Research Article

On an Integral Transform of a Class of Analytic Functions

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Abstract and Applied Analysis

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For $a, \gamma \geq 0$ and $\beta < 1$, let $K_{\delta}(a, \gamma)$ denote the class of all normalized analytic functions $f$ in the open unit disc $E = \{ z : |z| < 1 \}$ such that $Re^{\phi}((1 - a + 2\gamma)(f(z)/z) + (a - 2\gamma)f'(z) + \gamma zf''(z) - \beta) > 0$, $z \in E$ for some $\phi \in \mathbb{R}$. It is known (Noshiro (1934) and Warschawski (1935)) that functions in $K_{\delta}(1, 0)$ are close-to-convex and hence univalent for $0 \leq \beta < 1$. For $f \in K_{\delta}(a, \gamma)$, we consider the integral transform $F(z) = V_{\lambda}(f)(z) := \int_{0}^{1} \lambda(t)(f(tz)/t)dt$, where $\lambda$ is a nonnegative real-valued integrable function satisfying the condition $\int_{0}^{1} \lambda(t)dt = 1$. The aim of present paper is, for given $\delta < 1$, to find sharp values of $\beta$ such that (i) $V_{\lambda}(f) \in K_{\delta}(1, 0)$ whenever $f \in K_{\delta}(a, \gamma)$ and (ii) $V_{\lambda}(f) \in K_{\delta}(a, \gamma)$ whenever $f \in K_{\delta}(a, \gamma)$.

1. Introduction

Let $A$ denote the class of analytic functions $f$ defined in the open unit disc $E = \{ z : |z| < 1 \}$ with the normalizations $f(0) = f'(0) - 1 = 0$, and let $S$ be the subclass of $A$ consisting of functions univalent in $E$. For any two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $A$, the Hadamard product (or convolution) of $f$ and $g$ is the function $f \ast g$ defined by

$$ (f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (1.1) $$

For $f \in A$, Fournier and Ruscheweyh [1] introduced the integral operator

$$ F(z) = V_{\lambda}(f)(z) := \int_{0}^{1} \lambda(t) \frac{f(tz)}{t} dt, \quad (1.2) $$
where \( \lambda \) is a nonnegative real-valued integrable function satisfying the condition \( \int_0^1 \lambda(t) \, dt = 1. \) This operator contains some well-known operators such as Libera, Bernardi, and Komatu as its special cases. Fournier and Ruscheweyh [1] applied the famous duality theory to show that for a function \( f \) in the class

\[
P(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}(f'(z) - \beta) > 0, \ z \in E \right\},
\]

the linear integral operator \( V_{\lambda}(f) \) is univalent in \( E. \) Since then, this operator has been studied by a number of authors for various choices of \( \lambda(t). \) In another remarkable paper, Barnard et al. in [2] obtained conditions such that \( V_{\lambda}(f) \in P_1(\beta) \) whenever \( f \) is in the class

\[
P_1(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left((1 - \gamma\frac{f(z)}{z} + \gamma f'(z) - \beta) > 0, \ z \in E \right\},
\]

with \( \beta < 1, \gamma \geq 0. \) Note that for \( 0 \leq \beta < 1, \) functions in \( P_1(\beta) \equiv P(\beta) \) satisfy the condition \( \Re f'(z) > \beta \) in \( E \) and thus are close-to-convex in \( E. \) A domain \( D \) in \( \mathbb{C} \) is close-to-convex if its compliment in \( \mathbb{C} \) can be written as union of nonintersecting half lines.

In 2008, Ponnusamy and Rønning [3] discussed the univalence of \( V_{\lambda}(f) \) for the functions in the class

\[
R_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left(f'(z) + \gamma zf''(z) - \beta \right) > 0, \ z \in E \right\}.
\]

In a very recent paper, Ali et al. [4] studied the class

\[
\kappa(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left((1 - \alpha + 2\gamma\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma zf''(z) - \beta \right) > 0, \ z \in E \right\},
\]

where \( \alpha, \gamma \geq 0 \) and \( \beta < 1. \) In this paper, they obtained sufficient conditions so that the integral transform \( V_{\lambda}(f) \) maps normalized analytic functions \( f \in \kappa(\beta) \) into the class of starlike functions. It is evident that \( \kappa(1, 0) \equiv D(\beta), \kappa(\alpha, 0) \equiv D_a(\beta) \) and \( \kappa(1 + 2\gamma, \gamma) \equiv R_\gamma(\beta). \)

In the present paper, we shall mainly tackle the following problems.

1. For given \( \delta < 1, \) find sharp values of \( \beta = \beta(\delta, \alpha) \) such that \( V_{\lambda}(f) \in \kappa(1, 0) \) whenever \( f \in \kappa(\beta). \)

2. For given \( \delta < 1, \) find sharp values of \( \beta = \beta(\delta) \) such that \( V_{\lambda}(f) \in \kappa(\alpha, \gamma) \) whenever \( f \in \kappa(\beta). \)

To prove one of our results, we shall need the generalized hypergeometric function \( _pF_q, \) so we define it here.
Let \( \alpha_j (j = 1, 2, \ldots, p) \) and \( \beta_j (j = 1, 2, \ldots, q) \) be complex numbers with \( \beta_j \neq 0, -1, -2, \ldots (j = 1, 2, \ldots, q) \). Then the generalized hypergeometric function \( pF_q \) is defined by

\[
p_{F_q}(z) = p_{F_q}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \frac{\sum_{n=0}^{\infty} (\alpha_1)_n \cdots (\alpha_p)_n (\beta_1)_n \cdots (\beta_q)_n z^n}{n!} (p \leq q + 1),
\]

where \((a)_n\) is the Pochhammer symbol, defined in terms of the Gamma function, by

\[
(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 
1, & n = 0, \\
\alpha(a+1) \cdots (a+n-1), & n \in \mathbb{N}. 
\end{cases} 
\]

In particular, \( zF_1 \) is called the Gaussian hypergeometric function. We note that the \( pF_q \) series in (1.7) converges absolutely for \(|z| < \infty\) if \( p < q + 1 \) and for \( z \in E \) if \( p = q + 1 \).

We shall also need the following lemma.

**Lemma 1.1** (see [5]). Let \( \beta_1 < 1, \beta_2 < 1, \) and \( \eta \in \mathbb{R} \). Then, for \( p, q \) analytic in \( E \) with \( p(0) = q(0) = 1 \), the conditions \( \Re p(z) > \beta_1 \) and \( \Re q(z) > \beta_2 \) imply \( \Re ((p * q)(z) - \delta) > 0 \), where \( 1 - \delta = (1 - \beta_1)(1 - \beta_2) \).

### 2. Main Results

We use the notations introduced in [4]. Let \( \mu \geq 0 \) and \( \nu \geq 0 \) satisfy

\[
\mu + \nu = \alpha - \gamma, \quad \mu \nu = \gamma.
\]

When \( \gamma = 0 \), then \( \mu \) is chosen to be 0, in which case, \( \nu = \alpha \geq 0 \). When \( \alpha = 1 + 2\gamma, (2.1) \) yields \( \mu + \nu = 1 + \gamma = 1 + \mu \nu \) or \( (\mu - 1)(1 - \nu) = 0 \).

(i) For \( \gamma > 0 \), then choosing \( \mu = 1 \) gives \( \nu = \gamma \).

(ii) For \( \gamma = 0 \), then \( \mu = 0 \) and \( \nu = \alpha = 1 \).

**Theorem 2.1.** Let \( \mu \geq 0, \nu \geq 0 \) satisfy (2.1). Further, let \( \delta < 1 \) be given, and define \( \beta = \beta(\delta, \mu, \nu) \) by

\[
1 - \frac{1 - \delta}{2} \left\{ 1 - \frac{1}{\nu} \int_0^1 \frac{ds}{1 + ts^\nu} \right\} \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta}{1 + t^\eta \zeta^\mu} \right) dt, \quad \gamma \neq 0,
\]

\[
1 - \frac{1 - \delta}{2} \left\{ 1 - \frac{1}{\alpha} \int_0^1 \frac{\lambda(t)}{1 + t} dt + \left( \frac{1}{\alpha} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta}{1 + t^\eta \zeta^\mu} \right) dt \right\}^{-1}, \quad \gamma = 0 \quad (\mu = 0, \nu = \alpha > 0).
\]

If \( f \in \mathcal{H}_p(\alpha, \gamma) \), then \( F = V_1(f) \in \mathcal{H}_6(1, 0) \subset S \). The value of \( \beta \) is sharp.

**Proof.** The case \( \gamma = 0 \) \( (\mu = 0, \nu = \alpha > 0) \) corresponds to Theorem 1.5 in [2]. So we assume that \( \gamma > 0 \).
Define
\[(1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma zf''(z) = H(z). \quad (2.3)\]

Writing \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\), it follows that
\[H(z) = 1 + \sum_{n=1}^{\infty} a_{n+1} (n\nu + 1)(n\mu + 1) z^n. \quad (2.4)\]

It is a simple exercise to see that
\[f'(z) = H(z) * {}_3F_2\left(2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; z\right). \quad (2.5)\]

Let \(F(z) = V_1(f)(z)\), where \(V_1(f)\) is defined by (1.2). Then for \(\gamma \neq 0\), we can write
\[F'(z) = f'(z) * \int_0^1 \frac{\lambda(t)}{1-tz} dt
= H(z) * {}_3F_2\left(2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; z\right) * \int_0^1 \frac{\lambda(t)}{1-tz} dt \quad (2.6)\]
\[= H(z) * \int_0^1 \lambda(t) {}_3F_2\left(2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; tz\right) dt.\]

Since \(f \in \mathcal{K}_\mu(a, \gamma)\), it follows that \(\Re\{e^{i\phi}(H(z) - \beta)\} > 0\) for some \(\phi \in \mathbb{R}\). Now, for each \(\gamma > 0\), we first claim that
\[\Re\left[\int_0^1 \lambda(t) {}_3F_2\left(2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; tz\right) dt\right] > 1 - \frac{1 - \delta}{2(1 - \beta)}, \quad z \in E, \quad (2.7)\]

which, by Lemma 1.1, implies that \(F \in \mathcal{K}_0(1, 0)\). Therefore, it suffices to verify the inequality (2.7). Using the identity (which can be checked by comparing the coefficients of \(z^n\) on both sides)
\[\begin{align*}
{}_3F_2 (2, b, c; d, e; z) &= (d-1) {}_3F_2 (1, b, c; d-1, e; z) - (d-2) {}_3F_2 (1, b, c; d, e; z),
\end{align*} \quad (2.8)\]

it follows that
\[\begin{align*}
{}_3F_2\left(2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; z\right) &= \frac{1}{\nu} \int_0^1 \frac{ds}{1-zs^\mu} + \left(1 - \frac{1}{\nu}\right) \int_0^1 \frac{d\eta d\zeta}{1-z\eta^{\mu} \zeta^{\mu}}.
\end{align*} \quad (2.9)\]
Thus,

\[
\int_0^1 \frac{\lambda(t)}{t^{3/2}} \mathbf{F}_2 \left( \frac{2}{v}, \frac{1}{\mu}, \frac{1}{v+1}, \frac{1}{\mu+1}; tz \right) dt \\
= \int_0^1 \frac{1}{\nu} \int_0^1 \frac{ds}{1-tzs^\mu} + \left( 1 - \frac{1}{\nu} \right) \int_0^1 \frac{d\eta}{1-t\eta^n\zeta^\mu} \right) dt.
\]

Therefore, for \( \gamma > 0 \), we have

\[
\Re \left[ \int_0^1 \frac{\lambda(t)}{t^{3/2}} \mathbf{F}_2 \left( \frac{2}{v}, \frac{1}{\mu}, \frac{1}{v+1}, \frac{1}{\mu+1}; tz \right) dt \right] \\
> \frac{1}{\nu} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1+ts^\mu} \right) dt + \left( 1 - \frac{1}{\nu} \right) \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta}{1+t\eta^n\zeta^\mu} \right) dt
\]

(2.11)

= 1 - \frac{1 - \delta}{2(1 - \beta)},

in the view of (2.2).

To prove the sharpness, let \( f \in \mathcal{K}_\beta(\alpha, \gamma) \) be the function determined by

\[
(1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma zf''(z) = \beta + \left( 1 - \beta \right) \frac{1+z}{1-z}.
\]

(2.12)

Using a series expansion, we see that we can write

\[
f(z) = z + \sum_{n=0}^{\infty} \frac{2(1-\beta)}{(nv+1-\nu)(n\mu+1-\mu)} z^n.
\]

(2.13)

Then,

\[
F(z) = V_1(f)(z) = z + 2(1-\beta) \sum_{n=0}^{\infty} \frac{\eta_n}{(nv+1-\nu)(n\mu+1-\mu)} z^n,
\]

(2.14)
where \( q_n = \int_0^1 \lambda(t) t^{n-1} \, dt \). Equation (2.2) can be restated as

\[
\frac{1}{1 - \beta} = \frac{2}{1 - \delta} \left\{ 1 - \frac{1}{v} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1 + ts^u} \right) \, dt + \left( \frac{1}{v} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta d\zeta}{1 + tv^\nu z^\mu} \right) \, dt \right\}
\]

\[
= \frac{2}{1 - \delta} \left\{ 1 + \int_0^1 \lambda(t) \left( -\frac{1}{v} \int_0^1 \frac{ds}{1 + ts^u} + \left( \frac{1}{v} - 1 \right) \int_0^1 \frac{d\eta d\zeta}{1 + tv^\nu z^\mu} \right) \, dt \right\}
\]

\[
= \frac{2}{1 - \delta} \int_0^1 \lambda(t) \left\{ \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \nu^{n-1}}{(n\mu + 1 - \mu)} \left( -\frac{1}{v} + \left( \frac{1}{v} - 1 \right) \frac{1}{(nv + 1 - v)} \right) \right\} \, dt
\]

\[
= -\frac{2}{1 - \delta} \sum_{n=2}^{\infty} \frac{(-1)^{n-1} nq_n}{(nv + 1 - v)(n\mu + 1 - \mu)}.
\]

Finally,

\[
F'(z) = 1 + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{nq_n}{(nv + 1 - v)(n\mu + 1 - \mu)} z^{n-1},
\]

which for \( z = -1 \) takes the value

\[
F'(-1) = 1 + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{(-1)^{n-1} nq_n}{(nv + 1 - v)(n\mu + 1 - \mu)} = 1 + 2(1 - \beta) \left\{ \frac{-(1 - \delta)}{2(1 - \beta)} \right\} = \delta.
\]

This shows that the result is sharp.

Letting \( \gamma = 0 \) and \( \alpha = 1 \) in Theorem 1.1, we obtain the following result of Ruscheweyh [6].

**Corollary 2.2.** Let \( \delta < 1 \), and define \( \beta = \beta(\delta, 1) < 1 \) by

\[
\beta(\delta) = 1 - \frac{1 - \delta}{2} \left\{ 1 - \int_0^1 \frac{\lambda(t)}{1 + t} \, dt \right\}^{-1}.
\]

If \( f \in \mathcal{W}_\beta(1, 0) \equiv \mathcal{D}_1(\beta) \), then \( F = V_\lambda(f) \in \mathcal{W}_\delta(1, 0) \subset S \). The value of \( \beta \) is sharp.

**Theorem 2.3.** Let \( \delta < 1 \) and \( \alpha, \gamma \geq 0 \), and define \( \beta = \beta(\delta) < 1 \) by

\[
\frac{\beta}{1 - \beta} = -\int_0^1 \frac{\lambda(t) \left( (1 + \delta)/(1 - \delta) \right) t}{(1 + t)} \, dt.
\]

If \( f \in \mathcal{W}_\beta(\alpha, \gamma) \), then \( V_\lambda(f) \in \mathcal{W}_\delta(\alpha, \gamma) \). The value of \( \beta \) is sharp.
Abstract and Applied Analysis

Proof. The idea of the proof is similar to the one used to prove Theorem 2 in [1]. Let \( F(z) = V_\lambda(f)(z) = \int_0^1 \lambda(t)(f(tz)/t)\,dt \). Clearly,

\[
F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz}\,dt \ast f'(z). \tag{2.20}
\]

Since, \( f \in \mathcal{K}_\beta(\alpha, \gamma) \), so with

\[
g(z) = \frac{(1-\alpha+2\gamma)(f(z)/z) + (\alpha-2\gamma)f'(z) + \gamma zf''(z) - \beta}{1-\beta}, \tag{2.21}
\]

we have \( \Re[e^{i\phi}g(z)] > 0 \), where \( \phi \in \mathbb{R} \).

For \( \gamma \neq \alpha/2 \),

\[
f'(z) = \frac{1}{\alpha-2\gamma} \left( \beta + (1-\beta)g(z) \right) - \frac{1-\alpha+2\gamma}{\alpha-2\gamma} \frac{f(z)}{z} - \frac{\gamma}{\alpha-2\gamma}zf''(z). \tag{2.22}
\]

Putting this value in (2.20),

\[
F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz}\,dt \ast \left( \frac{1}{\alpha-2\gamma} \left( \beta + (1-\beta)g(z) \right) - \frac{1-\alpha+2\gamma}{\alpha-2\gamma} \frac{f(z)}{z} - \frac{\gamma}{\alpha-2\gamma}zf''(z) \right). \tag{2.23}
\]

Equivalently,

\[
F'(z) = \frac{1}{\alpha-2\gamma} g(z) \ast \left[ \beta + (1-\beta) \int_0^1 \frac{\lambda(t)}{1-tz}\,dt \right] - \frac{1-\alpha+2\gamma}{\alpha-2\gamma} \frac{F(z)}{z} - \frac{\gamma}{\alpha-2\gamma}zF''(z). \tag{2.24}
\]

Thus

\[
(1-\alpha+2\gamma)(F(z)/z) + (\alpha-2\gamma)F'(z) + \gamma zF''(z) = g(z) \ast \left[ \beta + (1-\beta) \int_0^1 \frac{\lambda(t)}{1-tz}\,dt \right]. \tag{2.25}
\]

In the case when \( \gamma = \alpha/2 \),

\[
g(z) = \frac{f(z)/z + \gamma zf''(z) - \beta}{1-\beta}. \tag{2.26}
\]

Since

\[
\frac{f(z)}{z} = \beta + (1-\beta)g(z) - \gamma zf''(z), \tag{2.27}
\]
This leads to,

\[ \frac{F(z)}{z} + \gamma zF''(z) = g(z) * \left[ \beta + (1 - \beta) \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] \tag{2.28} \]

which is clearly (2.25) with \( \gamma = \alpha/2 \).

Further \( F \in \mathcal{K}_0(\alpha, \gamma) \) if and only if \( G(z) := (F(z) - \delta z)/(1 - \delta) \in \mathcal{K}_0(\alpha, \gamma) \). Now using (2.25), we obtain

\[ (1 - \alpha + 2\gamma) \frac{G(z)}{z} + (\alpha - \gamma)G'(z) + \gamma zG''(z) = g(z) * \left[ \frac{\beta - \delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] \tag{2.29} \]

Since \( \Re e^\phi g(z) > 0 \) for some \( \phi \in \mathbb{R} \), it follows by duality principle [8, page 23] that

\[ (1 - \alpha + 2\gamma) \frac{G(z)}{z} + (\alpha - 2\gamma)G'(z) + \gamma zG''(z) \neq 0 \tag{2.30} \]

if, and only if,

\[ \Re \left[ \frac{\beta - \delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] > \frac{1}{2}. \tag{2.31} \]

Using \( \Re (1/(1 - tz)) > 1/(1 + t) \), we get

\[ \Re \left[ \frac{\beta - \delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] > 1 - \beta \left[ \frac{\beta - \delta}{1 - \delta} + \int_0^1 \frac{\lambda(t)}{1 + t} dt \right]. \tag{2.32} \]

By using (2.19), we have

\[ \frac{\beta - (1 + \delta)/2}{1 - \beta} = -\int_0^1 \frac{\lambda(t)}{(1 + t)} dt. \tag{2.33} \]

Thus,

\[ \frac{\beta - \delta}{1 - \beta} + \int_0^1 \frac{\lambda(t)}{1 + t} dt = \frac{1}{2} \frac{1 - \delta}{1 - \beta}. \tag{2.34} \]

which implies that

\[ \Re \left[ \frac{\beta - \delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] > 1 - \beta \left[ \frac{\beta - \delta}{1 - \delta} + \int_0^1 \frac{\lambda(t)}{1 + t} dt \right] = \frac{1}{2}. \tag{2.35} \]
Thus, we deduce, using duality principle, that $(1 - \alpha + 2\gamma)(G(z)/z) + (\alpha - \gamma)G'(z) + \gamma zG''(z)$ is contained in a half plane not containing the origin. So, $G \in \mathcal{K}_0(\alpha, \gamma)$ and hence $F \in \mathcal{K}_b(\alpha, \gamma)$.

To prove the sharpness, let $f(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} (z^n / (n\mu + 1 - \mu)(nv + 1 - \nu))$.

\[ F(z) = V_1(f)(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{z^n \omega_n}{(n\mu + 1 - \mu)(nv + 1 - \nu)}, \quad \text{where } \omega_n = \int_0^1 \frac{\lambda(t)t^{n-1}}{1 + t} \, dt. \tag{2.36} \]

Further,

\[ \frac{\beta}{1 - \beta} = -\frac{1}{\int_0^1 \frac{\lambda(t)(1 - ((1 + \delta)/(1 - \delta))t)}{1 + t} \, dt} \tag{2.37} \]

gives

\[ \frac{\beta}{1 - \beta} = -1 + \int_0^1 \frac{\lambda(t)(1 + (1 + \delta)/(1 - \delta))}{1 + t} \, dt, \tag{2.38} \]

or

\[ \frac{1}{1 - \beta} = \frac{2}{1 - \delta} \int_0^1 \frac{\lambda(t)1 + t}{1 + t} \, dt = \frac{2}{1 - \delta} \sum_{n=2}^{\infty} (-1)^n \omega_n. \tag{2.39} \]

Further, assume that

\[ H(z) = (1 - \alpha + 2\gamma) \frac{F(z)}{z} + (\alpha - \gamma)F'(z) + \gamma zF''(z). \tag{2.40} \]

Since $F(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} (\omega_n z^n / (n\mu + 1 - \mu)(nv + 1 - \nu))$, so,

\[ H(z) = 1 + 2(1 - \beta) \sum_{n=2}^{\infty} \omega_n z^{n-1}. \tag{2.41} \]

Therefore, for $z = -1$,

\[ H(-1) = 1 - 2(1 - \beta) \sum_{n=2}^{\infty} \omega_n (-1)^n = 1 - 2(1 - \beta) \frac{1 - \delta}{2(1 - \beta)} = \delta. \tag{2.42} \]

This shows that the result is sharp. \qed

Letting $\gamma = 0$ in Theorem 2.3 above, we obtain the following result of Kim and Rønning [9].
Corollary 2.4. Let $\delta < 1$ and $\alpha \geq 0$, and define $\beta = \beta(\delta)$ by
\[
\frac{\beta}{1 - \beta} = - \int_{0}^{1} \lambda(t) \frac{(1 - ((1 + \delta)/(1 - \delta)))t}{(1 + t)} dt.
\] (2.43)

If $f \in \mathcal{W}_{\rho}(\alpha, 0) \equiv \mathcal{P}_{\alpha}(\beta)$, then $V_{\lambda}(f) \in \mathcal{W}_{\rho}(\alpha, 0) \equiv \mathcal{P}_{\alpha}(\delta)$. The value of $\beta$ is sharp.

Upon setting $\lambda(t) = (1 + c)t^z$ with $-1 < c$, we have the following corollary.

Corollary 2.5. Let $\delta < 1$, $\alpha, \gamma \geq 0$, and $-1 < c \leq 0$ be given, and let $G(z)$ be defined by
\[
G(z) = \frac{(1 + c)}{z^\gamma} \int_{0}^{\gamma} u^{\gamma - 1} f(u) du.
\] (2.44)

Suppose that $f \in \mathcal{W}_{\rho}(\alpha, \gamma)$, then $G \in \mathcal{W}_{\rho}(\alpha, 0)$, where
\[
\beta = \frac{2(1 + c)}{2(1 + c)} F_{1}(1, 2 + c; 3 + c, -1) - (2 + c).
\] (2.45)

The constant $\beta$ is sharp.

The special case of Corollary 2.5 (with $\gamma = 0$) has been obtained by Aghalary et al. [11].

References
