lambda G(z)| < |G(z)| \leq |H(z)|$, for $|\lambda| < 1$, and $|z| = k$, then by Rouché's Theorem $H(z) - \lambda G(z)$ and $H(z)$ have the same number of zeros in $|z| < k$. On the other hand by inequality $|G(z)| \leq |H(z)|$ for $|z| = k$, any zero of $H(z)$, that lies on $|z| = k$, is the zero of $G(z)$. Therefore, $H(z) - \lambda G(z)$ has all its zeros in the closed disk $|z| \leq k$. Hence by Lemma 2.5, for all real or complex numbers α with $|\alpha| \geq k$ and $|z| = 1$, we have

$$|zD_\alpha (H(z) - \lambda G(z))| \geq n \frac{|\alpha| - k}{1 + k} |H(z) - \lambda G(z)|. \quad (2.34)$$

Now, consider a similar argument as used in the proof of Lemma 2.6, that for any value β with $|\beta| < 1$, we have

$$\begin{aligned} |zD_\alpha (H(z) - \lambda G(z))| & \geq n \frac{|\alpha| - k}{1 + k} |H(z) - \lambda G(z)| \\ & > n |\beta| \frac{|\alpha| - k}{1 + k} |H(z) - \lambda G(z)|, \end{aligned} \quad (2.35)$$

where $|z| = 1$, resulting in

$$T(z) = [zD_\alpha H(z) - \lambda zD_\alpha G(z)] + n\beta \frac{|\alpha| - k}{1 + k} [H(z) - \lambda G(z)] \neq 0, \quad (2.36)$$

where $|z| = 1$.

That is,

$$T(z) = \left[zD_\alpha H(z) + n\beta \frac{|\alpha| - k}{1 + k} H(z) \right] - \lambda \left[zD_\alpha G(z) + n\beta \frac{|\alpha| - k}{1 + k} G(z) \right] \neq 0, \quad (2.37)$$

for $|z| = 1$.

We also conclude that

$$\left| zD_\alpha H(z) + n\beta \frac{|\alpha| - k}{1 + k} H(z) \right| \geq \left| zD_\alpha G(z) + n\beta \frac{|\alpha| - k}{1 + k} G(z) \right| \quad (2.38)$$

for $|z| = 1$.

If (2.38) is not true, then there is a point $z = z_0$ with $|z_0| = 1$ such that

$$\left| z_0 D_\alpha H(z_0) + n\beta \frac{|\alpha| - k}{1 + k} H(z_0) \right| < \left| z_0 D_\alpha G(z_0) + n\beta \frac{|\alpha| - k}{1 + k} G(z_0) \right|. \quad (2.39)$$

Take

$$\lambda = \frac{z_0 D_\alpha H(z_0) + n\beta((|\alpha| - k)/(1 + k))H(z_0)}{z_0 D_\alpha G(z_0) + n\beta((|\alpha| - k)/(1 + k))G(z_0)}, \quad (2.40)$$

then $|\lambda| < 1$ and with this choice of λ , we have from (2.37), $T(z_0) = 0$ for $|z_0| = 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| = 1$. For β with $|\beta| = 1$, (2.38) follows by continuity. This completes the proof. $\square$

3. Proof of the Theorem

Proof of the Theorem 1.1. Under the assumption of Theorem 1.1, the polynomial $p(z) \neq 0$ in $|z| < k$, and thus if $m = \min_{|z|=k} |p(z)|$, then $m \leq |p(z)|$ for $|z| \leq k$. Now, for λ with $|\lambda| < 1$, we have

$$|\lambda m| < m \leq |p(z)|, \quad (3.1)$$

where $|z| = k$.

It follows by Rouché's Theorem that the polynomial $G(z) = p(z) - \lambda m$ has no zero in $|z| < k$. Therefore, the polynomial

$$H(z) = \left(\frac{z}{k} \right)^n \overline{G\left(\frac{k^2}{\bar{z}} \right)} = Q(z) - \bar{\lambda} m \left(\frac{z}{k} \right)^n, \quad (3.2)$$

will have all its zeros in $|z| \leq k$, where $Q(z) = (z/k)^n \overline{p(k^2/\overline{z})}$. Also $|G(z)| = |H(z)|$ for $|z| = k$. Applying Lemma 2.9 for the polynomials $H(z)$ and $G(z)$, we have

$$\left| zD_\alpha G(z) + n\beta \frac{|\alpha| - k}{1 + k} G(z) \right| \leq \left| zD_\alpha H(z) + n\beta \frac{|\alpha| - k}{1 + k} H(z) \right|, \quad (3.3)$$

where $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| = 1$. Substituting for $G(z)$ and $H(z)$ in the above inequality, we conclude that for every α , β , with $|\alpha| \geq k$, $|\beta| \leq 1$, and $|z| = 1$

$$\begin{aligned} & \left| zD_\alpha p(z) - \lambda nmz + n\beta \frac{|\alpha| - k}{1 + k} (p(z) - \lambda m) \right| \\ & \leq \left| zD_\alpha Q(z) - \bar{\lambda} \alpha nm \left(\frac{z}{k} \right)^n + n\beta \frac{|\alpha| - k}{1 + k} \left(Q(z) - \bar{\lambda} m \left(\frac{z}{k} \right)^n \right) \right|, \end{aligned} \quad (3.4)$$

that is,

$$\begin{aligned} & \left| zD_\alpha p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) - \lambda nm \left(z + \beta \frac{|\alpha| - k}{1 + k} \right) \right| \\ & \leq \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) - \bar{\lambda} nm \left(\frac{z}{k} \right)^n \left(\alpha + \beta \frac{|\alpha| - k}{1 + k} \right) \right|. \end{aligned} \quad (3.5)$$

Since all the zeros of $Q(z)$ lie in $|z| \leq k$ and $|p(z)| = |Q(z)|$ for $|z| = k$; therefore, by applying Lemma 2.6 to $Q(z)$, we have

$$\begin{aligned} \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| & \geq nk^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| \min_{|z|=k} |Q(z)| \\ & = nk^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| m. \end{aligned} \quad (3.6)$$

Then, for an appropriate choice of the argument of λ , we have

$$\begin{aligned} \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) - \bar{\lambda} nm \left(\frac{z}{k} \right)^n \left(\alpha + \beta \frac{|\alpha| - k}{1 + k} \right) \right| & = \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| \\ & \quad - |\lambda| nmk^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right|, \end{aligned} \quad (3.7)$$

where $|z| = 1$.

Then combining the right hand sides of (3.5) and (3.7), we can rewrite (3.5) as

$$\begin{aligned} & \left| zD_{\alpha}p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| - |\lambda|nm \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \\ & \leq \left| zD_{\alpha}Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| - |\lambda|nmk^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right|, \end{aligned} \quad (3.8)$$

where $|z| = 1$.

Equivalently,

$$\begin{aligned} & \left| zD_{\alpha}p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| \leq \left| zD_{\alpha}Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| \\ & \quad - |\lambda|nm \left\{ k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| - \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right\}. \end{aligned} \quad (3.9)$$

As $|\lambda| \rightarrow 1$ we have

$$\begin{aligned} & \left| zD_{\alpha}p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| \leq \left| zD_{\alpha}Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| \\ & \quad - nm \left\{ k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| - \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right\}. \end{aligned} \quad (3.10)$$

It implies for every real or complex number β with $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} & 2 \left| zD_{\alpha}p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| \leq \left| zD_{\alpha}p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| \\ & \quad + \left| zD_{\alpha}Q(z) + n\beta \frac{|\alpha| - k}{1 + k} Q(z) \right| \\ & \quad - nm \left\{ k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| - \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right\}. \end{aligned} \quad (3.11)$$

This in conjunction with Lemma 2.8 gives for $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} & 2 \left| zD_{\alpha}p(z) + n\beta \frac{|\alpha| - k}{1 + k} p(z) \right| \leq n \left\{ k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| + \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right\} \max_{|z|=1} |p(z)| \\ & \quad - n \left\{ k^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1 + k} \right| - \left| z + \beta \frac{|\alpha| - k}{1 + k} \right| \right\} \min_{|z|=k} |p(z)|. \end{aligned} \quad (3.12)$$

The proof is complete. $\square$

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