Research Article

Approximate $n$-Lie Homomorphisms and Jordan $n$-Lie Homomorphisms on $n$-Lie Algebras

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Using fixed point methods, we establish the stability of $n$-Lie homomorphisms and Jordan $n$-Lie homomorphisms on $n$-Lie algebras associated to the following generalized Jensen functional equation:

$$\mu f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \mu \sum_{j=2}^{n} f\left(\frac{\sum_{i=1, i \neq j}^{n} x_i}{(n - 1)x_j/n}\right) = f(\mu x_1)(n \geq 2).$$

1. Introduction

Let $n$ be a natural number greater or equal to 3. The notion of an $n$-Lie algebra was introduced by Filippov in 1985 [1]. The Lie product is taken between $n$ elements of the algebra instead of two. This new bracket is $n$-linear, antisymmetric and satisfies a generalization of the Jacobi identity. For $n = 3$ this product is a special case of the Nambu bracket, well known in physics, which was introduced by Nambu [2] in 1973, as a generalization of the Poisson bracket in Hamiltonian mechanics.

An $n$-Lie algebra is a natural generalization of a Lie algebra. Namely, a vector space $V$ together with a multilinear, antisymmetric $n$-ary operation $[\cdot]: \Lambda^n V \to V$ is called an $n$-Lie algebra, $n \geq 3$, if the $n$-ary bracket is a derivation with respect to itself, that is,

$$[[x_1, \ldots, x_n], x_{n+1}, \ldots, x_{2n-1}] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1} [x_i, x_{n+1}, \ldots, x_{2n-1}], \ldots, x_n], \quad (1.1)$$

where $x_1, x_2, \ldots, x_{2n-1} \in V$. Equation (1.1) is called the generalized Jacobi identity. The meaning of this identity is similar to that of the usual Jacobi identity for a Lie algebra (which is a 2-Lie algebra).
In [1] and several subsequent papers, [3–5] a structure theory of finite-dimensional \( n \)-Lie algebras over a field \( F \) of characteristic 0 was developed.

\( n \)-ary algebras have been considered in physics in the context of Nambu mechanics [2, 6] and, recently (for \( n = 3 \)), in the search for the effective action of coincident M2-branes in \( M \)-theory initiated by the Bagger-Lambert-Gustavsson (BLG) model [7, 8] (further references on the physical applications of \( n \)-ary algebras are given in [9]).

From now on, we only consider \( n \)-Lie algebras over the field of complex numbers. An \( n \)-Lie algebra \( A \) is a normed \( n \)-Lie algebra if there exists a norm \( ||| \cdot ||| \) on \( A \) such that
\[
||| [x_1, x_2, \ldots, x_n] ||| \leq ||| x_1 \! ||| \! \cdot \! ||| x_2 \! ||| \! \cdot \! \cdots \! \cdot \! ||| x_n \! |||
\]
for all \( x_1, x_2, \ldots, x_n \in A \). A normed \( n \)-Lie algebra \( A \) is called a Banach \( n \)-Lie algebra, if \( (A, ||| \cdot |||) \) is a Banach space.

Let \( (A, [\cdot]_A) \) and \( (B, [\cdot]_B) \) be two Banach \( n \)-Lie algebras. A \( \mathbb{C} \)-linear mapping \( H : (A, [\cdot]_A) \to (B, [\cdot]_B) \) is called an \( n \)-Lie homomorphism if
\[
H([x_1, x_2, \ldots, x_n]_A) = [H(x_1) H(x_2) \cdots H(x_n)]_B
\]
for all \( x_1, x_2, \ldots, x_n \in A \). A \( \mathbb{C} \)-linear mapping \( H : (A, [\cdot]_A) \to (B, [\cdot]_B) \) is called a Jordan \( n \)-Lie homomorphism if
\[
H([x x \cdots x]_A) = [H(x) H(x) \cdots H(x)]_B
\]
for all \( x \in A \).

The study of stability problems had been formulated by Ulam [10] during a talk in 1940. Under what condition does there exist a homomorphism near an approximate homomorphism? In the following year, Hyers [11] answered affirmatively the question of Ulam for Banach spaces, which states that if \( \varepsilon > 0 \) and \( f : X \to Y \) is a map with \( X \) a normed space, \( Y \) a Banach spaces such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \varepsilon
\]
for all \( x, y \in X \), then there exists a unique additive map \( T : X \to Y \) such that
\[
\| f(x) - T(x) \| \leq \varepsilon
\]
for all \( x \in X \). A generalized version of the theorem of Hyers for approximately linear mappings was presented by Rassias [12] in 1978 by considering the case when inequality (1.4) is unbounded. Due to that fact, the additive functional equation \( f(x + y) = f(x) + f(y) \) is said to have the generalized Hyers-Ulam-Rassias stability property. A large list of references concerning the stability of functional equations can be found in [13–32].

In 1982–1994, Rassias (see [26–28]) solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, Rassias considered the mixed product sum of powers of norms control function. For more details see [33–57].

In 2003 Cădariu and Radu applied the fixed-point method to the investigation of the Jensen functional equation [58]. They could present a short and a simple proof (different of the “direct method”, initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation [58] and for quadratic functional equation.
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Park and Rassias [59] proved the stability of homomorphisms in C*-algebras and Lie C*-algebras and also of derivations on C*-algebras and Lie C*-algebras for the Jensen-type functional equation

\[ \mu f \left( \frac{x + y}{2} \right) + \mu f \left( \frac{x - y}{2} \right) - f(\mu x) = 0 \]  

(1.6)

for all \( \mu \in \mathbb{T} := \{ \lambda \in \mathbb{C}; |\lambda| = 1 \} \).

In this paper, by using the fixed-point methods, we establish the stability of \( n \)-Lie homomorphisms and Jordan \( n \)-Lie homomorphisms on \( n \)-Lie Banach algebras associated to the following generalized Jensen type functional equation:

\[ \mu f \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) + \mu \sum_{j=2}^{n} f \left( \frac{\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j}{n} \right) - f(\mu x_1) = 0 \]  

(1.7)

for all \( \mu \in (\mathbb{T}_{\frac{1}{n_0}} := \{ e^{i\theta}; 0 \leq \theta \leq 2\pi / n_0 \} \cup \{ 1 \} \), where \( n \geq 2 \).

Throughout this paper, assume that \( (A, [ ]_A), (B, [ ]_B) \) are two \( n \)-Lie Banach algebras.

2. Main Results

Before proceeding to the main results, we recall a fundamental result in fixed point theory.

**Theorem 2.1** (see [60]). Let \( (\Omega, d) \) be a complete generalized metric space, and let \( T : \Omega \rightarrow \Omega \) be a strictly contractive function with Lipschitz constant \( L \). Then for each given \( x \in \Omega \), either

\[ d \left( T^mx, T^{m+1}x \right) = \infty \quad \forall m \geq 0, \]  

(2.1)

or other exists a natural number \( m_0 \) such that

(i) \( d(T^mx, T^{m+1}x) < \infty \) for all \( m \geq m_0 \);

(ii) the sequence \( \{T^mx\} \) is convergent to a fixed point \( y^* \) of \( T \);

(iii) \( y^* \) is the unique fixed point of \( T \) in \( \Lambda = \{ y \in \Omega : d(T^{m_0}x, y) < \infty \} \);

(iv) \( d(y, y^*) \leq (1/(1-L))d(y, Ty) \) for all \( y \in \Lambda \).

We start our work with the main theorem of the our paper.

**Theorem 2.2.** Let \( n_0 \in \mathbb{N} \) be a fixed positive integer number. Let \( f : A \rightarrow B \) be a function for which there exists a function \( \phi : A^n \rightarrow [0, \infty) \) such that

\[ \left\| \mu f \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) + \mu \sum_{j=2}^{n} f \left( \frac{\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j}{n} \right) - f(\mu x_1) \right\|_B \leq \phi(x_1, x_2, \ldots, x_n) \]  

(2.2)
for all \( \mu \in (\mathbb{T}^{1}_{1/n_{0}} := \{e^{i\theta} : 0 \leq \theta \leq 2\pi / n_{0}\} \cup \{1\}) \) and all \( x_{1}, \ldots, x_{n} \in A \), and that

\[
\|f([x_{1}, x_{2}, \ldots, x_{n}]) - [f(x_{1}) f(x_{2}) \cdots f(x_{n})]\|_{B} \leq \phi(x_{1}, x_{2}, \ldots, x_{n})
\]  

(2.3)

for all \( x_{1}, \ldots, x_{n} \in A \). If there exists an \( L < 1 \) such that

\[
\phi(x_{1}, x_{2}, \ldots, x_{n}) \leq nL\phi\left(\frac{x_{1}}{n}, \frac{x_{2}}{n}, \ldots, \frac{x_{n}}{n}\right)
\]  

(2.4)

for all \( x_{1}, \ldots, x_{n} \in A \), then there exists a unique \( n \)-Lie homomorphism \( H : A \to B \) such that

\[
\|f(x) - H(x)\| \leq \frac{L}{1-L} \phi(x, 0, \ldots, 0)
\]  

(2.5)

for all \( x \in A \).

Proof. Let \( \Omega \) be the set of all functions from \( A \) into \( B \) and let

\[
d(g,h) := \inf \{C \in \mathbb{R}^{+} : \|g(x) - h(x)\|_{B} \leq C \phi(x, 0, \ldots, 0), \forall x \in A\}.
\]  

(2.6)

It is easy to show that \((\Omega, d)\) is a generalized complete metric space [61].

Now we define the mapping \( J : \Omega \to \Omega \) by \( J(h)(x) = (1/n)h(nx) \) for all \( x \in A \). Note that for all \( g, h \in \Omega \),

\[
d(g,h) < C \Rightarrow \|g(x) - h(x)\| \leq C \phi(x, 0, \ldots, 0), \quad \forall x \in A,
\]

\[
\Rightarrow \left\| \frac{1}{n}g(nx) - \frac{1}{n}h(nx) \right\| \leq \frac{1}{|n|} C \phi(nx, 0, \ldots, 0), \quad \forall x \in A,
\]

\[
\Rightarrow \left\| \frac{1}{n}g(nx) - \frac{1}{n}h(nx) \right\| \leq L C \phi(x, 0, \ldots, 0), \quad \forall x \in A,
\]

\[
\Rightarrow d(J(g), J(h)) \leq LC.
\]  

(2.7)

Hence we see that

\[
d(J(g), J(h)) \leq Ld(g,h)
\]  

(2.8)

for all \( g, h \in \Omega \). It follows from (2.4) that

\[
\lim_{m \to \infty} \frac{1}{n^{m}} \phi(n^{m}x_{1}, n^{m}x_{2}, \ldots, n^{m}x_{n}) = 0
\]  

(2.9)

for all \( x_{1}, \ldots, x_{n} \in A \). Putting \( \mu = 1, \; x_{1} = x, \) and \( x_{j} = 0 \) \((j = 2, \ldots, n)\) in (2.2), we obtain

\[
\left\| n f\left(\frac{x}{n}\right) - f(x) \right\|_{B} \leq \phi(x, 0, \ldots, 0)
\]  

(2.10)
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for all \( x \in A \). Thus by using (2.4), we obtain that

\[
\left\| \frac{1}{n} f(nx) - f(x) \right\|_B \leq \frac{1}{n} \phi(nx, 0, \ldots, 0) \leq L\phi(x, 0, \ldots, 0)
\]

(2.11)

for all \( x \in A \), that is,

\[
d(f, J(f)) \leq L < \infty.
\]

(2.12)

By Theorem 2.1, \( J \) has a unique fixed point in the set \( X_1 := \{ h \in \Omega : d(f, h) < \infty \} \). Let \( H \) be the fixed point of \( J \). \( H \) is the unique mapping with

\[
H(nx) = nH(x)
\]

(2.13)

for all \( x \in A \), such that there exists \( C \in (0, \infty) \) satisfying

\[
\left\| f(x) - H(x) \right\|_B \leq C\phi(x, 0, \ldots, 0)
\]

(2.14)

for all \( x \in A \). On the other hand we have \( \lim_{m \to \infty} d(J^m(f), H) = 0 \), so

\[
\lim_{m \to \infty} \frac{1}{n^m} f(n^m x) = H(x)
\]

(2.15)

for all \( x \in A \). Also by Theorem 2.1, we have

\[
d(f, H) \leq \frac{1}{1 - L} d(f, J(f)).
\]

(2.16)

It follows from (2.12) and (2.16) that

\[
d(f, H) \leq \frac{L}{1 - L}.
\]

(2.17)

This implies the inequality (2.5). By (2.21), we have

\[
\begin{align*}
&\left\| H([x_1 x_2 \cdots x_n]_A) - [H(x_1)H(x_2)H(x_3) \cdots H(x_n)]_B \right\|_B \\
&= \lim_{m \to \infty} \left\| \frac{1}{n^m} H([n^m x_1 n^m x_2 \cdots n^m x_n]_A) - \frac{1}{n^m} ([H(n^m x_1)H(n^m x_2)H(n^m x_3) \cdots H(n^m x_n)]_B) \right\|_B \\
&\leq \lim_{m \to \infty} \frac{1}{n^m} \phi(n^m x_1, n^m x_2, \ldots, n^m x_n) = 0
\end{align*}
\]

(2.18)
for all \(x_1,\ldots,x_n \in A\). Hence

\[
H([x_1x_2\cdots x_n]_A) = [H(x_1)H(x_2)H(x_3)\cdots H(x_n)]_B
\]  
(2.19)

for all \(x_1,\ldots,x_n \in A\).

On the other hand, it follows from (2.2), (2.9), and (2.15) that

\[
\left\| H\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \sum_{j=2}^{n} H\left(\frac{\sum_{i=1,i\neq j}^{n} x_i - (n-1)x_j}{n}\right) - H(x_1) \right\|_B
\]

\[
= \lim_{m \to \infty} \frac{1}{h^m} \left\| f\left(n^{m-1} \sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(n^{m-1} \left(\sum_{i=1,i\neq j}^{n} x_i - (n-1)x_j\right)\right) - f(n^m x_1) \right\|_B
\]

\[
\leq \lim_{m \to \infty} \frac{1}{h^m} \phi(n^m x_1, n^m x_2, \ldots, n^m x_n) = 0
\]  
(2.20)

for all \(x_1,\ldots,x_n \in A\). Then

\[
H\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \sum_{j=2}^{n} H\left(\frac{\sum_{i=1,i\neq j}^{n} x_i - (n-1)x_j}{n}\right) = H(x_1)
\]  
(2.21)

for all \(x_1,\ldots,x_n \in A\). Putting \(s_1 = \sum_{i=1}^{n} x_i/n\) and \(s_j = \sum_{i=1,i\neq j}^{n} x_i - (n-1)x_j/n\) \((j = 2,3,\ldots,n)\) in (2.21), we obtain

\[
H\left(\sum_{j=1}^{n} s_j\right) = \sum_{j=1}^{n} H(s_j)
\]  
(2.22)

for all \(s_1,\ldots,s_n \in A\). Setting \(s_j = 0\) \((j = 3,4,\ldots,n)\) in (2.22) to get

\[
H(s_1 + s_2) = H(s_1) + H(s_2)
\]  
(2.23)

hence \(H\) is cauchy additive. Letting \(x_i = x\) for all \(i = 1,2,\ldots,n\) in (2.2), we obtain

\[
\left\| \mu f(x) - f(\mu x) \right\|_B \leq \phi(x,x,\ldots,x)
\]  
(2.24)

for all \(x \in A\). It follows that

\[
\left\| H(\mu x) - \mu H(x) \right\| = \lim_{m \to \infty} \frac{1}{h^m} \left\| f(n^m x) - \mu f(n^m x) \right\|_B
\]

\[
\leq \lim_{m \to \infty} \frac{1}{h^m} \phi(n^m x, n^m x, \ldots, n^m x) = 0
\]  
(2.25)
for all $\mu \in T^1_{1/n}$, and all $x \in A$. One can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear. Hence, $H : A \to B$ is an $n$-Lie homomorphism satisfying (2.5), as desired.

**Corollary 2.3.** Let $\theta$ and $p$ be nonnegative real numbers such that $p < 1$. Suppose that a function $f : A \to B$ satisfies

$$\|\mu f\left(\sum_{i=1}^{n} x_i/n\right) + \mu \sum_{j=2}^{n} f\left(\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j/n\right) - f(\mu x_1)\|_B \leq \theta \sum_{i=1}^{n} \|x_i\|^p_A$$

(2.26)

for all $\mu \in T^1$ and all $x_1, \ldots, x_n \in A$ and

$$\|f([x_1x_2 \cdots x_n]_A) - [f(x_1)f(x_2) \cdots f(x_n)]_B\|_B \leq \theta \sum_{i=1}^{n} \|x_i\|^p_A$$

(2.27)

for all $x_1, \ldots, x_n \in A$. Then there exists a unique $n$-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2p}{\ell(2-2^p)} \theta \|x\|^p_A$$

(2.28)

for all $x \in A$.

**Proof.** Put $\phi(x_1, x_2, \ldots, x_n) := \theta \sum_{i=1}^{n} \|x_i\|^p_A$ for all $x_1, \ldots, x_n \in A$ in Theorem 2.2. Then (2.9) holds for $p < 1$, and (2.28) holds when $L = 2^{(p-1)}$.

**Theorem 2.4.** Let $n_0 \in \mathbb{N}$ be a fixed positive integer number. Let $f : A \to B$ be a function for which there exists a function $\phi : A^n \to [0, \infty)$ such that

$$\|\mu f\left(\sum_{i=1}^{n} x_i/n\right) + \mu \sum_{j=2}^{n} f\left(\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j/n\right) - f(\mu x_1)\|_B \leq \phi(x_1, x_2, \ldots, x_n)$$

(2.29)

for all $\mu \in (T^1_{1/n_0} := \{e^{\theta} : 0 \leq \theta \leq 2\pi/n_0 \cup \{1\})$ and all $x_1, \ldots, x_n \in A$, and that

$$\|f([xx \cdots x]_A) - [f(x)f(x) \cdots f(x)]_B\|_B \leq \phi(x, x, \ldots, x)$$

(2.30)

for all $x \in A$. If there exists an $L < 1$ such that

$$\phi(x_1, x_2, \ldots, x_n) \leq nL\phi\left(\frac{x_1}{n}, \frac{x_2}{n}, \ldots, \frac{x_n}{n}\right)$$

(2.31)

for all $x_1, \ldots, x_n \in A$, then there exists a unique Jordan $n$-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{L}{1-L} \phi(x, 0, 0, \ldots, 0)$$

(2.32)

for all $x \in A$. 

Proof. By the same reasoning as the proof of Theorem 2.2, we can define the mapping

\[ H(x) = \lim_{m \to \infty} \frac{1}{n^m} f \left( n^m x \right) \]  

(2.33)

for all \( x \in A \). Moreover, we can show that \( H \) is \( \mathbb{C} \)-linear. The inequality (2.30) follows that

\[
\|H([xx \cdots x]_A) - [H(x)H(x) \cdots H(x)]_B\|_B \\
= \lim_{m \to \infty} \left\| \frac{1}{n^m} H([n^m x \cdots n^m x]_A) - \frac{1}{n^m} ([H(n^m x)H(n^m x) \cdots H(n^m x)]_B) \right\|_B \\
\leq \lim_{m \to \infty} \frac{1}{n^m} \phi(n^m x, n^m x, \ldots, n^m x) = 0
\]

for all \( x \in A \). So

\[ H([xx \cdots x]_A) = [H(x)H(x) \cdots H(x)]_B \]  

(2.35)

for all \( x \in A \). Hence \( H : A \to B \) is a Jordan \( n \)-Lie homomorphism satisfying (2.32). \( \square \)

**Corollary 2.5.** Let \( \theta \) and \( p \) be nonnegative real numbers such that \( p < 1 \). Suppose that a function \( f : A \to B \) satisfies

\[
\left\| \mu f \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) + \mu \sum_{j=2}^{n} f \left( \frac{\sum_{i=1, i \neq j}^{n} x_i - (n-1)x_j}{n} \right) - f(\mu x_1) \right\|_B \leq \theta \sum_{i=1}^{n} (\|x_i\|_A^p)
\]

(2.36)

for all \( \mu \in T^1 \) and all \( x_1, \ldots, x_n \in A \) and

\[
\left\| f([xx \cdots x]_A) - [f(x)f(x) \cdots f(x)]_B \right\|_B \leq n \theta \left( \|x\|_A^p \right)
\]

(2.37)

for all \( x \in A \). Then there exists a unique Jordan \( n \)-Lie homomorphism \( H : A \to B \) such that

\[
\left\| f(x) - H(x) \right\|_B \leq \frac{2^p}{\ell(2-2^p)} \theta \|x\|_A^p
\]

(2.38)

for all \( x \in A \).

**Proof.** It follows by Theorem 2.4 by putting \( \phi(x_1, x_2, \ldots, x_n) := \theta \sum_{i=1}^{n} (\|x_i\|_A^p) \) for all \( x_1, \ldots, x_n \in A \) and \( L = 2^{(p-1)} \). \( \square \)

**References**


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