Research Article

Existence and Nonexistence of Positive Solutions for Quasilinear Elliptic Problem

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Using variational arguments we prove some existence and nonexistence results for positive solutions of a class of elliptic boundary-value problems involving the p -Laplacian.

1. Introduction

In a recent paper, Rădulescu and Repovš [1] studied the existence and nonexistence of positive solutions of the nonlinear elliptic problem

$$\begin{aligned} -\Delta u &= \lambda k(x)u^q \pm h(x)u^p \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, \quad u > 0 \text{ in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $\lambda > 0$ is a parameter, $0 < q < 1 < p$, and h, k in $L^\infty(\Omega)$ such that

$$\operatorname{ess\,inf}_{x \in \Omega} k(x) > 0, \quad \operatorname{ess\,inf}_{x \in \Omega} h(x) > 0. \tag{1.2}$$

They showed using sub-supersolutions arguments and monotonicity methods that the problem $(1.1)_+$ has a minimal solution, provided that $\lambda > 0$ is small enough. The next result is concerned with problem $(1.1)_-$ and asserts that there is some $\lambda^* > 0$ such that $(1.1)_-$ has a nontrivial solution if $\lambda > \lambda^*$ and no solution exists provided that $\lambda < \lambda^*$.

In the present paper we consider that the corresponding quasilinear problem

$$\begin{aligned} -\Delta_p u &= \lambda k(x)u^q \pm h(x)u^r \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, \quad u > 0 \text{ in } \Omega, \end{aligned} \quad (1.3)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, denotes the p -Laplacian operator, $1 < p < \infty$, $\lambda > 0$, $0 \leq q < p-1 < r < p^*-1$, with $p^* = Np/(N-p)$ if $p < N$, and $p^* = +\infty$ otherwise, and h, k in $L^\infty(\Omega)$ such that

$$\operatorname{ess\,inf}_{x \in \Omega} k(x) > 0, \quad \operatorname{ess\,inf}_{x \in \Omega} h(x) > 0. \quad (1.4)$$

We are concerned with the existence of weak solutions of problems (1.3)₊ and (1.3)₋, that is, for functions $u \in W_0^{1,p}(\Omega)$ satisfying $\operatorname{ess\,inf}_K u > 0$ over every compact set $K \subset \Omega$ and

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \lambda \int_{\Omega} k(x)u^q \phi \, dx \pm \int_{\Omega} h(x)u^r \phi \, dx \quad (1.5)$$

for all $\phi \in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi : \Omega \rightarrow \mathbb{R}$ with compact support. Using variational methods, we will prove the following theorems.

Theorem 1.1. *Assume $0 \leq q < p-1 < r < p^*-1$. Then there exists a positive number Λ such that the following properties hold:*

- (1) *for all $\lambda \in (0, \Lambda)$ problem (1.3)₊ has a minimal solution u_λ ;*
- (2) *Problem (1.3)₊ has a solution if $\lambda = \Lambda$;*
- (3) *Problem (1.3)₊ does not have any solution if $\lambda > \Lambda$.*

Theorem 1.2. *Assume $0 \leq q < p-1 < r < p^*-1$. Then there exists a positive number Λ such that the following properties hold:*

- (1) *If $\lambda > \Lambda$, then problem (1.3)₋ has at least one solution;*
- (2) *If $\lambda < \Lambda$, then problem (1.3)₋ does not have any solution.*

2. Proof of Theorem 1.1

At first, we give the definition of weak supersolution and subsolution of (1.3)₊. By definition $u \in W_0^{1,p}(\Omega)$ is a weak subsolution to (1.3)₊ if $u > 0$ in Ω and

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx \leq \lambda \int_{\Omega} k(x)u^q \phi \, dx \pm \int_{\Omega} h(x)u^r \phi \, dx \quad (2.1)$$

for all $\phi \in C_c^\infty(\Omega)$. Similarly $u \in W_0^{1,p}(\Omega)$ is a weak supersolution to (1.3)₊ if in the above the reverse inequalities hold.

Let us define

$$\Lambda \stackrel{\text{def}}{=} \sup\{\lambda > 0 : (1.3)_+ \text{ has a weak solution}\} \quad (2.2)$$

and the energy functional $E_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} E_\lambda(u) = & \frac{1}{p} \int_{\Omega} |\nabla u|^p dx \\ & - \frac{\lambda}{q+1} \int_{\Omega} k(x) u^{q+1} dx - \frac{1}{r+1} \int_{\Omega} h(x) u^{q+1} dx \end{aligned} \quad (2.3)$$

in the Sobolev space $W_0^{1,p}(\Omega)$.

The proof of the theorem is organized in several steps.

Step 1 (existence of minimal solution for $0 < \lambda < \Lambda$). To show the existence of a solution to $(1.3)_+$, we construct a subsolution $\underline{u}_\lambda$, and a supersolution $\bar{u}_\lambda$, such that $\underline{u}_\lambda \leq \bar{u}_\lambda$.

We introduce the following Dirichlet problem:

$$\begin{aligned} -\Delta_p \tilde{u} &= \lambda k(x) \tilde{u}^q \quad \text{in } \Omega, \\ \tilde{u}|_{\partial\Omega} &= 0, \quad \tilde{u} > 0 \text{ in } \Omega. \end{aligned} \quad (2.4)$$

From [2] we know there exists a unique solution, say $\tilde{u}$, satisfying the problem (2.4). Define $\underline{u}_\lambda = \epsilon \tilde{u}$. Then $-\Delta_p(\underline{u}_\lambda) = \lambda k(x) \epsilon^{p-1} \tilde{u}^q$ and $\underline{u}_\lambda$ is a subsolution of the problem $(1.3)_+$ if

$$\lambda k(x) \epsilon^{p-1} \tilde{u}^q \leq \lambda k(x) \epsilon^q \tilde{u}^q + h(x) \epsilon^r \tilde{u}^r. \quad (2.5)$$

Indeed, for ϵ small enough we get

$$\lambda k(x) \epsilon^{p-1} \tilde{u}^q \leq \lambda k(x) \epsilon^q \tilde{u}^q \leq \lambda k(x) \epsilon^q \tilde{u}^q + h(x) \epsilon^r \tilde{u}^r. \quad (2.6)$$

(Since $q < p-1$ and for $\epsilon \in (0, 1)$). Then $\epsilon \tilde{u}$ is a subsolution of the problem $(1.3)_+$.

On the other hand, let v the solution to the following problem be:

$$\begin{aligned} -\Delta_p v &= \lambda + 1 \quad \text{in } \Omega, \\ v|_{\partial\Omega} &= 0, \quad v > 0 \text{ in } \Omega. \end{aligned} \quad (2.7)$$

Then $0 < v < K$ in Ω . By simplicity of writing we call

$$F(u) = \lambda k(x) u^q + h(x) u^r. \quad (2.8)$$

Define $\bar{u}_\lambda(x) = T v(x)$ where T is a constant that will be chosen in such a way that

$$-\Delta_p \bar{u}_\lambda \geq F(TM) \geq F(\bar{u}_\lambda), \quad (2.9)$$

where $M = \max\{1, \|v\|_\infty\}$. Now $-\Delta_p \bar{u}_\lambda = T^{p-1}(\lambda + 1)$ and

$$F(\bar{u}_\lambda) \equiv \lambda k(x) T^q v^q + T^r v^r \leq \lambda c_1 T^q M^q + c_2 T^r M^r, \quad (2.10)$$

where $c_1 = \|k\|_{L^\infty}$ et $c_2 = \|h\|_{L^\infty}$. Then, it is sufficient to find T such that

$$(\lambda + 1) \geq \lambda c_1 T^{q+1-p} M^q + c_2 T^{r+1-p} M^r. \quad (2.11)$$

We call

$$\varphi(T) = \lambda A T^{q+1-p} + B T^{r+1-p}, \quad (2.12)$$

with $A = c_1 M^q, B = c_2 M^r$. Then

$$\lim_{T \rightarrow 0^+} \varphi(T) = \lim_{T \rightarrow \infty} \varphi(T) = \infty, \quad (2.13)$$

because $q + 1 - p < 0 < r + 1 - p$; then φ attains a minimum in $[0, \infty)$. Elementary computations shows that this function attains its minimum for $T_0 = C \lambda^{1/(r-q)}$ where $C = [AB^{-1}(r - p + 1)(p - q - 1)^{-1}]^{1/(r-q)}$. For the validity of (2.11) it suffices that

$$\varphi(T_0) \leq \lambda + 1, \quad (2.14)$$

that is,

$$D \lambda^{(r+1-p)/(r-q)} < \lambda + 1, \quad (2.15)$$

where D is a constant, depends on p, q , and M . Then there exists λ_0 such that for $0 < \lambda < \lambda_0$, $\bar{u}(x) = T_0 v$ is a supersolution of problem (1.3)₊. It remains to show that $\epsilon \tilde{u} \leq T_0 v$. In turn, fix the supersolution, that is, T , for ϵ small enough, we get

$$-\Delta_p \underline{u}_\lambda = \lambda k(x) \epsilon^{p-1} \tilde{u}^q \leq \lambda \epsilon^{p-1} \leq -\Delta_p (\bar{u}_\lambda). \quad (2.16)$$

Consequently, we may apply the weak comparison principle (see Proposition 2.3 in [3]) in order to conclude that $\underline{u}_\lambda \leq \bar{u}_\lambda$. Thus, By the classical iteration method (1.3)₊ has a solution between the subsolution and supersolution.

Let us now prove that u_λ is a minimal solution of (1.3)₊. We use here the weak comparison principle (see Proposition 2.3 in Cuesta and Takáč [3]) and the following monotone iterative scheme:

$$\begin{aligned} -\Delta_p u_n &= \lambda k(x) u_{n-1}^q + h(x) u_{n-1}^r \quad \text{in } \Omega; \\ u_n|_{\partial\Omega} &= 0, \end{aligned} \quad (2.17)$$

where $u_0 = \underline{u}_\lambda$, the unique solution to (2.4). Note that u_0 is a weak subsolution to $(1.3)_+$ and $u_0 \leq U$ where U is any weak solution to $(1.3)_+$. Then, from the weak comparison principle, we get easily that $u_0 \leq u_1$ and $\{u_n\}_{n=1}^\infty$ is a nondecreasing sequence. Furthermore, $u_n \leq U$ and $\{u_n\}_{n=1}^\infty$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Hence, it is easy to prove that $\{u_n\}$ converges weakly in $W_0^{1,p}(\Omega)$ and pointwise to $\hat{u}_\lambda$, a weak solution to $(1.3)_+$. Let us show that $\hat{u}_\lambda$ is the minimal solution to $(1.3)_+$ for any $0 < \lambda < \Lambda$. Let v_λ a weak solution to $(1.3)_+$ for any $0 < \lambda < \Lambda$. Then, $u_0 = \underline{u}_\lambda \leq v_\lambda$. From the weak comparison principle, $u_n \leq v_\lambda$ for any $n \geq 0$. Letting $n \rightarrow \infty$, we get $\hat{u}_\lambda \leq v_\lambda$. This completes the proof of the Step 1.

Step 2 (there exists $\Lambda > 0$ such that $(1.3)_+$ has no positive solution for $\lambda > \Lambda$). From the definition of Λ , problem $(1.3)_+$ does not have any solution if $\lambda > \Lambda$. In what follows we claim that $\Lambda < \infty$. We argue by contradiction: suppose there exists a sequence $\lambda_n \rightarrow \infty$ such that $(1.3)_+$ admits a solution u_n . Denote

$$m := \min \left\{ \operatorname{ess\,inf}_{x \in \Omega} k(x), \operatorname{ess\,inf}_{x \in \Omega} h(x) \right\} > 0. \quad (2.18)$$

There exists $\lambda_* > 0$ such that

$$m(\lambda t^q + t^r) \geq (\lambda_1 + \epsilon) t^{p-1} \quad \forall t > 0, \epsilon \in (0, 1), \lambda > \lambda_*, \quad (2.19)$$

where λ_1 is the first Dirichlet eigenvalue of $-\Delta_p$ is positive and is given by

$$\lambda_1 = \min_{u \neq 0} \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^p} \quad (2.20)$$

(see Lindqvist [4]). Choose $\lambda_n > \lambda_*$. Clearly u_n is a supersolution of the problem

$$\begin{aligned} -\Delta_p u &= (\lambda_1 + \epsilon) u^{p-1} \quad \text{in } \Omega, \\ u &> 0, \quad u|_{\partial\Omega} &= 0 \end{aligned} \quad (2.21)$$

for all $\epsilon \in (0, 1)$. We now use the result in [2] to choose $\mu < \lambda_1 + \epsilon$ small enough so that $\mu \phi_1(x) < u_n(x)$ and $\mu \phi_1$ is a subsolution to problem (2.8). By a monotone iteration procedure we obtain a solution to (2.8) for any $\epsilon \in (0, 1)$, contradicting the fact that λ_1 is an isolated point in the spectrum of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ (see Anane [5]). This proves the claim and completes the proof of the Step 2.

Step 3 (there exists at least one positive-weak solution for $\lambda = \Lambda$ to $(1.3)_+$). Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be such that $\lambda_k \uparrow \Lambda$ as $k \rightarrow \infty$. Then, from Step 1, there exists $u_k = u_{\lambda_k} \geq \underline{u}_{\lambda_k}$ to a weak positive solution to $(1.3)_+$ for $\lambda = \lambda_k$. Therefore, for any $\phi \in C_c^\infty(\Omega)$, we have

$$\int_{\Omega} |\nabla u_k|^{p-2} \nabla u_k \nabla \phi \, dx = \lambda_k k(x) \int_{\Omega} (u_k)^q \phi \, dx + h(x) \int_{\Omega} u_k^r \phi \, dx. \quad (2.22)$$

Since $u_k \in W_0^{1,p}(\Omega)$ and $u_k \geq \underline{u}_{\lambda_k}$ it is easy to see that (2.22) holds also for $\phi \in W_0^{1,p}(\Omega)$. Moreover, from above

$$E_{\lambda_k}(u_k) \leq E_{\lambda_k}(\underline{u}_{\lambda_k}) < \frac{1}{p} \int_{\Omega} |\nabla \underline{u}_{\lambda_k}|^p dx - \frac{\lambda_k k(x)}{q+1} \int_{\Omega} \underline{u}_{\lambda_k}^{q+1} dx < 0, \quad (2.23)$$

it follows that

$$\sup_k \|u_k\|_p < \infty. \quad (2.24)$$

Hence, there exists $u_{\Lambda} \geq \underline{u}_{\lambda_k}$ such that $u_k \rightharpoonup u_{\Lambda}$ in $W_0^{1,p}(\Omega)$ as $k \rightarrow \infty$ and then by Sobolev imbedding and using the fact that $k, h \in L^{\infty}(\Omega)$:

$$u_k \rightharpoonup u \quad \text{in } L^q(\Omega) \text{ and point wise a.e. as } k \rightarrow \infty. \quad (2.25)$$

From (2.22), (2.24), and (2.25), we get for any $\phi \in W_0^{1,p}(\Omega)$

$$\int_{\Omega} |\nabla u_{\Lambda}|^{p-2} \nabla u_{\Lambda} \nabla \phi dx = \lambda \int_{\Omega} k(x) u_{\Lambda}^q \phi dx + \int_{\Omega} h(x) u_{\Lambda}^r \phi dx \quad (2.26)$$

which completes the proof of the Step 3 and gives the proof of Theorem 1.1.

3. Proof of Theorem 1.2

At first, we introduce some notation which will be used throughout the proof. The norm in $W_0^{1,p}(\Omega)$ will be denoted by

$$\|u\|_p \stackrel{\text{def}}{=} \left(\int_{\Omega} |\nabla u|^p dx \right)^{1/p}. \quad (3.1)$$

The norm in $L^{q+1}(\Omega)$ will be denoted by

$$\|u\|_{q+1} \stackrel{\text{def}}{=} \left(\int_{\Omega} |u|^{q+1} dx \right)^{1/(q+1)}. \quad (3.2)$$

The norm in $L^{r+1}(\Omega)$ will be denoted by

$$\|u\|_{r+1} \stackrel{\text{def}}{=} \left(\int_{\Omega} |u|^{r+1} dx \right)^{1/(r+1)}. \quad (3.3)$$

Let us define the energy functional $J_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{\lambda}{q+1} \int_\Omega k(x) u^{q+1} dx + \frac{1}{r+1} \int_\Omega h(x) u^{r+1} dx \quad (3.4)$$

in the Sobolev space $W_0^{1,p}(\Omega)$.

The proof of the theorem is organized in several steps.

Step 1 (coercivity of J_λ): For any $u \in W_0^{1,p}(\Omega)$ and all $\lambda > 0$

$$J_\lambda(u) \geq \frac{1}{p} \|u\|^p - C_1 \|u\|_{q+1}^{q+1} + C_2 \|u\|_{r+1}^{r+1}, \quad (3.5)$$

where $C_1 = \lambda \|k\|_{L^\infty} / (q+1)$ and $C_2 = (r+1)^{-1} \text{ess inf}_{x \in \Omega} h(x)$ are positive constants. We call

$$\phi(T) = AT^{q+1-p} - BT^{r+1-p}. \quad (3.6)$$

Then

$$\lim_{T \rightarrow 0^+} \phi(T) = \lim_{T \rightarrow \infty} \phi(T) = \infty, \quad (3.7)$$

because $q+1-p < 0 < r+1-p$; then ϕ attains a minimum $m < 0$ in $[0, \infty)$. By elementary computations shows that this function attains its minimum for $T = [A(q+1-p)/(Br+1-p)]^{1/(r-q)}$.

Returning to (3.5), we deduce that

$$J_\lambda(u) \geq \frac{1}{p} \|u\|^p + m. \quad (3.8)$$

Hence, from (3.8), we get that

$$J_\lambda(u) \longrightarrow +\infty \quad \text{as } \|u\| \longrightarrow \infty. \quad (3.9)$$

Let $n \mapsto u_n$ be a minimizing sequence of J_λ in $W_0^{1,p}(\Omega)$, which is bounded in $W_0^{1,p}(\Omega)$ by Step 1. Without loss of generality, we may assume that $(u_n)_n$ is nonnegative, converges weakly to some u in $W_0^{1,p}(\Omega)$, and converges also pointwise. Moreover, by the weak lower semicontinuity of the norm $\|\cdot\|$ and the boundedness of $(u_n)_n$ in $W_0^{1,p}(\Omega)$ we get

$$J_\lambda(u) \leq \liminf_{n \rightarrow \infty} J_\lambda(u_n). \quad (3.10)$$

Hence u is a global minimizer of J_λ in $W_0^{1,p}(\Omega)$, which completes the proof of the Step 1.

Step 2 (the weak limit u is a nonnegative weak solution of (1.3)₋ if $\lambda > 0$ is sufficiently large). Firstly, observe that $J_\lambda(0) = 0$. Thus, to prove that the nonnegative solution is nontrivial, it suffices to prove that there exists $\lambda^* > 0$ such that

$$\inf_{u \in W_0^{1,p}(\Omega)} J_\lambda(u) < 0 \quad \forall \lambda > \lambda^*. \quad (3.11)$$

For this, we consider the constrained minimization problem

$$\begin{aligned} \lambda^* \stackrel{\text{def}}{=} \inf \left\{ \frac{1}{p} \int_{\Omega} |\nabla w|^p dx + \frac{1}{r+1} \int_{\Omega} h(x) |w|^{r+1} dx : w \in W_0^{1,p}(\Omega) \text{ and } \frac{1}{q+1} \right. \\ \left. \times \int_{\Omega} k(x) |w|^{q+1} dx = 1 \right\}. \end{aligned} \quad (3.12)$$

Let $n \mapsto v_n$ be a minimizing sequence of (3.12) in $W_0^{1,p}(\Omega)$, which is bounded in $W_0^{1,p}(\Omega)$, so that we can assume, without loss of generality, that it converges weakly to some $v \in W_0^{1,p}(\Omega)$, with

$$\frac{1}{q+1} \int_{\Omega} k(x) |v|^{q+1} dx = 1, \quad \lambda^* = \frac{1}{p} \int_{\Omega} |\nabla v|^p dx + \frac{1}{r+1} \int_{\Omega} h(x) |v|^{r+1} dx. \quad (3.13)$$

Thus, $J_\lambda(v) = \lambda^* - \lambda < 0$ for any $\lambda > \lambda^*$.

Now put

$$\Lambda \stackrel{\text{def}}{=} \inf \{ \lambda > 0 : (1.3)_- \text{ admits a non trivial weak solution} \}. \quad (3.14)$$

From above $\lambda^* \geq \Lambda$ and that problem (1.3)₋ has a solution for all $\lambda > \lambda^*$. The proof of the Step 2 is now completed.

Step 3 (problem (1.3)₋ has a weak solution for any $\lambda > \Lambda$). By the definition of Λ , there exists $\mu \in (\Lambda, \lambda)$ such that J_μ has a nontrivial critical point $u_\mu \in W_0^{1,p}(\Omega)$. Since $\mu < \lambda$, u_μ is a subsolution of the problem (1.3)₋. In order to find a super-solution of the problem (1.3)₋ which dominates u_μ , we consider the constrained minimization problem

$$\inf \{ J_\lambda(w); w \in W_0^{1,p}(\Omega) \text{ and } w \geq u_\mu \}. \quad (3.15)$$

Arguments similar to those used in Step 2 show that the above minimization problem has a solution $u_\lambda \geq u_\mu$ which is also a weak solution of problem (1.3)₋, provided $\lambda > \Lambda$.

Using similar arguments as in [6]. Thus, from Theorem 2.2 in Pucci and Servadei [7], based on the Moser iteration, it is clear that $u \in L_{\text{loc}}^\infty$. Next, again by bootstrap regularity [Corollary on p. 830] due to DiBenedetto, [8] shows that the weak solution $u \in C^{1,\alpha}(\Omega)$ where $\alpha \in (0, 1)$. Finally, the nonnegative follows immediately by the strong maximum principle since u is a C^1 nonnegative weak solution of the differential inequality $\nabla(|\nabla u|^{p-2} \nabla u) - h(x)u^r \leq 0$ in Ω , with $p-1 < r$, see, for instance, Section 4.8 of Pucci and Serrin [9]. Thus, $u > 0$ in Ω . The proof of the Step 3 is now completed.

Step 4 (nonexistence for $\lambda > 0$ is small). The same monotonicity arguments as in Step 3 show that (1.3)₋ does not have any solution if $\lambda < \Lambda$, which completes the proof of the Theorem 1.2.

References

- [1] V. Rădulescu and D. Repovš, "Combined effects in nonlinear problems arising in the study of anisotropic continuous media," *Nonlinear Analysis*, vol. 75, no. 3, pp. 1524–1530, 2012.
- [2] C. A. Santos, "Non-existence and existence of entire solutions for a quasi-linear problem with singular and super-linear terms," *Nonlinear Analysis*, vol. 72, no. 9-10, pp. 3813–3819, 2010.
- [3] M. Cuesta and P. Takáč, "A strong comparison principle for positive solutions of degenerate elliptic equations," *Differential and Integral Equations*, vol. 13, no. 4–6, pp. 721–746, 2000.
- [4] P. Lindqvist, "On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$," *Proceedings of the American Mathematical Society*, vol. 109, no. 1, pp. 157–164, 1990.
- [5] A. Anane, "Simplicité et isolation de la première valeur propre du p -laplacien avec poids," *Comptes Rendus des Séances de l'Académie des Sciences I*, vol. 305, no. 16, pp. 725–728, 1987.
- [6] R. Filippucci, P. Pucci, and V. Rădulescu, "Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions," *Communications in Partial Differential Equations*, vol. 33, no. 4–6, pp. 706–717, 2008.
- [7] P. Pucci and R. Servadei, "Regularity of weak solutions of homogeneous or inhomogeneous quasilinear elliptic equations," *Indiana University Mathematics Journal*, vol. 57, no. 7, pp. 3329–3363, 2008.
- [8] E. DiBenedetto, " $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations," *Nonlinear Analysis*, vol. 7, no. 8, pp. 827–850, 1983.
- [9] P. Pucci and J. Serrin, "Maximum principles for elliptic partial differential equations," in *Handbook of Differential Equations: Stationary Partial Differential Equations*, M. Chipot, Ed., vol. 4, pp. 355–483, Elsevier, 2007.