Research Article

Parallel and Cyclic Algorithms for Quasi-Nonexpansives in Hilbert Space

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1. Introduction

Throughout this paper, we always assume that C is a nonempty, closed, and convex subset of a real Hilbert space H. Let A : C → H be a nonlinear mapping. Recall the following definitions.

(1) A is said to be monotone if

\[ \langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \]  \hspace{1cm} (1.1)

(2) A is said to be strongly positive if there exists a constant \( \gamma > 0 \) such that

\[ \langle Ax, x \rangle \geq \gamma \|x\|^2, \quad \forall x \in C. \]  \hspace{1cm} (1.2)
(3) $A$ is said to be strongly monotone if there exists a constant $\alpha > 0$ such that
\[ \langle Ax - Ay, x - y \rangle \geq \alpha \| x - y \|^2, \quad \forall x, y \in C. \quad (1.3) \]

For such a case, $A$ is said to be $\alpha$-strongly monotone.

(4) $A$ is said to be inverse strongly if there exists a constant $\alpha > 0$ such that
\[ \langle Ax - Ay, x - y \rangle \geq \alpha \| Ax - Ay \|^2, \quad \forall x, y \in C. \quad (1.4) \]

For such a case, $A$ is said to be $\alpha$-inverse-strongly-monotone ($\alpha$-ism).

Assume $A$ is strongly positive operator, that is, there is a constant $\gamma$ with the property
\[ \langle Ax, x \rangle \geq \gamma \| x \|^2, \quad \forall x \in H. \quad (1.5) \]

Remark 1.1. Let $F = A - \gamma f$, where $A$ is strongly positive operator, and $f$ is contraction mapping with coefficient $\beta \in (0, 1)$. It is a simple matter to see that the operator $F$ is $(\bar{\gamma} - \gamma \beta)$-strongly monotone over $C$, that is,
\[ \langle Fx - Fy, x - y \rangle \geq (\bar{\gamma} - \gamma \beta) \| x - y \|^2, \quad \forall (x, y) \in C \times C. \quad (1.6) \]

The classical variational inequality which is denoted by $\text{VI}(A, C)$ is to find $x \in C$ such that
\[ \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \quad (1.7) \]

The variational inequality has been extensively studied in literature; see, for example, [1, 2] and the reference therein. A mapping $T : C \to C$ is said to be a strict pseudocontraction [3] if there exists a constant $0 \leq k < 1$ such that
\[ \| Tx - Ty \|^2 \leq \| x - y \|^2 + k \| (I - T)x - (I - T)y \|^2, \quad (1.8) \]
for all $x, y \in C$ (If (1.8) holds, we also say that $T$ is a $k$-strict pseudo-contraction). These mappings are extensions of nonexpansive mappings which satisfy the inequality (1.8) with $k = 0$. That is, $T : C \to C$ is nonexpansive if
\[ \| Tx - Ty \| \leq \| x - y \|, \quad \forall x, y \in C. \quad (1.9) \]

In [4], Xu proved that the sequence $\{ x_n \}$ defined by the iterative method below with the initial guess $x_0 \in H$ chosen arbitrarily,
\[ x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad n \geq 0, \quad (1.10) \]
where the sequence $\{\alpha_n\}$ satisfies certain conditions, he proved the sequence $\{x_n\}$ converges strongly to the unique solution of the following minimization problem:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle.$$  \hfill (1.11)

In [5], Marino and Xu considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0,$$  \hfill (1.12)

they proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.12) converges strongly to the unique solution of the variational inequality

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad x \in C,$$  \hfill (1.13)

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x),$$  \hfill (1.14)

where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). Some people also study the applications of the iterative method (1.12) [6, 7].

Acedo and Xu [8] consider the following parallel and cyclic algorithms:

**Parallel Algorithm**

The sequence $\{x_n\}$ was generated by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n,$$  \hfill (1.15)

where $\{T_i\}_{i=1}^{N}$ are $N$ strict pseudocontractions defined on a closed convex subset $C$ of a Hilbert space $H$. Under the following assumptions on the sequences of the weights $\{\lambda_i^{(n)}\}_{i=1}^{N}$:

(a1) $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all $n$ and $\inf_{n \geq 1} \lambda_i^{(n)} > 0$, for all $1 \leq i \leq N$,

(a2) $\sum_{i=1}^{N} \sqrt{\sum_{n=1}^{N} |\lambda_i^{(n+1)} - \lambda_i^{(n)}|} < \infty$.

By (1.15), they will prove the weak convergence to a solution of the problem $x \in \bigcap_{i=1}^{N} F_{ix}(T_i)$. 
Cyclic Algorithm

They define the sequence \( \{ x_n \} \) cyclically by

\[
x_1 = \alpha_0 x_0 + (1 - \alpha_0) T_0 x_0;
\]

\[
x_2 = \alpha_1 x_1 + (1 - \alpha_1) T_1 x_1;
\]

\[
\vdots
\]

\[
x_N = \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1}) T_{N-1} x_{N-1};
\]

\[
x_{N+1} = \alpha_N x_N + (1 - \alpha_N) T_0 x_N;
\]

\[
\vdots
\]

They define the sequence \( \{ x_n \} \) cyclically by

\[
x_1 = \alpha_0 x_0 + (1 - \alpha_0) T_0 x_0;
\]

\[
x_2 = \alpha_1 x_1 + (1 - \alpha_1) T_1 x_1;
\]

\[
\vdots
\]

\[
x_N = \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1}) T_{N-1} x_{N-1};
\]

\[
x_{N+1} = \alpha_N x_N + (1 - \alpha_N) T_0 x_N;
\]

\[
\vdots
\]

In a more compact form, they are rewritten \( x_{n+1} \) as

\[
x_{N+1} = \alpha_N x_N + (1 - \alpha_N) T_N x_n,
\]

where \( \{ T_i \}_{i=1}^N \) are \( k_i \)-strict pseudo-contractions and \( T_N = T_i \) with \( i = n \) (mod \( N \)), \( 0 \leq i \leq N - 1 \). They show that this cyclic algorithm (1.17) is weakly convergent if the sequence \( \{ \alpha_n \} \) of parameters is appropriately chosen. On the other hand, Osilike and Shehu [9] also consider the cyclic algorithm (1.17), under appropriate assumptions on the sequences of \( \{ \alpha_n \} \), some strong convergence theorems are proved.

In this paper, we are concerned with the problem of finding a point \( x \) such that

\[
x \in \bigcap_{i=1}^N F_{ix}(T_{\omega_i}), \quad N \geq 1,
\]

where \( T_{\omega_i} = (1 - \omega_i) I + \omega_i T_i \), \( \{ \omega_i \}_{i=1}^N \in (0,1] \) and \( \{ T_i \}_{i=1}^N \) are quasi-nonexpansive mappings defined on a closed convex subset \( C \) of a Hilbert space \( H \). Here \( F_{ix}(T_{\omega_i}) = \{ q \in C : T_{\omega_i} q = q \} \) is the set of fixed points of \( T_i \), \( 1 \leq i \leq N \).

Let \( T \) be defined by

\[
T = \sum_{i=1}^N \lambda_i T_{\omega_i},
\]

where \( \lambda_i > 0 \) for all \( i \in (0,1) \) such that \( \sum_{i=1}^N \lambda_i = 1 \). Motivated and inspired by Acedo and Xu [8], we consider the following two general iterative algorithms for a family of quasi-nonexpansive mappings.
Algorithm 1.2.

\[ T = \sum_{i=1}^{N} \lambda_i T_{\omega_i^r} \]  \hspace{1cm} (1.20)

\[ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)Tx_n. \]

Algorithm 1.3.

\[ T = \sum_{i=1}^{N} \lambda_i^{(n)} T_{\omega_i^r} \]  \hspace{1cm} (1.21)

\[ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)Tx_n. \]

In (1.20), the weights \( \{\lambda_i\}_{i=1}^{N} \) are constant in the sense that they are independent of \( n \), the number of steps of the iteration process. In (1.21), we consider a more general case by allowing the weights \( \{\lambda_i^{(n)}\}_{i=1}^{N} \). Under appropriate assumptions on the sequences of the weights \( \{\lambda_i^{(n)}\}_{i=1}^{N}, \{\lambda_i\}_{i=1}^{N}, \{\alpha_n\} \) and \( \{\beta_n\} \). From (1.20) and (1.21), we will prove some strong convergence to a solution of the problem (1.18). In addition, we can also know that the condition \( \sum_{i=1}^{N} \sqrt{\sum_{i=1}^{N} |\lambda_i^{(n+1)} - \lambda_i^{(n)}|} < \infty \) in [8] is superfluous.

Another approach to the problem (1.18) is the cyclic algorithm (for convenience, we relabel the mappings \( \{T_{\omega_i^r}\}_{i=1}^{N} \) as \( \{T_{\omega_i}\}_{i=0}^{N-1} \)). This means that beginning with an \( x_0 \in C \), we define the sequence \( \{x_n\} \) cyclically by

\[ x_1 = \alpha_0 g(x_0) + \beta_0 x_0 + ((I - \beta_0)I - \alpha_0 A)T_{\omega_0}x_0, \]

\[ x_2 = \alpha_1 g(x_1) + \beta_1 x_1 + ((I - \beta_1)I - \alpha_1 A)T_{\omega_1}x_1, \]

\[ \vdots \]

\[ x_N = \alpha_{N-1} g(x_{N-1}) + \beta_{N-1} x_{N-1} + ((I - \beta_{N-1})I - \alpha_{N-1} A)T_{\omega_{N-1}}x_{N-1}, \]

\[ x_{N+1} = \alpha_N g(x_N) + \beta_N x_N + ((I - \beta_N)I - \alpha_N A)T_{\omega_N}x_N, \]

\[ \vdots \]

(1.22)

In a more compact form, \( x_{n+1} \) can be written as

\[ x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)T_{[n]}x_n, \]  \hspace{1cm} (1.24)

where \( T_{[n]} = T_{\omega_n^r}, T_{\omega_n} = (1 - \omega_i)I + \omega_i T_i, \{\omega_i\}_{i=1}^{N} \in (0, 1], \) with \( i = n \) (mod \(N\), \( 0 \leq i \leq N - 1 \)).

We will show that this cyclic algorithm (1.24) is also strongly convergent if the sequence \( \{\alpha_n\} \) of parameters is appropriately chosen.
2. Preliminaries

Throughout this paper, we write \( x_n \to x \) to indicate that the sequence \( \{x_n\} \) converges weakly to \( x \). \( x_n \to x \) implies that \( \{x_n\} \) converges strongly to \( x \). The following definitions and lemmas are useful for main results.

Definition 2.1. An operator \( T : H \to H \) is said to be quasi-nonexpansive if

\[
F_{ix}(T) \neq 0 \quad \text{and if } \|Tx - z\| \leq \|x - z\|, \quad \forall z \in F_{ix}(T), \forall x \in H.
\] (2.1)

Iterative methods for quasi-nonexpansive mappings have been extensively investigated; see [10, 11].

Remark 2.2. From the above definitions, It is easy to see that

(i) a nonexpansive mapping is a quasi-nonexpansive mapping;

(ii) the set of fixed points of \( T \) is the set \( F_{ix}(T) = \{x \in H : Tx = x\} \). We assume that \( F_{ix}(T) \neq \emptyset \), it is well known that \( F_{ix}(T) \) is closed and convex.

Remark 2.3 (see [10]). Let \( T_\alpha = (1 - \alpha)I + \alpha T \), where \( T \) is a quasi-nonexpansive on \( H, F_{ix}(T) \neq \emptyset \) and \( \alpha \in (0,1] \). Then the following statements are reached:

(i) \( F_{ix}(T) = F_{ix}(T_\alpha) \);

(ii) \( T_\alpha \) is quasi-nonexpansive;

(iii) \( \|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \alpha (1 - \alpha)\|Tx - x\|^2 \), for all \( (x, q) \in H \times F_{ix}(T) \);

(iv) \( \langle x - T_\alpha x, x - q \rangle \geq (\alpha/2)\|Tx - x\|^2 \), for all \( (x, q) \in H \times F_{ix}(T) \).

Example 2.4. Let \( X = \ell^2 \) with the norm \( \| \cdot \| \) defined by

\[
\|X\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}, \quad \forall x = (x_1, x_2, \ldots, x_n, \ldots) \in X,
\] (2.2)

and \( C = \{x = (x_1, x_2, \ldots, x_n, \ldots) \mid x_1 \leq 0, x_i \in R^1, i = 2, 3, \ldots\} \). Then \( C \) is a nonempty subset of \( X \).

Now, for any \( x = (x_1, x_2, \ldots, x_n, \ldots) \in C \), define a mapping \( T : C \to C \) as follows:

\[
T(x) = (0, 4x_1, 0, \ldots, 0, \ldots).
\] (2.3)

It is easy to see that \( T \) is a quasi-nonexpansive mapping. In fact, for any \( x = (x_1, x_2, \ldots, x_n, \ldots) \in X \), taking \( T(x) = x \), that is,

\[
(0, 4x_1, 0, \ldots, 0, \ldots) = (x_1, x_2, \ldots, x_n, \ldots),
\] (2.4)
we have $F(T) = \{0\}$ and

$$
\|T(x) - 0\| = \|(0, 4x_1, 0, \ldots, 0, \ldots) - (0, 0, 0, \ldots, 0, \ldots)\| = 4|x_1| \\
\leq 4 \sqrt{\sum_{i=1}^{\infty} x_i^2} \\
= \|(x_1, x_2, \ldots, x_n, \ldots) - (0, 0, 0, \ldots, 0, \ldots)\| \\
= \|x - 0\|.
$$

(2.5)

Lemma 2.5. Assume $C$ is a closed convex subset of a Hilbert space $H$.

(i) Given an integer $N \geq 1$, assume, for all $1 \leq i \leq N$, $T_i : C \to C$ is a quasi-nonexpansive. Let $\{\lambda_i\}_{i=1}^{N}$ be a positive sequence such that $\sum_{i=1}^{N} \lambda_i = 1$. Then $\sum_{i=1}^{N} \lambda_i T_i$ is a quasi-nonexpansive.

(ii) Let $\{T_i\}_{i=1}^{N}$ and $\sum_{i=1}^{N} \lambda_i = 1$ be given as in (i) above. Suppose that $\{T_i\}_{i=1}^{N}$ has a common fixed point. Then

$$
F_{ix}\left(\sum_{i=1}^{N} \lambda_i T_i\right) = \bigcap_{i=1}^{N} F_{ix}(T_i). 
$$

(2.6)

(iii) Assume $T_i : C \to C$ be quasi-nonexpansives, let $T_{a_i} = (1 - \alpha_i)I + \alpha_i T_i$, $1 \leq i \leq N$. If $\bigcap_{i=1}^{N} F_{ix}(T_i) \neq \emptyset$, then

$$
F_{ix}(T_{a_1} T_{a_2} \cdots T_{a_N}) = \bigcap_{i=1}^{N} F_{ix}(T_{a_i}).
$$

(2.7)

Proof. To prove (i) we only need to consider the case of $N = 2$ (the general case can be proved by induction). Set $T = (1 - \lambda)T_1 + \lambda T_2$, where $\lambda \in (0, 1)$ and for $i = 1, 2$, $T_i$ is a quasi-nonexpansive. We verify directly the following inequality: for all $z \in F_{ix}(T_1) \cap F_{ix}(T_2)$,

$$
\|Tx - z\| = \|(1 - \lambda)T_1 + \lambda T_2)x - z\| \\
\leq (1 - \lambda)\|T_1x - z\| + \lambda\|T_2x - z\| \\
\leq (1 - \lambda)\|x - z\| + \lambda\|x - z\| \\
\leq \|x - z\|,
$$

(2.8)

that is, $T$ is a quasi-nonexpansive.

To prove (ii) again we can assume $N = 2$. It suffices to prove that $F_{ix}(T) \subset F_{ix}(T_1) \cap F_{ix}(T_2)$, where $T = (1 - \lambda)T_1 + \lambda T_2$ with $\lambda \in (0, 1)$. Let $x \in F_{ix}(T)$.
Taking $z \in F_{ix}(T_1) \cap F_{ix}(T_2)$ to deduce that

$$
\|z - x\| = \|(1 - \lambda)(z - T_1x) + \lambda(z - T_2x)\|
$$

$$
\leq (1 - \lambda)\|z - T_1x\| + \lambda\|z - T_2x\|
$$

$$
\leq (1 - \lambda)\|z - x\| + \lambda\|z - x\|
$$

$$
\leq \|z - x\|.  \tag{2.9}
$$

By the strict convexity of $H$, it follows that $T_1(x) - z = T_2(x) - z = x - z$; that is, $T_1(x) = T_2(x) = x$, hence $x \in F_{ix}(T_1) \cap F_{ix}(T_2)$. According to induction, we can easily claim that (2.6) is holds.

To prove (iii) by induction, for $N = 2$, set $T_{a_i} = (1 - \alpha_i)I + \alpha_iT_i$ for all $i = 1, 2$. Obviously

$$
F_{ix}(T_{a_1}) \cap F_{ix}(T_{a_2}) \subset F_{ix}(T_{a_1}T_{a_2}).  \tag{2.10}
$$

Now we prove

$$
F_{ix}(T_{a_1}T_{a_2}) \subset F_{ix}(T_{a_1}) \cap F_{ix}(T_{a_2}).  \tag{2.11}
$$

For all $q \in F_{ix}(T_{a_1}T_{a_2})$, $T_{a_1}T_{a_2}q = q$, if $T_{a_1}q = q$, then $T_{a_2}q = q$, the conclusion holds. In fact, we can claim that $T_{a_1}q = q$. From Remark 2.3, we know that $T_{a_i}$ is quasi-nonexpansive and $F_{ix}(T_{a_1}) \cap F_{ix}(T_{a_2}) = F_{ix}(T_1) \cap F_{ix}(T_2) \neq \emptyset$. Take $p \in F_{ix}(T_{a_1}) \cap F_{ix}(T_{a_2})$, then

$$
\|p - q\|^2 = \|p - T_{a_1}T_{a_2}q\|^2
$$

$$
= \|p - [(1 - \alpha_1)T_{a_1}q + \alpha_1T_{a_1}T_{a_2}q]\|^2
$$

$$
= \|T_{a_1}(p - T_{a_1}q) + \alpha_1(p - T_1T_{a_2}q)\|^2
$$

$$
= (1 - \alpha_1)\|p - T_{a_2}q\|^2 + \alpha_1\|p - T_{a_1}T_{a_2}q\|^2 - \alpha_1(1 - \alpha_1)\|T_{a_1}q - T_1T_{a_2}q\|^2
$$

$$
\leq (1 - \alpha_1)\|p - T_{a_2}q\|^2 + \alpha_1\|p - T_{a_1}q\|^2 - \alpha_1(1 - \alpha_1)\|T_{a_1}q - T_1T_{a_2}q\|^2
$$

$$
= \|p - T_{a_1}q\|^2 - \alpha_1(1 - \alpha_1)\|T_{a_2}q - T_1T_{a_2}q\|^2.  \tag{2.12}
$$

From (2.12), we have

$$
\|T_{a_1}q - T_1T_{a_2}q\|^2 \leq 0,  \tag{2.13}
$$

namely, $T_{a_1}q = T_1T_{a_2}q$, that is,

$$
T_{a_1}q \in F_{ix}(T_1) = F_{ix}(T_{a_1}), \quad T_{a_1}q = T_{a_1}T_{a_2}q = q.  \tag{2.14}
$$
Suppose that the conclusion holds for $N = k$, we prove that

$$F_{ix}(T_{a_1}T_{a_2} \cdots T_{a_{k+1}}) = \bigcap_{i=1}^{k+1} F_{ix}(T_{a_i}).$$

(2.15)

It suffices to verify

$$F_{ix}(T_{a_1}T_{a_2} \cdots T_{a_{k+1}}) \subset \bigcap_{i=1}^{k+1} F_{ix}(T_{a_i}),$$

(2.16)

for all $q \in F_{ix}(T_{a_1}T_{a_2} \cdots T_{a_{k+1}})$, that is, $T_{a_1}T_{a_2} \cdots T_{a_{k+1}}q = q$. Using Remark 2.3 again, take $p \in \bigcap_{i=1}^{k+1} F_{ix}(T_{a_i})$, we obtain

$$\|p - q\|^2 = \|p - T_{a_1}T_{a_2} \cdots T_{a_{k+1}}q\|^2
= \|p - [(1 - \alpha_1)T_{a_2} \cdots T_{a_{k+1}}q - \alpha_1T_1T_{a_2} \cdots T_{a_{k+1}}q]\|^2
= \|[1 - \alpha_1](p - T_{a_2} \cdots T_{a_{k+1}}q) - \alpha_1(p - T_1T_{a_2} \cdots T_{a_{k+1}}q)\|^2
= (1 - \alpha_1)\|p - T_{a_2} \cdots T_{a_{k+1}}q\|^2 + \alpha_1\|p - T_1T_{a_2} \cdots T_{a_{k+1}}q\|^2
- \alpha_1(1 - \alpha_1)\|T_{a_2} \cdots T_{a_{k+1}}q - T_1T_{a_2} \cdots T_{a_{k+1}}q\|^2
\leq \|p - T_{a_2} \cdots T_{a_{k+1}}q\|^2 - \alpha_1(1 - \alpha_1)\|T_{a_2} \cdots T_{a_{k+1}}q - T_1T_{a_2} \cdots T_{a_{k+1}}q\|^2.$$

(2.17)

From (2.17), we obtain

$$\|T_{a_2} \cdots T_{a_{k+1}}q - T_1T_{a_2} \cdots T_{a_{k+1}}q\|^2 \leq 0,$$

(2.18)

this implies that

$$T_{a_2} \cdots T_{a_{k+1}}q \in F_{ix}(T_1) = F_{ix}(T_{a_1}),$$

(2.19)

namely,

$$T_{a_2} \cdots T_{a_{k+1}}q = T_{a_1}T_{a_2} \cdots T_{a_{k+1}}q = q.$$  

(2.20)

From (2.20) and inductive assumption, we have

$$q \in F_{ix}(T_{a_1}T_{a_2} \cdots T_{a_{k+1}}) \subset \bigcap_{i=2}^{k+1} F_{ix}(T_{a_i}),$$

(2.21)

therefore

$$T_{a_i}q = q, \quad i = 2, 3, \ldots, k + 1.$$  

(2.22)
Substituting it into (2.20), we obtain $T_q q = q$. Thus we assert that
\[ q \in \bigcap_{i=1}^{k+1} F_{iX}(T_{\alpha_i}). \quad (2.23) \]

**Definition 2.6.** A mapping $T$ is said to be demiclosed, if for any sequence $\{x_n\}$ which converges weakly to $y$, and if the sequence $\{Tx_n\}$ strongly converges to $z$, then $T(y) = z$.

**Lemma 2.7 (see [5]).** Assume $A$ is a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$.

**Lemma 2.8 (see [12]).** Let $H$ be a Hilbert space, $K$ a closed convex subset of $H$, and $T : K \to K$ a nonexpansive mapping with $F_X(T) \neq \emptyset$, if $\{x_n\}$ is a sequence in $K$ weakly converging to $x$ and if $\{(I - T)x_n\}$ converges strongly to $y$, then $(I - T)x = y$.

**Lemma 2.9 (see [13]).** Let $\{\tau_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\tau_{n_j}\}_{j \geq 1}$ of $\{\tau_n\}$ which satisfies $\tau_{n_j} < \tau_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\delta(n)\}_{n \geq n_0}$ defined by
\[ \delta(n) = \max\{k \leq n | \tau_k < \tau_{k+1}\}. \quad (2.24) \]
Then $\{\delta(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \to \infty} \delta(n) = \infty$, for all $n \geq n_0$, it holds that $\tau_{\delta(n)} < \tau_{\delta(n)+1}$ and one has
\[ \tau_n < \tau_{\delta(n)+1}. \quad (2.25) \]

**Lemma 2.10.** Let $K$ be a closed convex subset of a real Hilbert space $H$, given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality
\[ \langle x - y, y - z \rangle \geq 0, \quad \forall z \in K. \quad (2.26) \]

### 3. Parallel Algorithm

In this section, we discuss the parallel algorithm, respectively, for solving the variational inequality over the set of the common fixed points of finite quasi-nonexpansives.

Before stating our main convergence result, we establish the boundedness of the iterates given by following algorithm:
\[ T = \sum_{i=1}^{N} \lambda_i T_{\alpha_i}, \quad (3.1) \]
\[ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + ((I - \beta_n)(I - \alpha_n)A)Tx_n. \]

In (3.1), the weight $\{\lambda_i\}_{i=1}^{N}$ are constant in the sense that they are independent of $n$, the number of steps of the iteration process. Below we consider a more general case by allowing
the weights \( \{\lambda_i\}_{i=1}^N \) to be step dependent. That is, initializing with \( x_0 \), we define \( \{x_n\} \) by the algorithm

\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left( (I - \beta_n) I - \alpha_n A \right) \sum_{i=1}^N \lambda_i^{(n)} T_{\omega_i} x_n,
\]

(3.2)

From (3.1) and (3.2), the sequence \( \{x_n\} \) which converges strongly to the unique solution of variational inequality problem \( \text{VI}(\gamma f - A, \bigcap_{i=1}^N F_{ix}(T_{\omega_i})) \): find \( x^* \) in \( \bigcap_{i=1}^N F_{ix}(T_{\omega_i}) \) such that

\[
\forall v \in \bigcap_{i=1}^N F_{ix}(T_{\omega_i}), \quad \langle (\gamma f - A)x^*, v - x^* \rangle \leq 0,
\]

(3.3)

or equivalently

\[
x^* = \left( P_{\bigcap_{i=1}^N F_{ix}(T_{\omega_i})} \cdot C \right)(x^*),
\]

(3.4)

where \( P_{\bigcap_{i=1}^N F_{ix}(T_{\omega_i})} \) denotes the metric projection from \( H \) onto \( \bigcap_{i=1}^N F_{ix}(T_{\omega_i}) \) (see, [14] for more details on the metric projection).

**Lemma 3.1.** The sequence \( \{x_n\} \) is generated by (3.2), where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequence in \([0, 1]\), and \( \{T_i\}_{i=1}^N \) is a quasi-nonexpansive mapping on \( H \), is bounded and satisfies

\[
\|x_n - v\| \leq \max \left\{ \|x_1 - v\|, \frac{\|\gamma f(v) - Av\|}{\gamma - \gamma \beta} \right\}, \quad \forall n \geq 1,
\]

(3.5)

where \( v \) is any element in \( F_{ix}(T_i) \), \( 1 \leq i \leq N \).

**Proof.** Since \( \lim_{n \to \infty} \alpha_n = 0 \), we shall assume that \( \alpha_n \leq (1 - \beta_n) \|A\|^{-1} \) and \( 1 - \alpha_n (\gamma - \gamma \beta) > 0 \).

Observe that if \( \|u\| = 1 \), then

\[
\langle ((I - \beta_n) I - \alpha_n A) u, u \rangle = (1 - \beta_n) - \alpha_n \langle Au, u \rangle \geq (1 - \beta_n - \alpha_n \|A\|) \geq 0.
\]

(3.6)

By Lemma 2.7, we obtain

\[
\| (I - \beta_n) I - \alpha_n A \| \leq 1 - \beta_n - \alpha_n \gamma.
\]

(3.7)

Let \( B_n = \sum_{i=1}^N \lambda_i^{(n)} T_{\omega_i} \), for all \( n \geq 1 \). By Lemma 2.5, each \( B_n \) is a quasi-nonexpansive mapping on \( H \), and in light of Remark 2.3. Taking \( v \in F_{ix}(T) \), we have

\[
\left\| \sum_{i=1}^N \lambda_i^{(n)} T_{\omega_i} x_n - v \right\| \leq \left\| \sum_{i=1}^N \lambda_i^{(n)} T_{\omega_i} (x_n - v) \right\| \leq \sum_{i=1}^N \lambda_i^{(n)} \|x_n - v\| \leq \|x_n - v\|.
\]

(3.8)
From (3.1), we have

\[ \|x_{n+1} - v\| = \|\alpha_n (y f(x_n) - A v) + \beta_n (x_n - v) + ((I - \beta_n) I - \alpha_n A) T(x_n - v)\| \]
\[ \leq \alpha_n \|y f(x_n) - A v\| + \beta_n \|x_n - v\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \|x_n - v\| \]
\[ \leq \alpha_n \|y f(x_n) - f(v)\| + \alpha_n \|y f(v) - A v\| + (1 - \alpha_n \bar{\gamma}) \|x_n - v\| \]
\[ = [1 - \alpha_n (\bar{\gamma} - \gamma \beta)] \|x_n - v\| + \alpha_n \|y f(v) - A v\|. \]  

(3.9)

By simple inductions, we obtain

\[ \|x_n - v\| \leq \max\left\{ \|x_1 - v\|, \frac{\|y f(v) - A v\|}{\bar{\gamma} - \gamma \beta}\right\}, \quad \forall n \geq 1, \]  

(3.10)

which gives that the sequence \( \{x_n\} \) is bounded. \( \square \)

**Lemma 3.2.** Assume that \( \{x_n\} \) is defined by (3.2), if \( x^* \) is solution of (3.3) with \( T : C \to C \) demiclosed and \( \{y_n\} \subset H \) is a bounded sequence such that \( \|T y_n - y_n\| \to 0 \), then

\[ \liminf_{n \to \infty} \langle (A - y f)x^*, y_n - x^* \rangle \geq 0. \]  

(3.11)

**Proof.** Clearly, by \( \|T y_n - y_n\| \to 0 \) and \( T : H \to H \) demiclosed, we know that any weak cluster point of \( \{y_n\} \) belongs to \( F_T(T) \). It is also a simple matter to see that there exist \( \bar{y} \) and a subsequence \( \{y_n\} \) of \( \{y_n\} \) such that \( \lim_{j \to \infty} y_{n_j} \to \bar{y} \) (hence \( \bar{y} \in F_T(T) \)) and such that

\[ \liminf_{n \to \infty} \langle (A - y f)x^*, y_n - x^* \rangle = \lim_{j \to \infty} \langle (A - y f)x^*, y_{n_j} - x^* \rangle, \]  

(3.12)

it follows from (3.3), we can derive that

\[ \liminf_{n \to \infty} \langle (A - y f)x^*, y_n - x^* \rangle = \langle (A - y f)x^*, \bar{y} - x^* \rangle \geq 0, \]  

(3.13)

that is the desired result. \( \square \)

**Theorem 3.3.** Let \( C \) be a closed convex subset of a Hilbert space \( H \) and let \( T_i : C \to C \) be a quasi-non-expansive for \( T_i = (1 - \omega_i) I + \omega_i T, \omega_i \in (0, 1), i \in \{1, \ldots, N\} \) such that \( \bigcap_{i=1}^N F_T(T_i) \neq \emptyset \), \( f \) be a contraction with coefficient \( \beta \in (0, 1) \), and \( \lambda_i \) a positive constant such that \( \sum_{i=1}^N \lambda_i^{(a)} = 1 \) for all \( n \) and \( \inf_{n \geq 1} \lambda_i^{(a)} > 0 \) for all \( i \in [1, N] \). Let \( A \) be a strongly positive bounded linear operator with coefficient \( \bar{\gamma} \). Given the initial guess \( x_0 \in H \) chosen arbitrarily and given sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) in \( (0, 1) \), satisfying the following conditions:

(c1) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^\infty \alpha_n = \infty \),

(c2) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \).
Let \( \{x_n\} \) be the sequence generated by (3.2). Then \( \{x_n\} \) converges strongly to the unique \( x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) \), \( N \geq 1 \) verifying

\[
x^* = \left( P_{\bigcap_{i=1}^{N} F_{ix}(T_{\omega_i})} \cdot f \right) x^*
\]

which equivalently solves the following variational inequality problem:

\[
x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \quad \langle (\gamma f - A)x^*, \bar{x} - x^* \rangle \leq 0, \quad \forall \bar{x} \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}).
\]

**Proof.** Taking \( B_n = \sum_{i=1}^{N} \lambda_i^{(n)} T_{\omega_i} \), for all \( n \geq 1 \). By Lemma 2.5(i), each \( B_n \) is a quasi-nonexpansive mapping on \( C \), and (3.2) can be rewritten as

\[
x_{n+1} = x_n + \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)B_n x_n.
\]

Denote by \( \Omega \) the common fixed point of the mappings \( \{T_{\omega_i}\}_{i=1}^{N} \) (by Lemma 2.5(ii), we can easily know that \( \Omega = \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) = \bigcap_{i=1}^{N} F_{ix}(T_i) \)) and take \( x^* \in \Omega \) and from (3.16) we deduce that

\[
x_{n+1} - x_n + \alpha_n (Ax_n - \gamma f(x_n)) = (I - \beta_n - \alpha_n A)(B_n x_n - x_n),
\]

and hence

\[
\langle x_{n+1} - x_n + \alpha_n (Ax_n - \gamma f(x_n)), x_n - x^* \rangle = \langle (1 - \beta_n - \alpha_n A)B_n x_n - x_n, x_n - x^* \rangle
\]

\[
= (1 - \beta_n - \alpha_n) (\langle B_n x_n - x_n, x_n - x^* \rangle) + \alpha_n \langle (I - A)(B_n - I)x_n, x_n - x^* \rangle.
\]

Moreover, by \( x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) \) and using Remark 2.3(iv), we obtain

\[
\langle x_n - B_n x_n, x_n - x^* \rangle \geq \left\langle x_n - \sum_{i=1}^{N} \lambda_i^{(n)} T_{\omega_i} x_n, x_n - x^* \right\rangle
\]

\[
\geq \sum_{i=1}^{N} \lambda_i^{(n)} \langle x_n - T_{\omega_i} x_n, x_n - x^* \rangle
\]

\[
\geq \sum_{i=1}^{N} \lambda_i^{(n)} \omega_i \|x_n - T_i x_n\|^2,
\]

\[
\geq \sum_{i=1}^{N} \lambda_i^{(n)} \omega_i \|x_n - T_i x_n\|^2.
\]
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which combined with the (3.18) entails

\[ \langle x_{n+1} - x_n + \alpha_n (A - \gamma f)x_n, x_n - x^* \rangle \leq \frac{- (1 - \beta_n - \alpha_n)}{2} \sum_{i=1}^{N} \left( \lambda_i^{(n)} \omega_i \|x_n - T_ix_n\|^2 \right) \]

\[ + \alpha_n \langle (I - A)(B_n - I)x_n, x_n - x^* \rangle, \]  

(3.20)

or equivalently

\[ - \langle x_n - x_{n+1}, x_n - x^* \rangle \leq -\alpha_n \langle (A - \gamma f)x_n, x_n - x^* \rangle \]

\[ - \frac{(1 - \beta_n - \alpha_n)}{2} \sum_{i=1}^{N} \left( \lambda_i^{(n)} \omega_i \|x_n - T_ix_n\|^2 \right) \]

\[ + \alpha_n \langle (I - A)(B_n - I)x_n, x_n - x^* \rangle. \]  

(3.21)

Furthermore, using the following classical equality:

\[ \langle u, v \rangle = \frac{1}{2} \|u\|^2 - \frac{1}{2} \|u - v\|^2 + \frac{1}{2} \|v\|^2, \quad \forall u, v \in C, \]  

(3.22)

and setting \( \tau_n = \frac{1}{2} \|x_n - x^*\|^2 \), we have

\[ \langle x_n - x_{n+1}, x_n - x^* \rangle = \tau_n - \tau_{n+1} + \frac{1}{2} \|x_n - x_{n+1}\|^2. \]  

(3.23)

So that (3.21) can be equivalently rewritten as

\[ \tau_{n+1} - \tau_n - \frac{1}{2} \|x_n - x_{n+1}\|^2 \leq -\alpha_n \langle (A - \gamma f)x_n, x_n - x^* \rangle \]

\[ - \frac{(1 - \beta_n - \alpha_n)}{2} \sum_{i=1}^{N} \left( \lambda_i^{(n)} \omega_i \|x_n - T_ix_n\|^2 \right) \]

\[ + \alpha_n \langle (I - A)(B_n - I)x_n, x_n - x^* \rangle. \]  

(3.24)

Now using (3.16) again, we have

\[ \|x_{n+1} - x_n\|^2 = \|\alpha_n (\gamma f(x_n) - Ax_n) + (I - \beta_n - \alpha_n A)(B_n x_n - x_n)\|^2. \]  

(3.25)

Since \( A : H \rightarrow H \) is a strongly positive bounded linear operator with coefficient \( \tau > 0 \), hence it is a classical matter to see that

\[ \|x_{n+1} - x_n\|^2 \leq 2\alpha_n^2 \|\gamma f(x_n) - Ax_n\|^2 + 2(1 - \beta_n - \alpha_n \tau)^2 \|B_n x_n - x_n\|^2. \]  

(3.26)
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from

\[ \|B_n x_n - x_n\|^2 = \left\| \sum_{i=1}^{N} \lambda_i^{(n)} T_{\omega_i} x_n - x_n \right\|^2 \]

\[ = \left\| \sum_{i=1}^{N} \lambda_i^{(n)} (T_{\omega_i} x_n - x_n) \right\|^2 \]

\[ \leq 2 \sum_{i=1}^{N} (\lambda_i^{(n)})^2 \alpha_n \|x_n - T_i x_n\|^2 \]

\[ \leq 2 \sum_{i=1}^{N} \alpha_i \|x_n - T_i x_n\|^2 \]

and \((1 - \beta_n - \alpha_n y)^2 \leq (1 - \beta_n - \alpha_n y)\) yields

\[ \frac{1}{2} \|x_{n+1} - x_n\|^2 \leq \alpha_n \|y f(x_n) - Ax_n\|^2 + (1 - \beta_n - \alpha_n y) \sum_{i=1}^{N} (\lambda_i^{(n)} \omega_i \|x_n - T_i x_n\|^2). \] (3.28)

Then from (3.24) and (3.28), we have

\[ \mathcal{T}_{n+1} - \mathcal{T}_n + \left[ \frac{(1 - \beta_n - \alpha_n)}{2} - (1 - \beta_n - \alpha_n y) \right] \sum_{i=1}^{N} (\lambda_i^{(n)} \omega_i \|x_n - T_i x_n\|^2) \]

\[ \leq \alpha_n \left( \alpha_n \|y f(x_n) - Ax_n\|^2 - \langle (A - y f) x_n, x_n - x^* \rangle + \langle (I - A) (B_n - I) x_n, x_n - x^* \rangle \right). \] (3.29)

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists \( n_0 \) such that \( \{\mathcal{T}_n \}_{n=n_0} \) is nonincreasing. In this situation, \( \{\mathcal{T}_n\} \) is then convergent because it is also nonnegative (hence it is bounded from below), so that \( \lim_{n \to \infty} (\mathcal{T}_{n+1} - \mathcal{T}_n) = 0 \); hence, in light of (3.29) together with \( \lim_{n \to \infty} \alpha_n = 0 \), \( 0 < \lim_{n \to \infty} \beta_n \leq \lim_{n \to \infty} \beta_n < 1 \), and the boundedness of \( \{x_n\} \), we obtain

\[ \lim_{n \to \infty} \sum_{i=1}^{N} (\lambda_i^{(n)} \omega_i \|x_n - T_i x_n\|^2) = 0. \] (3.30)

By (3.27) and (3.30), we can easily claim that

\[ \lim_{n \to \infty} \|B_n x_n - x_n\| = 0. \] (3.31)

It also follows from (3.29) that

\[ \mathcal{T}_n - \mathcal{T}_{n+1} \geq \alpha_n \left( -\alpha_n \|y f(x_n) - Ax_n\|^2 + \langle (A - y f) x_n, x_n - x^* \rangle + \langle (I - A) (B_n - I) x_n, x_n - x^* \rangle \right). \] (3.32)
Then, by $\sum_{n=0}^{\infty} \alpha_n = \infty$, we obviously deduce that

$$
\liminf_{n \to \infty} \left( -\alpha_n \| y f(x_n) - Ax_n \|^2 + \langle (A - \gamma f)x_n, x_n - x^* \rangle + \langle (I - A)(B_n - I)x_n, x_n - x^* \rangle \right) \leq 0.
$$

(3.33)

Since $\{ f(x_n) \}$ and $\{ x_n \}$ are both bounded, $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} \| B_n x_n - x_n \| = 0$, we obtain

$$
\liminf_{n \to \infty} \langle (A - \gamma f)x_n, x_n - x^* \rangle \leq 0.
$$

(3.34)

Moreover, by Remark 1.1, we have

$$
2(\bar{\gamma} - \gamma \beta) \lim_{n \to \infty} \mathcal{T}_n + \langle (A - \gamma f)x^*, x_n - x^* \rangle \leq \langle (A - \gamma f)x_n, x_n - x^* \rangle,
$$

(3.35)

which by (3.34) entails

$$
\liminf_{n \to \infty} (2(\bar{\gamma} - \gamma \beta) \lim_{n \to \infty} \mathcal{T}_n + \langle (A - \gamma f)x^*, x_n - x^* \rangle) \leq 0,
$$

(3.36)

hence, recalling that $\lim_{n \to \infty} \mathcal{T}_n$ exists, we equivalently obtain

$$
2(\bar{\gamma} - \gamma \beta) \lim_{n \to \infty} \mathcal{T}_n + \liminf_{n \to \infty} \langle (A - \gamma f)x^*, x_n - x^* \rangle \leq 0,
$$

(3.37)

namely,

$$
2(\bar{\gamma} - \gamma \beta) \lim_{n \to \infty} \mathcal{T}_n \leq -\liminf_{n \to \infty} \langle (A - \gamma f)x^*, x_n - x^* \rangle.
$$

(3.38)

From (3.30) and invoking Lemma 3.2, we obtain

$$
\liminf_{n \to \infty} \langle (A - \gamma f)x^*, \tilde{x} - x^* \rangle \geq 0, \quad \tilde{x} \in \bigcap_{n=1}^{N} F_{ix}(T_{\omega_i}),
$$

(3.39)

which by (3.38) yields $\lim_{n \to \infty} \mathcal{T}_n = 0$, so that $\{ x_n \}$ converges strongly to $x^*$.

Case 2. Suppose there exists a subsequence $\{ \mathcal{T}_{\delta(n)} \}_{k \geq 0}$ of $\{ \mathcal{T}_n \}_{n \geq 0}$ such that $\mathcal{T}_{\delta(n)} \leq \mathcal{T}_{\delta(n+1)}$ for all $k \geq 0$. In this situation, we consider the sequence of indices $\{ \delta(n) \}$ as defined in Lemma 2.9. It follows that $\mathcal{T}_{\delta(n+1)} - \mathcal{T}_{\delta(n)} > 0$, which by (3.29) amounts to

$$
\left[ \frac{(1 - \beta_{\delta(n)}) - \alpha_{\delta(n)}}{2} - (1 - \beta_{\delta(n)}) - \alpha_{\delta(n)} \bar{\gamma} \right] \sum_{i=1}^{N} (\lambda_i^{(n)} \omega_i \| x_{\delta(n)} - T_i x_{\delta(n)} \|^2) \\
\leq \alpha_{\delta(n)} \left( \| y f(x_{\delta(n)}) - Ax_{\delta(n)} \|^2 - \langle (A - \gamma f)x_{\delta(n)}, x_n - x^* \rangle \right),
$$

(3.40)
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hence, by the boundedness of \( \{x_n\} \) and \( \lim_{n \to \infty} \alpha_n = 0 \), we immediately obtain

\[
\lim_{n \to \infty} \sum_{i=1}^{N} (\omega_i \|x_{\delta(n)} - T_i x_{\delta(n)}\|^2) = 0. \tag{3.41}
\]

From (3.28) we have

\[
\frac{1}{2} \|x_{\delta(n)+1} - x_{\delta(n)}\|^2 \leq \alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - Ax_{\delta(n)}\|^2 + (1 - \beta_{\delta(n)} - \alpha_{\delta(n)} \gamma^2) \sum_{i=1}^{N} (\omega_i \|x_{\delta(n)} - T_i x_{\delta(n)}\|^2)
\]

\[
\leq \alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - Ax_{\delta(n)}\|^2 + (1 - \beta_{\delta(n)} - \alpha_{\delta(n)} \gamma^2) \sum_{i=1}^{N} (\omega_i \|x_{\delta(n)} - T_i x_{\delta(n)}\|^2), \tag{3.42}
\]

which together with (3.41), \( \lim_{n \to \infty} \alpha_n = 0 \) and \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1 \) yields

\[
\lim_{n \to \infty} \|x_{\delta(n)+1} - x_{\delta(n)}\| = 0. \tag{3.43}
\]

Now by (3.40), we clearly have

\[
\alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - \mu B x_{\delta(n)}\|^2 \geq \langle (A - \gamma f)x_{\delta(n)}, x_{\delta(n)} - x^* \rangle, \tag{3.44}
\]

which in the light of (3.38) yields

\[
2(\gamma - \gamma^2) \mathcal{T}_{\delta(n)} + \langle (A - \gamma f)x, x_{\delta(n)} - x^* \rangle \leq \alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - Ax_{\delta(n)}\|^2, \tag{3.45}
\]

hence (as \( \lim_{n \to \infty} \alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - Ax_{\delta(n)}\|^2 = 0 \)) it follows that

\[
2(\gamma - \gamma^2) \lim \sup_{n \to \infty} \mathcal{T}_{\delta(n)} \leq -\lim \inf_{n \to \infty} \langle (A - \gamma f)x, x_{\delta(n)} - x^* \rangle. \tag{3.46}
\]

From (3.41) and invoking Lemma 3.2, we obtain

\[
\lim_{n \to \infty} \langle (A - \gamma f)x, \hat{x} - x^* \rangle \geq 0, \quad \hat{x} \in \bigcap_{n=1}^{N} F_{\alpha_i}(T_{\omega_i}), \tag{3.47}
\]

which by (3.46) yields \( \lim \sup_{n \to \infty} \mathcal{T}_{\delta(n)} = 0 \), so that \( \lim_{n \to \infty} \mathcal{T}_{\delta(n)} = 0 \). Combining (3.43), we have \( \lim_{n \to \infty} \mathcal{T}_{\delta(n)+1} = 0 \). Then, recalling that \( \mathcal{T}_n \subset \mathcal{T}_{\delta(n)+1} \) (by Lemma 2.9), we get
In this case, by the upper semicontinuity of \( ds \) and the equality over the set of the common fixed points of finite quasi-nonexpansive mappings, we introduce the following cyclic algorithm for solving the variational inequality problem:

\[
\langle (I - A + \gamma f)x^* - x^*, \bar{x} - x^* \rangle \leq 0, \quad \bar{x} \in \bigcap_{n=1}^{N} F_{ix}(T_{\omega_i}). \tag{3.48}
\]

So, by the Lemma 2.10, it is equivalent to the fixed point equation

\[
x^* = P_{\bigcap_{n=1}^{N} F_{ix}(T_{\omega_i})}(I - A + \gamma f)x^* = \left( P_{\bigcap_{n=1}^{N} F_{ix}(T_{\omega_i})} \cdot f \right)x^*. \tag{3.49}
\]

If the sequences of the weights \( \{\lambda_i^{(n)}\}_{i=1}^{N} = \{\lambda_i\}_{i=1}^{N} \) in (3.2), according to the proof of Theorem 3.3, we can obtain the following corollary.

**Corollary 3.4.** Let \( C \) be a closed convex subset of a Hilbert space \( H \) and let \( T_i : C \to C \) be a quasi-nonexpansive mapping for \( T_{\omega_i} = (1 - \omega_i)I + \omega_i T_i, \omega_i \in (0, 1), i \in (1, \ldots, N) \) such that \( \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) \neq \emptyset \). Let \( A \) be a strongly positive bounded linear operator with coefficient \( \overline{\gamma} \). Given the initial guess \( x_0 \in H \) chosen arbitrarily and given sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) in \( (0, 1) \), satisfying the following conditions:

1. \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \);
2. \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \).

Let \( \{x_n\} \) be the sequence generated by (3.1). Then \( \{x_n\} \) converges strongly to the unique a element \( x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) \), \( N \geq 1 \) verifying

\[
x^* = \left( P_{\bigcap_{i=1}^{N} F_{ix}(T_{\omega_i})} \cdot f \right)x^*, \tag{3.50}
\]

which equivalently solves the following variational inequality problem:

\[
x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \quad \langle f - A)x^*, \bar{x} - x^* \rangle \leq 0, \quad \forall \bar{x} \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}). \tag{3.51}
\]

### 4. Cyclic Algorithm

In this section, we discuss the cyclic algorithm, respectively, for solving the variational inequality over the set of the common fixed points of finite quasi-nonexpansive mappings and introduce quasi-shrinking mappings and quoted its definition from [11]. Hereafter, for nonempty closed set \( S \subset H \) and \( r \geq 0 \), we use the notations \( d_S : H \ni u \mapsto d(u, S) := \inf_{x \in S} ||u - x||, \quad \phi(S, r) := \{u \in H \mid d(u, S) = r\}, \quad \Psi(S, r) := \{u \in H \mid d(u, S) \leq r\}, \quad \Psi(S, r) := \{u \in H \mid d(u, S) \geq r\} \). In this case, by the upper semicontinuity of \( d_S \) (see e.g., [14, Theorem 1.3.3]), \( \Psi(S, r) \) is closed. Moreover, for a nonempty bounded closed convex set \( C \subset H \) and \( r \geq 0 \), it is not hard to verify that (i) \( \phi(C, r) \) and \( \Psi(C, r) \) are also closed; (ii) \( \phi(C, r) \) and \( \Psi(C, r) \) are bounded; (iii) \( \Psi(C, r) \) is convex.
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Definition 4.1 (see [11]). Suppose that $T : H \to H$ is quasi-nonexpansive with $F_{ix}(T) \cap C \neq \emptyset$ for some closed convex set $C$. Then $T : H \to H$ is called quasi-shrinking on $C$ if

$$D : r \in [0, \infty) \mapsto \begin{cases} \inf_{u \in \mathbb{L}(F_{ix}(T), r) \cap C} d(u, F_{ix}(T)) - d(T(u), F_{ix}(T)), \\ \infty \quad \text{otherwise} \end{cases}$$

(4.1)

satisfies $D(r) = 0 \Leftrightarrow r = 0$. In particular, if $T$ is quasi-shrinking on $H$, then $T$ is just called quasi-shrinking.

Let $C$ be a closed convex subset of a Hilbert space $H$ and let $\{T_i\}_{i=1}^{N-1}$ be quasi-nonexpansives defined on $C$ such that the common fixed point set

$$F := \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \quad N \geq 1,$$

(4.2)

where $T_{\omega_i} = (1 - \omega_i)I + \omega_i T_i$, $\{\omega_i\}_{i=1}^{N} \in (0, 1)$. Let $x_0 \in C$, let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ sequences in $(0, 1)$. The cyclic algorithm generates a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$x_1 = \alpha_0 \gamma f(x_0) + \beta_0 x_0 + ((I - \beta_0)I - \alpha_0 A)T_{\omega_0} x_0,$$

$$x_2 = \alpha_1 \gamma f(x_1) + \beta_1 x_1 + ((I - \beta_1)I - \alpha_1 A)T_{\omega_1} x_1,$$

$$\vdots$$

$$x_N = \alpha_{N-1} \gamma f(x_{N-1}) + \beta_{N-1} x_{N-1} + ((I - \beta_{N-1})I - \alpha_{N-1} A)T_{\omega_{N-1}} x_{N-1},$$

$$x_{N+1} = \alpha_{N-1} \gamma f(x_{N-1}) + \beta_N x_N + ((I - \beta_N)I - \alpha_N A)T_{\omega_N} x_N,$$

$$\vdots$$

(4.3)

In general, $x_{n+1}$ is defined by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)T_{[n]} x_n,$$

(4.4)

where $T_{[n]} = T_{\omega_i} = (1 - \omega_i)I + \omega_i T_i$, with $i = n \pmod N$, $0 \leq i \leq N - 1$.

Lemma 4.2 (see [11]). Let $\varphi(x) : [0, \infty) \to [0, \infty)$ satisfy

(i) $x_1 > x_2 \Rightarrow \varphi(x_1) > \varphi(x_2)$,

(ii) $\varphi(x) = 0 \Leftrightarrow x = 0$.

Let $\{z_n\}_{n=1}^{\infty}$ satisfy $\lim_{n \to \infty} z_n = 0$. Then any sequence $\{b_n\}_{n=1}^{\infty} \subset [0, \infty)$ satisfying

$$b_{n+1} \leq b_n - \varphi(b_n) + z_{n+1}, \quad n = 0, 1, 2, \ldots$$

(4.5)

converges to 0.
Lemma 4.3 (see [15]). Assume that \( \{\alpha_n\}_{n=0}^{\infty} \) is a sequence of nonnegative real numbers such that
\[
\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\delta_n, \quad n \geq 0,
\]
where \( \{\gamma_n\}_{n=0}^{\infty} \subset [0, 1] \) and \( \{\delta_n\}_{n=0}^{\infty} \) satisfy the following conditions:

(i) \( \sum_{n=0}^{\infty} \gamma_n = \infty \) and \( \lim_{n \to \infty} \gamma_n = 0 \),

(ii) \( \limsup_{n \to \infty} \delta_n \leq 0 \) or \( \sum_{n=0}^{\infty} |\gamma_n\delta_n| < \infty \).

Then \( \lim_{n \to \infty} \alpha_n = 0 \).

Theorem 4.4. Let \( C \) be a closed convex subset of a Hilbert space \( H \) and let \( T_i : H \to H \) be quasi-nonexpansive for \( T_{\omega_i} = (1 - \omega_i)I + \omega_i T_i, \omega_i \in (0, 1), i \in \{1, 2, \ldots, N\} \) such that \( F := \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) \neq \emptyset \) and \( f \) a contraction with coefficient \( \beta \in (0, 1) \). Let \( A \) be a strongly positive bounded linear operator with coefficient \( \gamma \). Given the initial guess \( x_0 \in H \) chosen arbitrarily and given sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) in \( (0, 1) \), satisfying the following conditions:

(4.1a) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \);

(4.1b) \( \sum_{n=0}^{\infty} \|\alpha_{n+1} - \alpha_n\| < \infty \) or \( \lim_{n \to \infty} \alpha_n / \alpha_{n+1} = 1 \);

(4.1c) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \).

Let \( \{x_n\} \) be the sequence generated by (4.4). Then \( \{x_n\} \) converges strongly to the unique a element \( x^* \in F := \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \) \( N \geq 1 \) verifying
\[
x^* = \left( \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) \cdot f \right) x^*,
\]
which equivalently solves the following variational inequality problem:
\[
x^* \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}), \quad \langle (\gamma f - A) x^*, \bar{x} - x^* \rangle \leq 0, \quad \forall \bar{x} \in \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}).
\]

Proof. Take a \( p \in F := \bigcap_{i=1}^{N} F_{ix}(T_{\omega_i}) \). We break the proof process into several steps.

Step 1. \( \{x_n\} \) is bounded. In light of the Remark 2.3, we obtain
\[
\|T_\phi(x_n - p)\| = \|T_\phi(x_n - p)\| \leq \|x_n - p\|.
\]

From (4.4), we have
\[
\|x_{n+1} - p\| = \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((I - \beta_n)I - \alpha_n A) T_{\phi}(x_n - p)\|
\leq \alpha_n\|\gamma f(x_n) - Ap\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\gamma)\|x_n - p\|
\leq \alpha_n\|\gamma f(x_n) - f(p)\| + \beta_n\|\gamma f(p) - Ap\| + (1 - \alpha_n\gamma)\|x_n - p\|
= [1 - \alpha_n(\gamma - \gamma\beta)\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|.
\]
By simple inductions, we obtain
\[
\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \left\| \frac{r f(p) - Ap}{\bar{\gamma} - \gamma \bar{\beta}} \right\| \right\}, \quad \forall n \geq 1,
\]
(4.11)

which gives that the sequence \( \{x_n\} \) is bounded; we also know that \( \{T_n x_n\} \) and \( \{f(x_n)\} \) are bounded.

**Step 2.** Moreover if \( T_n : H \to H \) is quasi-shrinking on the set \( C \), we obtain the following statements:

(a) \( \lim_{n \to \infty} d(x_n, F) = 0 \);

(b) \( \lim_{n \to \infty} \|T_n x_n - x_n\| = 0 \);

(c) \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \).

By the boundedness of \( \{x_n\} \), \( \{T_n x_n\} \), and \( \{f(x_n)\} \), there exists \( M > 0 \) satisfying
\[
\max_{n \geq 0} \{\|x_n\|, \|T_n x_n\|, \|f(x_n)\|\} \leq M.
\]
(4.12)

By a simple inspection, we deduce
\[
d(x_{n+1}, F) \leq \|x_{n+1} - P_F(T_n x_n)\|
= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)T_n x_n - P_F(T_n x_n)\|
\leq \alpha_n \gamma \|f(x_n)\| + \beta_n \|x_n - T_n x_n\| + \|T_n x_n - P_F(T_n x_n)\|
\leq \alpha_n (\gamma \|f(x_n)\| + \|AT_n x_n\|) + \beta_n (\|x_n\| + \|T_n x_n\|) + \|T_n x_n - P_F(T_n x_n)\|
\leq d(T_n x_n, F) + 2(\alpha_n + \beta_n) M.
\]
(4.13)

By \( \{x_n\}_{n \geq 0} \subset C \), we can assume the boundedness of the sequence \( b_n := d(x_n, F) \geq 0 \ (n \in \mathbb{N}) \). Moreover, by Definition 4.1 and (4.13), it follows that
\[
D(b_n) \leq b_n - d(T_n x_n, F)
\leq b_n - b_{n+1} + 2(\alpha_n + \beta_n) M, \quad \forall n \geq 0.
\]
(4.14)

Now application of Lemma 4.2 to (4.14) yields \( \lim_{n \to \infty} b_n = 0 \), hence (a) is proved.

The statements (b) and (c) are verified by
\[
\|T_n x_n - x_n\| = \|T_n x_n - P_F(x_n) + P_F(x_n) - x_n\|
\leq \|T_n x_n - P_F(x_n)\| + \|P_F(x_n) - x_n\|
\leq \|x_n - P_F(x_n)\| + \|P_F(x_n) - x_n\|
= 2d(x_n, F) \to 0, \quad (n \to \infty),
\]
(4.15)
\[ \|x_{n+1} - x_n\| = \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((I - \beta_n)I - \alpha_n A)T_{[n]}x_n - x_n\| \]
\[ \leq \|\alpha_n \gamma f(x_n) - \beta_n x_n - ((I - \beta_n)I - \alpha_n A)T_{[n]}x_n\| \]
\[ \rightarrow 0, \quad (n \rightarrow \infty). \]

**Step 3.** Let \( n \rightarrow \infty \), we obtain that
\[ \|x_{n+1} - x_n\| = 0. \]

From (4.4) and (4.16), we obtain
\[ \|x_{n+1} - x_n\| = \|\alpha_n \gamma f(x_{n+1}) + \beta_n x_{n+1} + ((I - \beta_n)I - \alpha_n A)T_{[n+1]}x_{n+1} - x_n\| \]
\[ - \|\alpha_n \gamma f(x_n) - \beta_n x_n - ((I - \beta_n)I - \alpha_n A)T_{[n]}x_n\| \]
\[ = \|\alpha_n \gamma f(x_{n+1}) - \alpha_n A\| \]
\[ + \|\alpha_n \gamma f(x_n) - \beta_n x_n - ((I - \beta_n)I - \alpha_n A)T_{[n]}x_n\| \]
\[ \leq \|\alpha_n \gamma f(x_{n+1}) - \alpha_n A\| \]
\[ + \|\alpha_n \gamma f(x_n) - \beta_n x_n - ((I - \beta_n)I - \alpha_n A)T_{[n]}x_n\| \]
\[ \rightarrow 0, \quad (n \rightarrow \infty). \]

By conditions (4.1a), (4.1b), (4.1c), (4.15), and (4.16), \( \{x_n\} \) and \( \{T_{[n]}x_n\} \) are bounded we obtain that
\[ \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \]

**Step 4.** Let \( n \rightarrow \infty \), we obtain that
\[ \|x_{n+1} - T_{[n]}x_n\| = \|\alpha_n \gamma f(x_n) - \beta_n x_n - T_{[n]}x_n\|. \]
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It follows from the condition (4.1a), (4.1c), (4.15), and the boundedness of \( \{ f(x_n) \} \) and \( \{ T_n x_n \} \) that

\[
\| x_{n+1} - T_{[n]} x_n \| \to 0, \quad (n \to \infty). \tag{4.20}
\]

Recursively,

\[
\begin{align*}
\| x_{n+N} - T_{[n+N]} x_{n+N-1} \| & \to 0, \quad (n \to \infty), \\
\| x_{n+N-1} - T_{[n+N-1]} x_{n+N-2} \| & \to 0, \quad (n \to \infty). \tag{4.21}
\end{align*}
\]

By Remark 2.3, \( T_{[n+N]} \) is quasi-nonexpansive, we obtain

\[
\| T_{[n+N]} x_{n+N-1} - T_{[n+N]} T_{[n+N-1]} x_{n+N-2} \| \to 0, \quad (n \to \infty). \tag{4.22}
\]

Proceeded accordingly, we obtain

\[
\begin{align*}
\| T_{[n+N]} T_{[n+N-1]} x_{n+N-2} - T_{[n+N]} T_{[n+N-1]} T_{[n+N-2]} x_{n+N-3} \| & \to 0, \quad (n \to \infty), \\
\vdots \\
\| T_{[n+N]} \cdots T_{[n+2]} x_{n+1} - T_{[n+N]} \cdots T_{[n+1]} x_n \| & \to 0, \quad (n \to \infty). \tag{4.23}
\end{align*}
\]

Note that

\[
\begin{align*}
\| x_{[n+N]} - T_{[n+N]} \cdots T_{[n+1]} x_n \| & \leq \| x_{n+N} - T_{[n+N]} x_{n+N-1} \| \\
& + \| T_{[n+N]} x_{n+N-1} - T_{[n+N]} T_{[n+N-1]} x_{n+N-2} \| \\
& + \cdots \\
& + \| T_{[n+N]} \cdots T_{[n+2]} x_{n+1} - T_{[n+N]} \cdots T_{[n+1]} x_n \|.
\end{align*} \tag{4.24}
\]

From all the expressions above, we have

\[
\| x_{[n+N]} - T_{[n+N]} \cdots T_{[n]} x_n \| \to 0, \quad (n \to \infty). \tag{4.25}
\]

Since

\[
\| x_n - T_{[n+N]} \cdots T_{[n+1]} x_n \| \leq \| x_n + x_{n+N} \| + \| x_{n+N} - T_{[n+N]} \cdots T_{[n]} x_n \|, \tag{4.26}
\]

it is concluded that

\[
\lim_{n \to \infty} \| x_n - T_{[n+N]} \cdots T_{[n+1]} x_n \| = 0. \tag{4.27}
\]
Using Remark 2.3 and Lemma 2.5, we obtain

\begin{equation}
\lim_{j \to \infty} \left\| x_{n_j} - T_{[n_j+N]} \cdots T_{[n_j+1]}x_{n_j} \right\| = 0.
\end{equation}

(4.28)

Observe that, for each \( n_j, T_{[n_j+N]}, \ldots, T_{[n_j+1]} \) is some permutation of the mappings \( T_{[1]}, \ldots, T_{[N]} \), since \( T_{[1]}, \ldots, T_{[N]} \) are finite, all the full permutation are \( N! \), there must be some permutation that appears infinite times. Without loss of generality, suppose that this permutation is \( T_{[1]}, \ldots, T_{[N]} \), we can take a subsequence \( \{x_{n_k}\} \subset \{x_{n_j}\} \) such that

\begin{equation}
\lim_{j \to \infty} \left\| x_{n_j} - T_{[1]} \cdots T_{[N]}x_{n_j} \right\| = 0.
\end{equation}

(4.29)

It is easy to prove that \( T_{[1]}, \ldots, T_{[N]} \) is quasi-nonexpansive. By Lemma 2.5, we have

\[ \hat{x} = T_{[1]} \cdots T_{[N]}x. \]

(4.30)

Using Remark 2.3 and Lemma 2.5, we obtain

\[ \hat{x} \in F_{ix}(T_{[1]} \cdots T_{[N]}) = \bigcap_{n=1}^{N} F_{ix}(T_{[n]}) = \bigcap_{n=1}^{N} F_{ix}(T_{n}). \]

(4.31)

\textbf{Step 6.} \( \lim \inf_{n \to \infty} \langle (y - A)x^*, x_n - x^* \rangle \leq 0 \). Indeed, there exists a subsequence \( \{x_{n_j}\} \subset \{x_n\} \) such that

\begin{equation}
\lim_{n \to \infty} \inf \langle (y - A)x^*, x_n - x^* \rangle = \lim_{j \to \infty} \left( \langle (y - A)x^*, x_{n_j} - x^* \rangle \right). \end{equation}

(4.32)

Without loss of generality, we may further assume that \( x_{n_j} \to \hat{x} \). It follows from (4.31) that \( \hat{x} \in \bigcap_{n=1}^{N} F(T_{n}) \). Since \( x^* \) is the unique solution of (4.8), we have

\begin{equation}
\lim_{n \to \infty} \inf \langle (y - A)x^*, x_n - x^* \rangle = \lim_{j \to \infty} \left( \langle (y - A)x^*, x_{n_j} - x^* \rangle \right) = \langle (y - A)x^*, \hat{x} - x^* \rangle \leq 0.
\end{equation}

(4.33)

In addition, the variational inequality (4.33) can be written as

\[ \langle (I - A + yf)x^* - x^*, \hat{x} - x^* \rangle \leq 0, \quad \hat{x} \in \bigcap_{n=1}^{N} F_{ix}(T_{n}). \]

(4.34)

So, by the Lemma 2.10, it is equivalent to the fixed point equation

\[ x^* = P_{\bigcap_{n=1}^{N} F_{ix}(T_{n})} (I - A + yf)x^* = \left( P_{\bigcap_{n=1}^{N} F_{ix}(T_{n})} \cdot f \right)x^*. \]

(4.35)
Step 7. lim_{n→∞}∥x_n − x^*∥ = 0. From (4.4), we obtain

\[
∥x_{n+1} − x^*∥^2 = ∥\alpha_n f(x_n) + β_n x_n + ((I − β_n)I − α_n A)T_{[n]}x_n − x^*∥^2
= \langle \alpha_n (γ f(x_n) − Ax^*) + β_n (x_n − x^*) + ((I − β_n)I − α_n A)(T_{[n]}x_n − x^*), x_{n+1} − x^* \rangle
≤ \alpha_n (γ f(x_n) − Ax^*, x_{n+1} − x^*) + β_n (x_n − x^*, x_{n+1} − x^*)
+ (1 − β_n − α_n)(T_{[n]}x_n − x^*, x_{n+1} − x^*) + α_n (I − A)(T_{[n]}x_n − x^*, x_{n+1} − x^*)
\]
\[
≤ \alpha_n (γ f(x_n) − f(x^*), x_{n+1} − x^*) + (γ f(x^*) − Ax^*, x_{n+1} − x^*)
+ (I − A)(T_{[n]}x_n − x^*, x_{n+1} − x^*) + β_n ∥x_n − x^*∥∥x_{n+1} − x^*∥
+ (1 − β_n − α_n γ)∥T_{[n]}x_n − x^*∥∥x_{n+1} − x^*∥
\]
\[
≤ \alpha_n γ∥x_n − x^*∥∥x_{n+1} − x^*∥ + α_n (γ f(x^*) − Ax^*, x_{n+1} − x^*)
α_n (1 − γ)∥T_{[n]}x_n − x^*∥∥x_{n+1} − x^*∥ + β_n ∥x_n − x^*∥∥x_{n+1} − x^*∥
+ (1 − β_n − α_n γ)∥x_n − x^*∥∥x_{n+1} − x^*∥
\]
\[
≤ \frac{1}{2}(1 − γ) (∥x_n − x^*∥^2 + ∥x_{n+1} − x^*∥^2) + α_n M
\]
\[
\leq \frac{1}{2}(1 − γ) (∥x_n − x^*∥^2 + \frac{1}{2}∥x_{n+1} − x^*∥^2) + α_n M,
\]
(4.36)

where M = (1 − γ)∥T_{[n]}x_n − x^*∥∥x_{n+1} − x^*∥ + (γ f(x^*) − Ax^*, x_{n+1} − x^*). It follows that

\[
∥x_{n+1} − x^*∥^2 \leq (1 − (1 − γ))∥x_n − x^*∥^2 + 2α_n M.
(4.37)

By using Lemma 4.3, we can obtain the desired conclusion easily.

5. Application

In this section, we constructed a numerical example to compare the parallel algorithm and cyclic algorithm which is simple.

Let x = (x_1, x_2) ∈ R^2 and f(x) = (1/2)(\sin(x_1), \cos(x_2)) be a contraction mapping with coefficient 1/2. Let T_1(x) = (0, 4x_1) and T_2(x) = (4x_2, 0) be quasi-nonexpansive mappings. Let α_n = β_n = 1/3, A = I and γ = λ_1 = λ_2 = 1/2. According to (1.20) and (1.24), we can obtain the following parallel algorithm and cyclic algorithm:

**Parallel Algorithm**

\[
x_{n+1} = \frac{1}{6} f(x_n) + \frac{1}{2} x_n + \frac{1}{12}(T_1 + T_2) x_n.
(5.1)
\]
Cyclic Algorithm

\[
x_1 = \frac{1}{6} f(x_0) + \frac{1}{2} x_0 + \frac{1}{6} (T_1) x_0;
\]

\[
x_2 = \frac{1}{6} f(x_1) + \frac{1}{2} x_1 + \frac{1}{6} (T_2) x_1;
\]

\[
x_3 = \frac{1}{6} f(x_2) + \frac{1}{2} x_2 + \frac{1}{6} (T_1) x_2;
\]

\[
\vdots
\]

\[
x_{n-1} = \frac{1}{6} f(x_{n-2}) + \frac{1}{2} x_{n-2} + \frac{1}{6} (T_2) x_{n-2};
\]

\[
x_n = \frac{1}{6} f(x_{n-1}) + \frac{1}{2} x_{n-1} + \frac{1}{6} (T_1) x_{n-1}.
\]

From Theorems 3.3 and 4.4, we can easily know that parallel algorithm (5.1) and cyclic algorithm (5.2) are converge to the unique point in \( R^2 \). Let \( x_0 = (5, 2) \) and \( |x_{n+1} - x_n|^2 \leq 10^{-9} \), and let \( x_p \) and \( x_X \) be the fixed point of the parallel algorithm and cyclic algorithm. Using the software of MATLAB, we obtain \( x_p = (0.6821, 0.7080) \) and \( x_X = (1.9325, 0.8729) \). From the computed results of \( x_p \) and \( x_X \), we can easily know that parallel algorithm (5.1) is simpler than cyclic algorithm (5.2). On the other hand, we need to explain that those algorithms do not converge a common fixed point, because parallel algorithm (5.1) and cyclic algorithm (5.2) have the different algorithm structure.

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References


