Research Article

On a Fixed Point for Generalized Contractions in Generalized Metric Spaces

Wasfi Shatanawi, 1 Ahmed Al-Rawashdeh, 2 Hassen Aydi, 3 and Hemant Kumar Nashine 4

1 Department of Mathematics, The Hashemite University, Zarqa 13115, Jordan
2 Department of Mathematical Sciences, UAEU, Al Ain 17551, UAE
3 Institut Supérieure d’Informatique et des Technologies de Communication de Hammam Sousse, Université de Sousse, Route GP1, 4011 H. Sousse, Tunisia
4 Department of Mathematics, Disha Institute of Management and Technology, Satya Vihar, Vidhansabha-Chandrakhuri Marg, Mandir Hasaud, (Chhattisgarh), Raipur 492101, India

Correspondence should be addressed to Ahmed Al-Rawashdeh, aalrawashdeh@uaeu.ac.ae

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Lakzian and Samet (2010) studied some fixed-point results in generalized metric spaces in the sense of Branciari. In this paper, we study the existence of fixed-point results of mappings satisfying generalized weak contractive conditions in the framework of a generalized metric space in sense of Branciari. Our results modify and generalize the results of Laksian and Samet, as well as, our results generalize several well-known comparable results in the literature.

1. Introduction and Preliminaries

Branciari in [1] initiated the notion of a generalized metric space as a generalization of a metric space in such a way that the triangle inequality is replaced by the “quadrilateral inequality,” $d(x, y) \leq d(x, a) + d(a, b) + d(b, y)$ for all pairwise distinct points $x, y, a,$ and $b$ of $X$. Afterwards, many authors initiated and studied many existing fixed-point theorems in such spaces. For more details about fixed-point theory in generalized metric spaces, we refer the reader to [1–13].

The following definitions will be needed in the sequel.

Definition 1.1 (see [1]). Let $X$ be a nonempty set and $d : X \times X \rightarrow [0, +\infty)$ such that for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from $x$ and $y$, one has
(p1): $x = y \iff d(x, y) = 0$,

(p2): $d(x, y) = d(y, x)$,

(p3): $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then, $(X, d)$ is called a generalized metric space (or shortly g.m.s).

Any metric space is a generalized metric space, but the converse is not true [1].

Definition 1.2 (see [1]). Let $(X, d)$ be a g.m.s, $\{x_n\}$ a sequence in $X$, and $x \in X$. We say that $\{x_n\}$ is g.m.s convergent to $x$ if and only if $d(x_n, x) \to 0$ as $n \to +\infty$. We denote this by $x_n \to x$.

Definition 1.3 (see [1]). Let $(X, d)$ be a g.m.s and $\{x_n\}$ a sequence in $X$. We say that $\{x_n\}$ is a g.m.s Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number $N$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N$.

Definition 1.4 (see [1]). Let $(X, d)$ be a g.m.s. Then, $(X, d)$ is called a complete g.m.s if every g.m.s Cauchy sequence is g.m.s convergent in $X$.

Very recently, Lakzian and Samet [9] proved the following nice result.

Theorem 1.5. Let $(X, d)$ be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ is such that for all $x, y \in X$

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad (1.1)$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing with $\psi(t) = 0$ if and only if $t = 0$, and $\phi : [0, \infty) \to [0, \infty)$ is continuous and $\phi(t) = 0$ if and only if $t = 0$. Then, there exists a unique point $u \in X$ such that $u = Tu$.

Note that Theorem 1.5 extends a result of Dutta and Choudhury [14] to the set of generalized metric spaces. Moreover, its proof is more technical compared with that of [9].

In this paper, we generalize in some cases Theorem 1.5 by replacing in (1.1) the term $d(x, y)$ by the quantity $\max\{d(x, y), d(x, Tx), d(y, Ty)\}$ and the continuity of $\phi$ by lower semicontinuity. Also, we derive some useful corollaries of this result.

2. Main Results

Let $X$ be a nonempty set and $T : X \to X$ a given mapping. For all $x, y \in X$, set

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \quad (2.1)$$

Also, let $\Psi = \{\psi \mid \psi : [0, \infty) \to [0, \infty)\}$ be continuous, nondecreasing, and $\psi(t) = 0$ if and only if $t = 0$, and $\Phi = \{\phi \mid \phi : [0, \infty) \to [0, \infty)\}$ is lower semi continuous, $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$. Note that, if $\psi \in \Psi$, $\psi$ is called an altering distance function [15].

The notion of a periodic point of a given mapping $T : X \to X$ is crucial for proving our main theorem. So we need the following definition.
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Definition 2.1. Let $X$ be a nonempty set. A given mapping $T : X \rightarrow X$ admits a periodic point if there exists $u \in X$ such that $u = T^p u$ for some $p \geq 1$. If $p = 1$, $u$ is a fixed point.

Hence, each fixed point is also a periodic point of $T$.

Now, in the following, let us prove our main result.

Theorem 2.2. Let $(X,d)$ be a Hausdorff and complete generalized metric space. Suppose that $T : X \rightarrow X$ is such that for all $x,y \in X$

$$\psi(d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)),$$

(2.2)

where $\psi \in \Psi$, $\phi \in \Phi$, and $M(x,y)$ is defined by (2.1). Then, there exists a unique point $u \in X$ such that $u = Tu$.

Proof. First, it is obvious that $M(x,y) = 0$ if and only if $x = y$ is a fixed point of $T$. Let $x_0 \in X$ an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \forall n \geq 0.$$ (2.3)

Step 1. We claim that

$$\lim_{n \rightarrow \infty} d(x_n,x_{n+1}) = 0.$$ (2.4)

Substituting $x = x_n$ and $y = x_{n-1}$ in (2.2) and using properties of functions $\psi$ and $\phi$, we obtain

$$\psi(d(x_{n+1},x_n)) = \psi(d(Tx_n,Tx_{n-1}))$$

$$\leq \psi(M(x_n,x_{n-1})) - \phi(M(x_n,x_{n-1}))$$

(2.5)

$$\leq \psi(M(x_n,x_{n-1}))$$

which implies that

$$d(x_{n+1},x_n) \leq M(x_n,x_{n-1}) \quad \forall n \geq 1.$$ (2.6)

Note that

$$M(x_n,x_{n-1}) = \max\{d(x_n,x_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_{n-1})\}$$

$$= \max\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\}.$$ (2.7)

If for some $n \geq 1$, $d(x_{n-1},x_n) < d(x_n,x_{n+1})$, then $M(x_n,x_{n-1}) = d(x_n,x_{n+1}) > 0$ and $\phi(d(x_{n+1},x_n)) > 0$ by a property of $\phi$, so (2.5) becomes

$$0 < \psi(d(x_{n+1},x_n)) \leq \psi(d(x_{n+1},x_n)) - \phi(d(x_{n+1},x_{n+1})) < \psi(d(x_{n+1},x_n))$$ (2.8)
a contradiction. Thus, for all \( n \geq 1 \),

\[
d(x_{n+1}, x_n) \leq d(x_{n-1}, x_n) = M(x_{n-1}, x_n).
\] (2.9)

From (2.9), the sequence \( \{d(x_n, x_{n+1})\} \) is monotone nonincreasing and so bounded below. So there exists \( r \geq 0 \) such that

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \lim_{n \to \infty} M(x_{n-1}, x_n) = r.
\] (2.10)

Letting \( \lim_{n \to \infty} r \) in (2.5) and using the above limits with the continuity of \( \varphi \) and the lower semicontinuity of \( \phi \), we get \( \varphi(r) \leq \varphi(r) - \phi(r) \), which implies that \( \phi(r) = 0 \), so \( r = 0 \) by a property of \( \phi \). Thus, (2.4) is proved.

**Step 2.** We shall prove that

\[
\lim_{n \to \infty} d(x_n, x_{n+2}) = 0.
\] (2.11)

By (2.2), we have

\[
\varphi(d(x_{n+2}, x_n)) = \varphi(d(Tx_{n+1}, Tx_{n-1})) \\
\leq \varphi(M(x_{n+1}, x_{n-1})) - \phi(M(x_{n+1}, x_{n-1})) \\
\leq \varphi(M(x_{n+1}, x_{n-1}))
\] (2.12)

which implies that

\[
d(x_{n+2}, x_n) \leq M(x_{n+1}, x_{n-1}) \quad \forall n \geq 1,
\] (2.13)

where

\[
M(x_{n+1}, x_{n-1}) = \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, Tx_{n+1}), d(x_{n-1}, Tx_{n-1})\} \\
= \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_n)\} \\
= \max\{d(x_{n+1}, x_{n-1}), d(x_{n-1}, x_n)\}.
\] (2.14)

Set \( \alpha_n = d(x_{n+2}, x_n) \) and \( \beta_n = d(x_n, x_{n+1}) \). Thus, by (2.12), one can write

\[
\varphi(\alpha_n) \leq \varphi(\max\{\alpha_{n-1}, \beta_{n-1}\}) - \phi(\max\{\alpha_{n-1}, \beta_{n-1}\}) \quad \forall n \geq 1
\] (2.15)

which implies that

\[
\alpha_n \leq \max\{\alpha_{n-1}, \beta_{n-1}\}.
\] (2.16)
On the other hand, having in mind that the sequence \( \{d(x_n, x_{n+1})\} = \{\beta_n\} \) is monotone nonincreasing, so
\[
\beta_n \leq \beta_{n-1} \leq \max\{\alpha_{n-1}, \beta_{n-1}\}. \tag{2.17}
\]

From (2.16) and (2.17), we have
\[
\max\{\alpha_n, \beta_n\} \leq \max\{\alpha_{n-1}, \beta_{n-1}\} \quad \forall n \geq 1. \tag{2.18}
\]

Therefore, the sequence \( \{\max\{\alpha_n, \beta_n\}\} \) is monotone nonincreasing, so it converges to some \( t \geq 0 \). Assume that \( t > 0 \). Now, by (2.4), it is obvious that
\[
\lim\sup_{n \to \infty} \alpha_n = \lim\sup_{n \to \infty} \max\{\alpha_n, \beta_n\} = \lim_{n \to \infty} \max\{\alpha_n, \beta_n\} = t. \tag{2.19}
\]

Taking the \( \lim\sup_{n \to \infty} \) in (2.15) and using (2.19) and the properties of \( \psi \) and \( \phi \), we obtain
\[
\psi(t) = \psi\left(\lim_{n \to \infty} \sup \alpha_n\right) = \lim_{n \to \infty} \sup \psi(\alpha_n) \leq \lim_{n \to \infty} \sup \psi\left(\max\{\alpha_{n-1}, \beta_{n-1}\}\right) - \lim_{n \to \infty} \inf \phi\left(\max\{\alpha_{n-1}, \beta_{n-1}\}\right) \leq \psi\left(\lim_{n \to \infty} \max\{\alpha_{n-1}, \beta_{n-1}\}\right) - \phi\left(\lim_{n \to \infty} \max\{\alpha_{n-1}, \beta_{n-1}\}\right) = \psi(t) - \phi(t) \tag{2.20}
\]

which implies that \( \phi(t) = 0 \), so \( t = 0 \), a contradiction. Thus, from (2.19),
\[
\lim_{n \to \infty} \sup \alpha_n = 0, \tag{2.21}
\]
and hence \( \lim_{n \to \infty} \alpha_n = 0 \), so (2.11) is proved.

**Step 3.** We claim that \( T \) has a periodic point.

We argue by contradiction. Assume that \( T \) has no periodic point. Then, \( \{x_n\} \) is a sequence of distinct points, that is, \( x_n \neq x_m \) for all \( m \neq n \). We will show that, in this case, \( \{x_n\} \) is g.m.s Cauchy. Suppose to the contrary. Then, there is an \( \varepsilon > 0 \) such that for an integer \( k \) there exist integers \( m(k) > n(k) > k \) such that
\[
d(x_{n(k)}, x_{m(k)}) > \varepsilon. \tag{2.22}
\]
For every integer \( k \), let \( m(k) \) be the least positive integer exceeding \( n(k) \) satisfying (2.22) and such that

\[
d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon.
\]  

(2.23)

Now, using (2.22), (2.23), and the rectangular inequality (because \( \{x_n\} \) is a sequence of distinct points), we find that

\[
\varepsilon < d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})
\]

\[
\leq d(x_{m(k)}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + \varepsilon.
\]  

(2.24)

Then, by (2.4) and (2.11), it follows that

\[
\lim_{k \to +\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon.
\]  

(2.25)

Now, by rectangular inequality, we have

\[
d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})
\]

\[
d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}).
\]  

(2.26)

Letting \( k \to \infty \) in the above inequalities, using (2.4) and (2.25), we obtain

\[
\lim_{k \to \infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon.
\]  

(2.27)

Therefore, by (2.4) and (2.27), we get that

\[
M(x_{m(k)-1}, x_{n(k)-1}) = \max\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)})\} \to \varepsilon
\]

as \( k \to \infty \).  

(2.28)

Applying (2.2) with \( x = x_{m(k)-1} \) and \( y = x_{n(k)-1} \), we have

\[
\psi(d(x_{m(k)}, x_{n(k)})) = \psi(Tx_{m(k)-1}, Tx_{n(k)-1}) \leq \psi(M(x_{m(k)-1}, x_{n(k)-1})) - \phi(M(x_{m(k)-1}, x_{n(k)-1})).
\]  

(2.29)

Letting \( k \to \infty \) in the above inequality and using (2.25) and (2.28), we obtain

\[
\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)
\]

(2.30)

which yields that \( \phi(\varepsilon) = 0 \), so \( \varepsilon = 0 \), which is a contradiction.
Hence, \( \{ x_n \} \) is g.m.s Cauchy. Since \((X, d)\) is a complete g.m.s, there exists \( u \in X \) such that \( x_n \to u \). Applying (2.2) with \( x = x_n \) and \( y = u \), we obtain

\[
\varphi(d(x_{n+1}, Tu)) = \varphi(d(Tx, Tu)) \leq \varphi(M(x, u)) - \phi(M(x, u)) \leq \varphi(M(x, u))
\]  

(2.31)

which implies that

\[
d(x_{n+1}, Tu) \leq M(x, u),
\]

(2.32)

where

\[
M(x, u) = \max\{d(x, u), d(x, x_{n+1}), d(u, Tu)\}.
\]

(2.33)

Since \( \lim_{n \to \infty} d(x_n, u) = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \), so we obtain that

\[
\lim_{n \to \infty} M(x_n, u) = d(u, Tu).
\]

(2.34)

It follows that

\[
\limsup_{n \to \infty} d(x_{n+1}, Tu) \leq d(u, Tu).
\]

(2.35)

Next, we shall find a contradiction of the fact that \( T \) has no periodic point in each of the two following cases.

(i) If, for all \( n \geq 2 \), \( x_n \neq u \) and \( x_n \neq Tu \), then by rectangular inequality

\[
d(u, Tu) \leq d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu),
\]

(2.36)

and, using (2.4), we get that

\[
d(u, Tu) \leq \limsup_{n \to \infty} d(x_{n+1}, Tu).
\]

(2.37)

From (2.35) and (2.37),

\[
\limsup_{n \to \infty} d(x_{n+1}, Tu) = d(u, Tu).
\]

(2.38)

Taking the \( \limsup_{n \to \infty} \) in (2.31) and using (2.34), (2.38), and the properties of \( \varphi \) and \( \phi \), we obtain

\[
\varphi(d(u, Tu)) \leq \varphi(d(u, Tu)) - \phi(d(u, Tu))
\]

(2.39)
which implies that \(d(u, Tu) = 0\), so \(u = Tu\), that is, \(u\) is a fixed point of \(T\), so \(u\) is a periodic point of \(T\). It contradicts the fact that \(T\) has no periodic point.

(ii) Let for some \(q \geq 2\), \(x_q = u\) or \(x_q = Tu\). Since \(T\) has no periodic point, then obviously \(u \neq x_0\). Indeed, if \(x_q = u = x_0\), so \(T^q x_0 = x_0\), that is, \(x_0\) is a periodic point of \(T\), while if \(x_q = Tu\) and \(x_0 = u\), so \(Tx_0 = Tu = x_q = T^r x_0 = T^{q-1}(Tx_0)\), that is, \(Tx_0\) is a periodic point of \(T\).

For all \(n \geq 0\), we have

\[
d(T^n u, u) = d(T^n x, u) = d(x_{n+q}, u) \quad \text{or} \quad d(T^n u, u) = d(T^{n-1} Tu, u) = d(T^{n-1} x, u) = d(x_{n+q-1}, u). \tag{2.40}
\]

In the two precedent identities, the integer \(q \geq 2\) is fixed, and so \(\{x_{n+q}\}\) and \(\{x_{n+q-1}\}\) are subsequences from \(\{x_n\}\), and since \(\{x_n\}\) g.m.s. converges to \(u\) in \((X, d)\) which is assumed to be Hausdorff, so the two subsequences g.m.s. converge to same unique limit \(u\), that is,

\[
\lim_{n \to \infty} d(x_{n+q}, u) = \lim_{n \to \infty} d(x_{n+q-1}, u) = 0. \tag{2.41}
\]

Thus,

\[
\lim_{n \to \infty} d(T^n u, u) = 0. \tag{2.42}
\]

Again, since \((X, d)\) is Hausdorff, then by (2.42),

\[
\lim_{n \to \infty} d(T^{n+2} u, u) = 0. \tag{2.43}
\]

On the other hand, since \(T\) has no periodic point, it follows that

\[
T^s u \neq T^r u \quad \text{for any} \ s, r \in \mathbb{N}, \ s \neq r. \tag{2.44}
\]

Using (2.44) and the rectangular inequality, we may write

\[
\left| d(T^{n+1} u, Tu) - d(u, Tu) \right| \leq d(T^{n+1} u, T^{n+2} u) + d(T^{n+2} u, u). \tag{2.45}
\]

Letting \(n \to \infty\) in the above limit and proceeding as (2.4) (since the point \(x_0\) is arbitrary), using (2.43), we obtain

\[
\lim_{n \to \infty} d(T^{n+1} u, Tu) = d(u, Tu). \tag{2.46}
\]

Now, by (2.2),

\[
\psi \left( d(T^{n+1} u, Tu) \right) \leq \psi (M(T^n u, u)) - \phi (M(T^n u, u)), \tag{2.47}
\]
where
\[
M(T^n u, u) = \max \left\{ d(T^n u, u), d(T^n u, T^{n+1} u), d(u, Tu) \right\} \to d(u, Tu) \quad \text{as } n \to \infty.
\] (2.48)

Letting \( n \to \infty \) in (2.47) and using (2.46) and the above limit, we get that
\[
\varphi(d(u, Tu)) \leq \varphi(d(u, Tu)) - \phi(d(u, Tu))
\] (2.49)

which holds only if \( d(u, Tu) = 0 \), that is, \( Tu = u \), which implies that \( u \) is a periodic point of \( T \). This contradicts the fact that \( T \) has no periodic point.

Consequently, \( T \) admits a periodic point, that is, there exists \( u \in X \) such that \( u = T^p u \) for some \( p \geq 1 \).

**Step 4. Existence of a fixed point of** \( T \).  
If \( p = 1 \), then \( u = Tu \), that is, \( u \) is a fixed point of \( T \). Suppose now that \( p > 1 \). We will prove that \( a = T^{p-1} u \) is a fixed point of \( T \). Suppose that it is not the case, that is, \( T^{p-1} u \neq T^p u \). Then, \( d(T^{p-1} u, T^p u) > 0 \) and \( \phi(d(T^{p-1} u, T^p u)) > 0 \), which implies that \( \phi(M(T^{p-1} u, T^p u)) > 0 \). Now, using inequality (2.2), we obtain
\[
\varphi(d(u, Tu)) = \varphi\left(d\left(T^p u, T^{p+1} u\right)\right)
\]
\[
= \varphi\left(d\left(T\left(T^{p-1} u\right), T(T^p u)\right)\right)
\]
\[
\leq \varphi\left(M\left(T^{p-1} u, T^p u\right)\right) - \phi\left(M\left(T^{p-1} u, T^p u\right)\right)
\]
\[
< \varphi\left(M\left(T^{p-1} u, T^p u\right)\right)
\] (2.50)

which by the monotone nondecreasing property of \( \varphi \) implies
\[
d(u, Tu) < M\left(T^{p-1} u, T^p u\right),
\] (2.51)

where
\[
M\left(T^{p-1} u, T^p u\right) = \max \left\{ d\left(T^{p-1} u, T^p u\right), d\left(T^{p-1} u, T^p u\right), d\left(T^p u, T^{p+1} u\right) \right\}
\]
\[
= \max \left\{ d\left(T^{p-1} u, T^p u\right), d(u, Tu) \right\} = d\left(T^{p-1} u, T^p u\right)
\] (2.52)

because otherwise we get a contradiction with (2.51). Thus, (2.51) becomes
\[
d(u, Tu) < d\left(T^{p-1} u, T^p u\right).
\] (2.53)
Again, using (2.2), we have

\[
\varphi\left(\mathcal{d}\left(T^{p-1}u, T^p u\right)\right) = \varphi\left(d\left(T\left(T^{p-2}u\right), T\left(T^{p-1}u\right)\right)\right)
\leq \varphi\left(M\left(T^{p-2}u, T^{p-1}u\right)\right) - \phi\left(M\left(T^{p-2}u, T^{p-1}u\right)\right)
< \varphi\left(M\left(T^{p-2}u, T^{p-1}u\right)\right)
\]  

(2.54)

Again, this implies that

\[
d\left(T^{p-1}u, T^p u\right) < M\left(T^{p-2}u, T^{p-1}u\right),
\]  

(2.55)

Where

\[
M\left(T^{p-2}u, T^{p-1}u\right) = \max\left\{d\left(T^{p-2}u, T^{p-1}u\right), d\left(T^{p-2}u, T^{p-1}u\right), d\left(T^{p-1}u, T^{p}u\right)\right\}
\]

\[
= \max\left\{d\left(T^{p-2}u, T^{p-1}u\right), d\left(T^{p-1}u, T^{p}u\right)\right\} = d\left(T^{p-2}u, T^{p-1}u\right)
\]  

(2.56)

because of (2.55). Thus, from (2.55),

\[
d\left(T^{p-1}u, T^p u\right) < d\left(T^{p-2}u, T^{p-1}u\right).
\]  

(2.57)

Continuing this process as (2.53) and (2.57), we find that

\[
d(u, Tu) < d\left(T^{p-1}u, T^p u\right) < d\left(T^{p-2}u, T^{p-1}u\right) < \cdots < d(u, Tu)
\]  

(2.58)

which is a contradiction. We deduce that \(a = T^{p-1}u\) is a fixed point of \(T\).

**Step 5. Uniqueness of the fixed point of \(T\).**

Suppose that there are two distinct points \(b, c \in X\) such that \(Tb = b\) and \(Tc = c\). Then, \(M(b, c) = \max\{d(b, c), d(b, Tb), d(c, Tc)\} = d(b, c)\) and \(\phi(d(b, c)) > 0\). By (2.2), we obtain

\[
\varphi(d(b, c)) = \varphi(d(Tb, Tc)) \leq \varphi(M(b, c)) - \phi(M(b, c))
\]

\[
= \varphi(d(b, c)) - \phi(d(b, c)) < \varphi(d(b, c))
\]  

(2.59)

a contradiction. Thus, \(T\) has a unique fixed point. This completes the proof of Theorem 2.2.

\[\Box\]

Now, we state some corollaries of Theorem 2.2, which are given in the following.

**Corollary 2.3.** Let \((X, d)\) be a Hausdorff and complete generalized metric space. Suppose that \(T : X \rightarrow X\) is such that, for all \(x, y \in X\), there exists \(k \in [0, 1)\) and

\[
d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty)\},
\]  

(2.60)

then \(T\) has a unique fixed point.
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Proof. It suffices to take \( q(t) = t \) and \( \phi(t) = (1 - k)t \) in Theorem 2.2.

Corollary 2.4. Let \((X, d)\) be a Hausdorff and complete generalized metric space. Suppose that \( T : X \to X \) is such that, for all \( x, y \in X \), there exists \( \alpha \in [0, 1/2) \) and

\[
(d(Tx, Ty)) \leq \alpha [d(x, Tx) + d(y, Ty)],
\]

then \( T \) has a unique fixed point.

Proof. Let \( k = 2\alpha \), so \( k \in [0, 1) \). Also, if (2.61) holds, so

\[
(d(Tx, Ty)) \leq \alpha [d(x, Tx) + d(y, Ty)] = k \frac{d(x, Tx) + d(y, Ty)}{2}
\]

\[
\leq k \max\{d(x, y), d(x, Tx), d(y, Ty)\}.
\]

Then, it suffices to apply Corollary 2.3.

Another easy consequence of Corollary 2.3 (a Reich contraction type) is the following.

Corollary 2.5. Let \((X, d)\) be a Hausdorff and complete generalized metric space. Suppose that \( T : X \to X \) is such that, for all \( x, y \in X \), there exists \( k \in [0, 1/3) \) and

\[
(d(Tx, Ty)) \leq k [d(x, y) + d(x, Tx) + d(y, Ty)],
\]

then \( T \) has a unique fixed point.

Corollary 2.6. Let \( T \) satisfy the conditions of Theorem 2.2, except that condition (2.2) is replaced by the following: there exist positive Lebesgue integrable functions \( u \) and \( v \) on \( \mathbb{R} \) such that \( \int_0^\epsilon u(t)dt > 0 \) and \( \int_0^\epsilon v(t)dt > 0 \) for each \( \epsilon > 0 \) and that

\[
\int_0^{\Phi(d(Tx, Ty))} u(t)dt \leq \int_0^{\Phi(M(x, y))} u(t)dt - \int_0^{\Phi(M(x, y))} v(t)dt.
\]

Then, \( T \) has a unique fixed point.

Proof. Consider the functions

\[
\varphi_0(x) = \int_0^x u(t)dt, \quad \varphi_1(x) = \int_0^x v(t)dt.
\]

Then, (2.64) becomes

\[
(\varphi_0 \circ \varphi)(d(Tx, Ty)) \leq (\varphi_0 \circ \varphi)(M(x, y)) - (\varphi_1 \circ \varphi)(M(x, y)),
\]

And, putting \( q_0 = \varphi_0 \circ \varphi \) and \( \Phi_0 = \varphi_1 \circ \varphi \) and applying Theorem 2.2, we obtain the proof of Corollary 2.6 (it is easy to verify that \( q_0 \in \Psi \) and \( \Phi_0 \in \Phi \)).
Corollary 2.7. Let \((X, d)\) be a Hausdorff and complete generalized metric space. Let \(T : X \to X\). Assume there exist positive Lebesgue integrable functions \(u\) and \(v\) on \(\mathbb{R}_+\) such that \(\int_0^\infty u(t)dt > 0\) and \(\int_0^\infty v(t)dt > 0\) for each \(\varepsilon > 0\) and for all \(x, y \in X\), and
\[
\int_0^{d(Tx,Ty)} u(t)dt \leq \int_0^{M(x,y)} u(t)dt - \int_0^{M(x,y)} v(t)dt,
\]
then \(T\) has a unique fixed point.

Proof. It follows by taking \(\varphi(t) = \psi(t) = t\) in Corollary 2.6. \(\square\)

Corollary 2.8. Let \((X, d)\) be a Hausdorff and complete generalized metric space. Let \(T : X \to X\). Assume there exist \(k \in [0, 1)\) and a positive Lebesgue integrable function \(u\) on \(\mathbb{R}_+\) such that \(\int_0^\infty u(t)dt > 0\) for each \(\varepsilon > 0\) and for all \(x, y \in X\), and
\[
\int_0^{d(Tx,Ty)} u(t)dt \leq k \int_0^{\max\{d(x,y),d(x,Tx),d(y,Ty)\}} u(t)dt,
\]
then \(T\) has a unique fixed point.

Proof. It suffices to take \(v(t) = (1-k)u(t)\) in Corollary 2.7. \(\square\)

Finally, let us finish this paper by noticing the following remark.

Remark 2.9. (i) Theorem 2.2 extends Theorem 3.1 of Lakzian and Samet [9].
(ii) Corollary 2.3 extends the results of Branciari [1], Azam and Arshad [2], and Sarma et al. [13].
(iii) Corollary 2.8 extends Theorem 2 of Samet [11].
(iv) Several publications attempting to generalize fixed-point theorems in metric spaces to g.m.s are plagued by the use of some false properties given in [1] (see, e.g., [2–5]). This was observed by Das and Dey [7] who proved a fixed-point theorem without using the false properties. Subsequently, but independently, this was also observed by Samet [12] and Sarma et al. [13] who proved fixed-point theorems assuming that the generalized metric space is Hausdorff. Here, we give a rigorous proof of Theorem 2.2 by taking the same assumption.

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References

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