Research Article

Characteristic Functions and Borel Exceptional Values of $E$-Valued Meromorphic Functions

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Abstract

The main purpose of this paper is to investigate the characteristic functions and Borel exceptional values of $E$-valued meromorphic functions from the $C_R = \{z : |z| < R\}$, $0 < R \leq +\infty$ to an infinite-dimensional complex Banach space $E$ with a Schauder basis. Results obtained extend the relative results by Xuan, Wu and Yang, Bhoosnurmath, and Pujari.

1. Introduction and Preliminaries


In [4], Xuan and Wu also proved Chuang’s inequality (see, e.g., [5]) of $E$-valued meromorphic mapping $f(z)$ in the whole complex plane, which compares the relationship between $T(r, f)$ and $T(r, f')$, and also obtained that the order and the lower order of $E$-valued meromorphic mapping $f(z)$ and those of its derivative $f'(z)$ are the same. In Section 2, we...
shall prove that Chuang’s inequality is valid for \( E \)-valued meromorphic mapping \( f(z) \) in the unit disc and prove that for any infinite-order \( E \)-valued meromorphic function \( f(z) \) defined in the unit disc has the same Xiong’s proximate order as its derivative \( f'(z) \).

In [5], Yang obtained much stronger results than those of Gopalakrishna and Bhoosnurmath [6] for the Borel exceptional values of meromorphic functions dealing with multiple values. In Section 3, we shall extend Le Yang’s result to \( E \)-valued meromorphic functions of finite and infinite orders in

\[
\mathbb{C}_R := \{ z : |z| < R \}, \quad 0 < R \leq +\infty. \tag{1.1}
\]

In the following, we introduce the definitions, notations, and results of [3, 4] which will be used in this paper.

Let \( (E, \| \cdot \|) \) be an infinite dimension complex Banach space with Schauder basis \( \{ e_j \} \) and the norm \( \| \cdot \| \). Thus, an \( E \)-valued meromorphic function \( f(z) \) defined in \( \mathbb{C}_R, 0 < R \leq +\infty \) can be written as

\[
f(z) = (f_1(z), f_2(z), \ldots, f_k(z), \ldots). \tag{1.2}
\]

Let \( E_n \) be an \( n \)-dimensional projective space of \( E \) with a basis \( \{ e_j \}_1^n \). The projective operator \( P_n : E \to E_n \) is a realization of \( E_n \) associated with basis.

The elements of \( E \) are called vectors and are usually denoted by letters from the alphabet: \( a, b, c, \ldots \). The symbol 0 denotes the zero vector of \( E \). We denote vector infinity, complex number infinity, and the norm infinity by \( \infty, \infty, \) and \( +\infty \), respectively. A vector-valued mappings is called holomorphic (meromorphic) if all \( f_j(z) \) are holomorphic (some of \( f_j(z) \) are meromorphic). The \( j \)-th derivative \( j = 1, 2, \ldots \) of \( f(z) \) is defined by

\[
f^{(j)}(z) = (f_1^{(j)}(z), f_2^{(j)}(z), \ldots, f_k^{(j)}(z), \ldots). \tag{1.3}
\]

A point \( z_0 \in \mathbb{C}_r \) is called a “pole” (or \( \infty \) point) of

\[
f(z) = (f_1(z), f_2(z), \ldots, f_k(z), \ldots) \tag{1.4}
\]

if \( z_0 \) is a pole (or \( \infty \) point) of at least one of the component functions \( f_k(z) \) \( (k = 1, 2, \ldots) \). A point \( z_0 \in \mathbb{C}_r \) is called a “zero” of \( f(z) = (f_1(z), f_2(z), \ldots, f_k(z), \ldots) \) if \( z_0 \) is a zero of all the component functions \( f_k(z) \) \( (k = 1, 2, \ldots) \). A point \( z_0 \in \mathbb{C}_r \) is called a pole or an \( \infty \)-point of \( f(z) \) of multiplicity \( q \in \mathbb{N}^* \), meaning that in such a point \( z_0 \) at least one of the meromorphic component functions \( f_j(z) \) has a pole of this multiplicity in the ordinary sense of function theory. A point \( z_0 \in \mathbb{C}_r \) is called a zero of \( f(z) \) of multiplicity \( q \in \mathbb{N}^* \), meaning that in such a point \( z_0 \) all component functions \( f_j(z) \) vanish, each with at least this multiplicity.
Let $n(r, f)$ or $n(r, \infty)$ denote the number of poles of $f(z)$ in $|z| \leq r$ and let $n(r, a, f)$ denote the number of $a$-points of $f(z)$ in $|z| \leq r$, counting with multiplicities. Define the volume function associated with $E$-valued meromorphic function $f(z)$ by

\[
V(r, \infty, f) = V(r, f) = \frac{1}{2\pi} \int_{\mathbb{C}} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi)\| \, dx \wedge dy, \quad \xi = x + iy,
\]

\[
V(r, a, f) = \frac{1}{2\pi} \int_{\mathbb{C}} \log \left| \frac{r}{\xi} \right| \Delta \log \|f(\xi) - a\| \, dx \wedge dy, \quad \xi = x + iy,
\]

and the counting function of finite or infinite $a$-points by

\[
N(r, f) = n(0, f) \log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt,
\]

\[
N(r, \infty) = n(0, \infty) \log r + \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} \, dt,
\]

\[
N(r, a, f) = n(0, a, f) \log r + \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} \, dt,
\]

respectively. Next, we define

\[
m(r, f) = m(r, \infty, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\| f(re^{i\theta}) \right\| d\theta,
\]

\[
m(r, a) = m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\left\| f(re^{i\theta}) - a \right\|} d\theta,
\]

\[
T(r, f) = m(r, f) + N(r, f).
\]

Let $\overline{n}(r, f)$ or $\overline{n}(r, \infty)$ denote the number of poles of $f(z)$ in $|z| \leq r$, and let $\overline{n}(r, a, f)$ denote the number of $a$-points of $f(z)$ in $|z| \leq r$, ignoring multiplicities. Similarly, we can define the counting functions $\overline{N}(r, f)$, $\overline{N}(r, \infty)$, and $\overline{N}(r, a, f)$ of $\overline{n}(r, f)$, $\overline{n}(r, \infty)$, and $\overline{n}(r, a, f)$.

If $f(z)$ is an $E$-valued meromorphic function in the whole complex plane, then the order and the lower order of $f(z)$ are defined by

\[
\lambda(f) = \lim \sup_{r \to +\infty} \frac{\log^+ T(r, f)}{\log r},
\]

\[
\mu(f) = \lim \inf_{r \to +\infty} \frac{\log^+ T(r, f)}{\log r}.
\]
If $f(z)$ is an $E$-valued meromorphic function in $\mathbb{C}_R$, $0 < R < +\infty$, then the order and the lower order of $f(z)$ are defined by

$$
\lambda(f) = \limsup_{r \to R^-} \frac{\log T(r, f)}{\log^+(1/(R - r))},
$$

(1.11)

$$
\mu(f) = \liminf_{r \to R^-} \frac{\log T(r, f)}{\log^+(1/(R - r))}.
$$

Lemma 1.1. Let $B(x)$ be a positive and continuous function in $[0, +\infty)$ which satisfies $\limsup_{x \to +\infty} (\log B(x)/\log x) = \infty$. Then there exists a continuously differentiable function $\rho(x)$, which satisfies the following conditions.

(i) $\rho(x)$ is continuous and nondecreasing for $x \geq x_0$ ($x_0 > 0$) and tends to $+\infty$ as $x \to +\infty$.

(ii) The function $U(x) = x^{\rho(x)}$ ($x \geq x_0$) satisfies the following:

$$
\lim_{x \to +\infty} \frac{\log U(X)}{\log U(x)} = 1, \quad X = x + \frac{x}{\log U(x)}.
$$

(1.12)

(iii) $\limsup_{x \to +\infty} (\log B(x)/\log U(x)) = 1$.

Lemma 1.1 is due to K. L. Hiong (also Qinglai Xiong) and $\rho(x)$ is called the proximate order of Hiong. A simple proof of the existence of $\rho(r)$ was given by Chuang [7]. Suppose that $f(z)$ is an $E$-valued meromorphic function of infinite order in the unit disk $\mathbb{C}_1$. Let $x = 1/(1 - r)$ and $X = 1/(1 - R)$. From (ii) and (iii) in Lemma 1.1, we have

$$
\lim_{r \to 1^-} \frac{\log U(1/(1 - R))}{\log U(1/(1 - r))} = 1, \quad R = \frac{r \log U(1/(1 - r)) + 1}{\log U(1/(1 - R)) + 1},
$$

(1.13)

$$
\limsup_{x \to 1^-} \frac{\log T(r, f)}{\log U(1/(1 - R))} = 1.
$$

Here, the functions $\rho(1/(1 - r))$ and $U(1/(1 - r))$ are called the proximate order and type function of $f(z)$, respectively.

Definition 1.2. An $E$-valued meromorphic function $f(z)$ in $\mathbb{C}_R$, $0 < R \leq +\infty$ is of compact projection, if for any given $\varepsilon > 0$, $\|P_n(f(z)) - f(z)\| < \varepsilon$ has sufficiently large $n$ in any fixed compact subset $D \subset \mathbb{C}_R$.

Throughout this paper, we say that $f(z)$ is an $E$-valued meromorphic function meaning that $f(z)$ is of compact projection. C.-G. Hu and Q. Hu [3] established the following Nevanlinna’s first and second main theorems of $E$-valued meromorphic functions.
Theorem 1.3. Let $f(z)$ be a nonconstant $E$-valued meromorphic function in $\mathbb{C}_R$, $0 < R \leq +\infty$. Then for $0 < r < R$, $a \in E$, $f(z) \neq a$,

$$T(r, f) = V(r, a) + N(r, a) + m(r, a) + \log^+ \|c_q(a)\| + \varepsilon(r, a).$$

(1.14)

Here, $\varepsilon(r, a)$ is a function satisfying that

$$|\varepsilon(r, a)| \leq \log^+ \|a\| + \log 2, \quad \varepsilon(r, 0) \equiv 0,$$

(1.15)

and $c_q(a) \in E$ is the coefficient of the first term in the Laurent series at the point $a$.

Theorem 1.4. Let $f(z)$ be a nonconstant $E$-valued meromorphic function in $\mathbb{C}_R$, $0 < R \leq +\infty$ and $a^{[k]} \in E \cup \{\hat{\infty}\}$ ($k = 1, 2, \ldots, q$) be $q \geq 3$ distinct points. Then for $0 < r < R$,

$$(q - 2)T(r, f) \leq \sum_{k=1}^{q} \left[ V(r, a^{[k]}) + N(r, a^{[k]}) \right] + S(r, f).$$

(1.16)

If $R = +\infty$, then

$$S(r, f) = O\left(\log T(r, f) + \log r\right)$$

(1.17)

holds as $r \to +\infty$ without exception if $f(z)$ has finite order and otherwise as $r \to +\infty$ outside a set $J$ of exceptional intervals of finite measure $\int_J dr < +\infty$. If the order of $f(z)$ is infinite and $\rho(r)$ is the proximate order of $f(z)$, then

$$S(r, f) = O\left(\log \Upsilon(r)\right)$$

(1.18)

holds as $r \to +\infty$ without exception.

If $0 < R < +\infty$, then

$$S(r, f) = O\left(\log T(r, f) + \log \frac{1}{R - r}\right)$$

(1.19)

holds as $r \to R$ without exception if $f(z)$ has finite order and otherwise as $r \to R$ outside a set $J$ of exceptional intervals of finite measure $\int_J d((r/(R - r))) < +\infty$.

In all cases, the exceptional set $J$ is independent of the choice of $a^{[k]}$. 
2. Characteristic Function of $E$-Valued Meromorphic Functions in the Unit Disc $\mathbb{C}_1$

In [4], Xuan and Wu proved the following.

**Theorem A.** Let $f(z)$ ($z \in \mathbb{C}$) be a nonconstant $E$-valued meromorphic function and $f(0) \neq \infty$. Then for $\tau > 1$ and $0 < r < R$, one has

$$T(r, f) < C_{\tau} T(\tau r, f') + \log^+ \tau r + 4 + \log^+ \|f(0)\|, \quad (2.1)$$

where $C_{\tau}$ is a positive constant.

**Theorem B.** Let $f(z)$ ($z \in \mathbb{C}$) be a nonconstant $E$-valued meromorphic function. Then we have

$$T(r, f') < 2T(r, f) + O(\log r + \log^+ T(r, f)). \quad (2.2)$$

**Theorem C.** For a nonconstant $E$-valued meromorphic function $f(z)$ ($z \in \mathbb{C}$) of order $\lambda(f) < +\infty$, one has $\lambda(f') = \lambda(f)$, $\mu(f) = \mu(f')$.

In this section, we shall prove that Theorems A, B, and C are valid for $E$-valued meromorphic function in the unit disc $\mathbb{C}_1$.

**Lemma 2.1.** Let $f(z)$ be an $E$-valued meromorphic function defined in the unit disc, and $f(0) \neq \infty$. If $0 < R < R' < 1$, then there exists a $\theta_0 \in [0, 2\pi)$, such that for any $0 \leq r \leq R$, one has

$$\log^+ \|f(e^{i\theta_0})\| \leq \frac{R' + R}{R' - R} m(R', f) + n(R', f) \log 4 + N(R', f). \quad (2.3)$$

**Lemma 2.2.** Let $f(z)$ be an $E$-valued meromorphic function defined in the unit disc, and let $0 < R < R' < R'' < 1$. Then there exists a positive number $R \leq \rho \leq R'$, such that for $|z| = \rho$, one has

$$\log^+ \|f(e^{i\theta_0})\| \leq \frac{R'' + R'}{R'' - R} m(R'', f) + n(R'', f) \log \frac{8eR''}{R'' - R}. \quad (2.4)$$

Lemmas 2.1 and 2.2 are due to Xuan and Wu [4] for the $E$-valued meromorphic function defined in the whole complex plane. From the proof of Xuan and Wu [4], we know that Lemmas 2.1 and 2.2 are also valid for the $E$-valued meromorphic function defined in the unit disc $\mathbb{C}_1$.

**Lemma 2.3.** Let $f(z)$ ($z \in \mathbb{C}_1$) be a nonconstant $E$-valued meromorphic function and $f(0) \neq \infty$. Suppose that $h(r) \geq 1$, $R = (1 + rh(r)) / (1 + h(r))$, then when $r$ sufficiently tends to 1, one has

$$n(r, f) \leq \frac{6h(r)}{1 - r} N(R, f). \quad (2.5)$$
Proof.

\[ N(R, f) = n(0, f) \log r + \int_0^R \frac{n(t, f) - n(0, f)}{t} dt = \int_0^R \frac{n(t, f)}{t} dt \]

\[ \geq \int_r^R \frac{n(t, f)}{t} dt \geq n(r, f) \log \frac{R}{r} \]

\[ = n(r, f) \log \left(1 + \frac{1-r}{r(1+h(r))}\right) \geq n(r, f) \left(\frac{1-r}{r(1+h(r))} - \frac{(1-r)/r(h(r))^2}{2}\right) \]

\[ \geq n(r, f) \left(\frac{(1-r)/r(1+h(r))}{2}\right) \geq n(r, f) \frac{1-r}{6h(r)}. \]

(2.6)

\[ \square \]

Lemma 2.4 (see [4]). Let \( f(z) \ (z \in \mathbb{C}, 0 < R \leq +\infty) \) be a nonconstant \( E \)-valued meromorphic function and \( f(0) \neq \infty \), and \( \gamma \) a curve from the origin along the segment \( \arg z = \theta \) to \( pe^{i\theta} \), and along \( \{|z| = \rho < r\} \) turn a rotation to \( pe^{i\theta} \). Then for any \( \{|z| = \rho \leq r\} \), one has

\[ \log^+ \|f(z)\| \leq \log^+ M + O(1), \]  

(2.7)

where \( M = \max\{\|f'(z)\|, z \in \gamma\} \).

Lemma 2.5 (see [3]). Let \( f(z) \) be a nonconstant \( E \)-valued meromorphic function in \( \mathbb{C}_1 \). Then for \( 0 < r < 1 \),

\[ \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left\| f'(re^{i\theta}) \right\| d\theta < K \left( \log T(r, f) + \log \frac{1}{1-r} \right). \]  

(2.8)

where \( K \) is a sufficiently large constant.

We are now in the position to establish the main results of this section.

Theorem 2.6. Let \( f(z) \ (z \in \mathbb{C}_1) \) be a nonconstant \( E \)-valued meromorphic function and \( f(0) \neq \infty \). Then for \( \varepsilon > 1 \) and any real function \( h(x) \geq 1 \), when \( r \) sufficiently tend to 1, one has

\[ T(r, f) < \frac{ch^{1+\varepsilon}(r)}{(1-r)^{1+\varepsilon}} T(R, f'), \quad R = \frac{1+rh(r)}{1+h(r)}. \]  

(2.9)
Proof. Denote \( R_1 = (R + 2r)/3, R_2 = (r + 2R)/3 \), we can get

\[
\begin{align*}
 r < R_1 < R_2 < R, \quad R_1 - r &= R_2 - R_1 = R - R_2 = \frac{R - r}{3}, \\
 R &= \frac{1 - 3R_2 h(r)}{1 + 3h(r)}, \quad R_2 + R_1 = r + R < 2, \quad 1 - R_2 = \frac{(1 - r)(1 + 3h(r))}{3(1 + h(r))} \geq \frac{1 - r}{2}; \quad (2.10) \\
 R - r &= \frac{1 - r}{1 + h(r)} \geq \frac{1 - r}{2h(r)}.
\end{align*}
\]

Applying Lemma 2.1 to \( f'(z) \) and combining Lemma 2.3, we can find a real number \( \theta_0 \in [0, 2\pi) \) such that for any \( 0 \leq t \leq R_1 \), one has

\[
\begin{align*}
\log^+ \|f'(te^{i\theta_0})\| &\leq \frac{R_2 + R_1}{R_2 - R_1} m(R_2, f') + n(R_2, f') \log 4 + N(R_2, f') \\
&\leq \left( \frac{6}{R - r} + \frac{2h(r)}{1 - R_2} \log 4 + 1 \right) T(R_2, f') \\
&\leq \left( \frac{6 + 6h(r)}{1 - r} + \frac{12h(r)}{1 - r} \log 4 + \frac{1 - r}{1 - r} \right) T(R, f') \\
&\leq \frac{6 + 6h(r) + 24h(r) + 1 - r}{1 - r} T(R, f') \leq \frac{40h(r)}{1 - r} T(R, f').
\end{align*}
\]

In view of Lemma 2.2, there is a \( \rho \in [r, R_1] \) such that for any \( z \in \{|z| = \rho\} \), one has

\[
\begin{align*}
\log^+ \|f'(z)\| &\leq \frac{R_2 + R_1}{R_2 - R_1} m(R_2, f') + n(R_2, f') \log \frac{8eR_2}{R_1 - R} \\
&\leq \left( \frac{6}{R - r} + \frac{6h(r)}{1 - R_2} \log \frac{48e h(r)}{1 - r} \right) T(R_2, f') \\
&\leq \left( \frac{6 + 6h(r)}{1 - r} + \frac{12h(r)}{1 - r} \log \frac{144h(r)}{1 - r} \right) T(R, f') \\
&\leq \left( \frac{12h(r)}{1 - r} \left( 9 + \log \frac{h(r)}{1 - r} \right) \right) T(R, f') \\
&\leq \left( \frac{12h(r)}{1 - r} \left( 9 + \left( \frac{h(r)}{1 - r} \right)^\epsilon \right) \right) T(R, f') \\
&\leq 120 \left( \frac{h(r)}{1 - r} \right)^{1+\epsilon} T(R, f').
\end{align*}
\]

From the origin along the segment \( \arg z = \theta_0 \) to \( pe^{i\theta_0} \) and along \( \{|z| = \rho\} \), turn a rotation to \( pe^{i\theta_0} \). We denote this curve by \( L \). In virtue of Lemma 2.4, we have

\[
\log^+ \|f(z)\| \leq \log^+ M + O(1) \quad (2.13)
\]
Abstract and Applied Analysis

holds for any \(|z = r \leq \rho|\), where \(M = \max\{\|f'(z)\|, z \in L\}\). In virtue of (2.11), (2.12), and (2.13), we have

\[
m(r, f) \leq m(\rho, f) \leq m(\rho, f') \leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ M d\theta \leq 121 \left( \frac{h(r)}{1 - r} \right)^{1+\varepsilon} T(R, f').
\]  

(2.14)

Hence,

\[
T(r, f) = m(r, f) + N(r, f) \leq m(r, f) + 2N(r, f') \leq 123 \left( \frac{h(r)}{1 - r} \right)^{1+\varepsilon} T(R', f').
\]

(2.15)

Theorem 2.7. Let \(f(z) (z \in \mathbb{C}_1)\) be a nonconstant \(E\)-valued meromorphic function and \(f(0) \neq 0, \infty\). Then for any \(0 < r < R < 1\), one has

\[
T(r, f') < 2T(r, f) + O\left( \log^+ \frac{1}{1 - r} + \log^+ T(r, f) \right).
\]

(2.16)

Proof. By Lemma 2.5, we have

\[
T(r, f') = m(r, f') + N(r, f')
\]

\[
\leq m(r, f) + m\left( r, \frac{f'}{f} \right) + 2N(r, f)
\]

\[
\leq 2T(r, f) + m\left( r, \frac{f'}{f} \right)
\]

\[
\leq 2T(r, f) + O\left( \log^+ \frac{1}{1 - r} + \log^+ T(r, f) \right).
\]

(2.17)

Theorem 2.8. For a nonconstant \(E\)-valued meromorphic function \(f(z) (z \in \mathbb{C}_1)\) of order \(\lambda(f) < +\infty\), one has \(\lambda(f) = \lambda(f'), \mu(f) = \mu(f')\).

Theorem 2.8 only discussed the \(E\)-valued meromorphic function of finite order. In fact, for any \(E\)-valued meromorphic function of infinite order, we have the following.

Theorem 2.9. If \(f(z) (z \in \mathbb{C}_1)\) is a nonconstant \(E\)-valued meromorphic function of order \(\lambda(f) = +\infty\), then the proximate orders of \(f(z)\) and \(f'(z)\) are the same.

Proof. Let \(h(r) = \log U(1/(1 - r))\), in view of Theorems 2.6 and 2.7, we can easily derive Theorem 2.9.
3. E-Valued Borel Exceptional Values of Meromorphic Functions in $\mathbb{C}_R$

Some definitions in this section can be found in [8].

**Definition 3.1.** Let $f(z) (z \in \mathbb{C}_R, 0 < R \leq +\infty)$ be an $E$-valued meromorphic function and $a \in E \cup \{\infty\}$, if $k$ is a positive integer, let $\overline{n}_k(r, f)$ or $\overline{n}_k(r, \infty)$ denote the number of distinct poles of $f(z)$ of order $\leq k$ in $|z| \leq r$, and let $\overline{n}_k(r, a)$ denote the number of distinct $a$-points of $f(z)$ of order $\leq k$ in $|z| \leq r$. Similarly, we can define the counting functions $\overline{N}_k(r, f)$, $\overline{N}_k(r, \infty)$, and $\overline{N}_k(r, a)$ of $\overline{n}_k(r, f)$, $\overline{n}_k(r, \infty)$, and $\overline{n}_k(r, a)$.

**Definition 3.2.** Let $f(z) (z \in \mathbb{C}_R, 0 < R \leq +\infty)$ be an $E$-valued meromorphic function and $a \in E \cup \{\infty\}$. If $R = +\infty$, we define

$$
\overline{p}_k(a, f) = \limsup_{r \to +\infty} \frac{\log^+ \left[V(a, f) + \overline{N}_k(r, a)\right]}{\log r},
$$

$$
\overline{p}(a, f) = \limsup_{r \to +\infty} \frac{\log^+ \left[V(a, f) + \overline{N}(r, a)\right]}{\log r},
$$

$$
\rho(a, f) = \limsup_{r \to +\infty} \frac{\log^+ \left[V(a, f) + N(r, a)\right]}{\log r}.
$$

If $R < +\infty$, we define

$$
\overline{p}_k(a, f) = \limsup_{r \to R} \frac{\log^+ \left[V(a, f) + \overline{N}_k(r, a)\right]}{\log(1/(R - r))},
$$

$$
\overline{p}(a, f) = \limsup_{r \to R} \frac{\log^+ \left[V(a, f) + \overline{N}(r, a)\right]}{\log(1/(R - r))},
$$

$$
\rho(a, f) = \limsup_{r \to R} \frac{\log^+ \left[V(a, f) + N(r, a)\right]}{\log(1/(R - r))}.
$$

**Definition 3.3.** Let $f(z) (z \in \mathbb{C}_R, 0 < R \leq +\infty)$ be an $E$-valued meromorphic function and $a \in E \cup \{\infty\}$ and $k$ is a positive integer, we say that $a$ is an

(i) E-valued evB (exceptional value in the sense of Borel) for $f$ for distinct zeros of order $\leq k$ if $\overline{p}_k(a, f) < \lambda(f)$;

(ii) E-valued evB for $f$ for distinct zeros if $\overline{p}(a, f) < \lambda(f)$;

(iii) E-valued evB for $f$ (for the whole aggregate of zeros) if $\rho(a, f) < \lambda(f)$.

In [5], Yang proved the following result.
Theorem D. Let \( f(z) (z \in \mathbb{C}, R = +\infty) \) be a meromorphic function with finite order \( \lambda > 0 \) and \( k_j (j = 1, 2, \ldots, q) \) be \( q \) positive integers. \( a \) is called a pseudo-Borel exceptional value of \( f(z) \) of order \( k \) if

\[
\lim \sup_{r \to +\infty} \frac{\log^* n_k(r, a)}{\log r} < \lambda(f).
\]

(3.3)

If \( f(z) \) has \( q \) distinct pseudo-Borel exceptional values \( a_j \) of order \( k_j \) \((j = 1, 2, \ldots, q)\), then

\[
\sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) \leq 2.
\]

(3.4)

It is natural to consider whether there exists a similar result, if meromorphic function \( f \) is replaced by \( E \)-valued meromorphic function \( f \). In this section, we extend the above theorem to \( E \)-valued meromorphic function in \( \mathbb{C}_R, 0 < R \leq +\infty \).

Theorem 3.4. Let \( f(z) (z \in \mathbb{C}_R, 0 < R \leq +\infty) \) be an \( E \)-valued meromorphic function with finite order \( \lambda > 0 \), \( a^{[j]}(j = 1, 2, \ldots, q) \) any system of distinct elements in \( E \cup \{ \hat{\infty} \} \), and \( k_j (j = 1, 2, \ldots, q) \) any system such that \( k_j \) is a positive integer or \( +\infty \). If \( a^{[j]} \) is an \( E \)-valued evB for \( f \) for distinct zeros of order \( \leq k_j \) \((j = 1, 2, \ldots, q)\), then

\[
\sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) \leq 2.
\]

(3.5)

Proof. By Theorem 1.4, we have

\[
(q - 2)T(r, f) \leq \sum_{j=1}^{q} \left[ V(r, a^{[j]}) + \overline{N}(r, a^{[j]}) \right] + S(r, f)
\]

(3.6)

holds for \( 0 < r < R \). For any \( j = 1, 2, \ldots, q \), we have

\[
\overline{N}(r, a^{[j]}) \leq \frac{1}{k_j + 1} \left\{ k_j \overline{N}(r, a^{[j]}) + N(r, a^{[j]}) \right\},
\]

(3.7)

\[
N(r, a^{[j]}) \leq T(r, f) - V(r, a^{[j]}) + O(1).
\]
Using (3.7) and (7) in (3.6), we get

\[(q - 2)T(r, f) \leq \sum_{j=1}^{q} \left( V(r, a^{[j]}) + \frac{1}{k_j + 1} \left\{ k_j \overline{N}_{k_j} (r, a^{[j]}) + N(r, a^{[j]}) \right\} \right) + S(r, f) \]

\[= \sum_{j=1}^{q} \left( V(r, a^{[j]}) + \frac{k_j}{k_j + 1} \overline{N}_{k_j} (r, a^{[j]}) + \frac{1}{k_j + 1} N(r, a^{[j]}) \right) + S(r, f) \]

\[\leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left( V(r, a^{[j]}) + \overline{N}_{k_j} (r, a^{[j]}) \right) + \sum_{j=1}^{q} \frac{1}{k_j + 1} T(r, f) + S(r, f). \tag{3.8} \]

Therefore, we have

\[
\left[ \sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right] T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left( V(r, a^{[j]}) + \overline{N}_{k_j} (r, a^{[j]}) \right) + S(r, f). \tag{3.9} \]

By hypothesis, we have

\[
\overline{p}_{k_j} (a^{[j]}, f) < \lambda, \quad j = 1, 2, \ldots, q. \tag{3.10} \]

If \( R = +\infty \), then there is a positive number \( \rho < \lambda \), such that for \( j = 1, 2, \ldots, q \), we can get

\[V(r, a^{[j]}) + \overline{N}_{k_j} (r, a^{[j]} \leq r^\rho). \tag{3.11} \]

Using (3.11) to (3.9), we have

\[
\left[ \sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right] T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} r^\rho + S(r, f). \tag{3.12} \]

If \( \sum_{j=1}^{q} \left( 1 - (1/(k_j + 1)) \right) > 2 \), then by Theorem 1.4 and (3.12), we can get a contradiction \( \lambda \leq \rho \). So

\[
\sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) \leq 2. \tag{3.13} \]

If \( R = +\infty \), then there is a positive number \( \rho < \lambda \), such that for \( j = 1, 2, \ldots, q \), we can get

\[V(r, a^{[j]}) + \overline{N}_{k_j} (r, a^{[j]} \leq \left( \frac{1}{R - r} \right)^\rho. \tag{3.14} \]
Using (3.14) to (3.9), we have

\[
\left[\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) - 2\right] T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left(\frac{1}{R - r}\right)^\rho + S(r, f). \tag{3.15}
\]

If \(\sum_{j=1}^{q} (1 - (1/(k_j + 1))) > 2\), then by Theorem 1.4 and (3.15), we can get a contradiction \(\lambda \leq \rho\). So

\[
\sum_{j=1}^{q} \left(1 - \frac{1}{k_j + 1}\right) \leq 2. \tag{3.16}
\]

From the proof of Theorem 3.4, we can get the following.

**Corollary 3.5.** Let \(f(z) (z \in \mathbb{C}_R, 0 < R \leq +\infty)\) be a nonconstant \(E\)-valued meromorphic function. Then for any system \(a^{[j]} (j = 1, 2, \ldots, t)\) of distinct elements in \(E \cup \{\hat{\infty}\}\) and any system \(k_j (j = 1, 2, \ldots, t)\) such that \(k_j\) is a positive integer or \(+\infty\), we have the following:

1. if all of \(a^{[j]} (j = 1, 2, \ldots, q)\) in \(E\), then

\[
\left(q - \sum_{j=1}^{q} \frac{1}{k_j + 1} - 2\right) T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left(V\big(r, a^{[j]}, f\big) + \overline{N}_{k_j}\big(r, a^{[j]}, f\big)\right) + S(r, f), \tag{3.17}
\]

2. if one of \(a^{[j]} (j = 1, 2, \ldots, q)\) is \(\hat{\infty}\), say \(a^{[q]} = \hat{\infty}\). Then,

\[
\left(q - \sum_{j=1}^{q} \frac{1}{k_j + 1} - 2\right) T(r, f) \leq \sum_{j=1}^{q-1} \frac{k_j}{k_j + 1} \left(V\big(r, a^{[j]}, f\big) + \overline{N}_{k_j}\big(r, a^{[j]}, f\big)\right) + \frac{k_q}{k_q + 1} \overline{N}_{k_q}(r, f) + S(r, f). \tag{3.18}
\]

**Remark 3.6.** If \(R = +\infty\), let \(q = r + t + s\) and \(k_j \equiv k (j = 1, 2, \ldots, r), k_j \equiv l (j = r + 1, \ldots, r + t)\) and \(k_j \equiv m (j = r + t + 1, \ldots, r + t + s)\) in Theorem 3.4. We can get the following result by Bhoosnumth and Pujari [8].

**Theorem E.** Let \(f(z) (z \in \mathbb{C}_R, 0 < R \leq +\infty)\) be an \(E\)-valued meromorphic function of order \(\lambda(f), 0 < \lambda(f) \leq +\infty\). If there exist distinct elements

\[
a^{[1]}, a^{[2]}, \ldots, a^{[r]}, \quad b^{[1]}, b^{[2]}, \ldots, b^{[l]}, \quad c^{[1]}, c^{[2]}, \ldots, c^{[s]} \tag{3.19}
\]
in $E \cup \{\infty\}$ such that $a^{[1]}, a^{[2]}, \ldots, a^{[r]}$ are $E$-valued evB for $f$ for distinct zeros of order $\leq k$, $b^{[1]}, b^{[2]}, \ldots, b^{[l]}$ are $E$-valued evB for $f$ for distinct zeros of order $\leq l$, $c^{[1]}, c^{[2]}, \ldots, c^{[s]}$ are $E$-valued evB for $f$ for distinct zeros of order $\leq m$, where $k, l,$ and $m$ are positive integers, then

$$\frac{rk}{k+1} + \frac{tl}{l+1} + \frac{sm}{m+1} \leq 2. \quad (3.20)$$

Bhoosnurmath and Pujari [8] pointed out that Theorem E is valid for $0 \leq \lambda(f) \leq +\infty$. In fact, Definition 3.3 is not well in the case of $\lambda(f) = 0$. In the case of $\lambda(f) = +\infty$, $a$ is an $E$-valued evB for $f$ if and only if $\tilde{\rho}_k(a, f)$ is finite. When $\tilde{\rho}_k(a, f)$ is infinite, we shall give the following definitions.

**Definition 3.7.** Let $f(z)$ ($z \in \mathbb{C}$) be an $E$-valued meromorphic function of infinite order and $\rho(r)$ is a proximate order of $f$ and $a \in E \cup \{\infty\}$. We say that $a$ is an

(i) $E$-valued evB (exceptional value in the sense of Borel) for $f$ for distinct zeros of order $\leq k$ if

$$\limsup_{r \to +\infty} \frac{\log^+ \left[ V(a, f) + N_k(r, a) \right]}{\log U(r)} < 1; \quad (3.21)$$

(ii) $E$-valued evB for $f$ for distinct zeros if

$$\limsup_{r \to +\infty} \frac{\log^+ \left[ V(a, f) + N(r, a) \right]}{\log U(r)} < 1; \quad (3.22)$$

(iii) $E$-valued evB for $f$ (for the whole aggregate of zeros) if

$$\limsup_{r \to +\infty} \frac{\log^+ \left[ V(a, f) + N(r, a) \right]}{\log U(r)} < 1. \quad (3.23)$$

**Theorem 3.8.** Let $f(z)$ ($z \in \mathbb{C}$) be an $E$-valued meromorphic function of infinite order and $\rho(r)$ is a proximate order of $f$, $a^{[j]}(j = 1, 2, \ldots, q)$ any system of distinct elements in $E \cup \{\infty\}$, and $k_j(j = 1, 2, \ldots, q)$ any system such that $k_j$ is a positive integer or $+\infty$. If $a^{[j]}$ is an $E$-valued evB for $f$ for distinct zeros of order $\leq k_j(j = 1, 2, \ldots, q)$, then

$$\sum_{j=1}^{q} \left( 1 - \frac{1}{k_j + 1} \right) \leq 2. \quad (3.24)$$

**Proof.** By Corollary 3.5, we have

$$\left( q - \sum_{j=1}^{q} \frac{1}{k_j + 1} - 2 \right) T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \left( V\left( r, a^{[j]} \right) + \tilde{N}_{k_j}\left( r, a^{[j]} \right) \right) + S(r, f). \quad (3.25)$$
By hypothesis, there exists a positive number $\eta < 1$ such that

$$V(r, a^{[j]}) + N_k(r, a^{[j]}) < U(r), \quad j = 1, 2, \ldots, q.$$  \hfill (3.26)

Using (3.25) to (3.26), we have

$$\left[ \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right] T(r, f) \leq \sum_{j=1}^q \frac{k_j}{k_j + 1} U(r) + S(r, f).$$  \hfill (3.27)

If $\sum_{j=1}^q (1 - (1/(k_j + 1))) > 2$, then by Theorem 1.4 and (3.27), we can get a contradiction. So

$$\sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) \leq 2.$$  \hfill (3.28)

\[\square\]

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**References**


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