Research Article

Some Properties on the q -Euler Numbers and Polynomials

T. Kim¹ and S.-H. Lee²

¹ Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

² Division of General Education, Kwangwoon University, Seoul 139-701, Republic of Korea

Correspondence should be addressed to T. Kim, tkkim@kw.ac.kr

Received 11 January 2012; Revised 28 March 2012; Accepted 11 April 2012

Academic Editor: Natig Atakishiyev

Copyright © 2012 T. Kim and S.-H. Lee. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We give some new identities on q -Euler numbers and polynomials by using the fermionic p -adic integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p -adic rational integers, the field of p -adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. The p -adic absolute value $|\cdot|_p$ is defined by $|p|_p = 1/p$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. As is well known, the fermionic p -adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} (-1)^x f(x), \quad (1.1)$$

where $f \in C(\mathbb{Z}_p)$ = the space of continuous functions on $\mathbb{Z}_p$ (see [1]).

From (1.1), we note that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{where } f_1(x) = f(x+1). \quad (1.2)$$

The q -Euler polynomials are defined by

$$\frac{2}{qe^t + 1} e^{xt} = e^{E_q(x)t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}, \quad (1.3)$$

with the usual convention about replacing E_q^n by $E_{n,q}$ (see [2, 3]).

Let us take $f(y) = q^y e^{t(x+y)}$. Then, by (1.2), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} q^y d\mu_{-1}(y) = \frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \quad (1.4)$$

By (1.3) and (1.4), we get the Witt's formula for the q -Euler polynomials as follows:

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(y), \quad n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \quad (1.5)$$

In the special case, $x = 0$, $E_{n,q}(0) = E_{n,q}$ are called the n -th q -Euler numbers.

From (1.3), we can derive the following recurrence relation for the q -Euler numbers $E_{n,q}$:

$$E_{0,q} = \frac{2}{[2]_q}, \quad q(E_q + 1)^n + E_{n,q} = 2\delta_{0,n}, \quad (1.6)$$

with the usual convention about replacing E_q^n by $E_{n,q}$ (see [4]).

By (1.5), we easily see that

$$E_{n,q}(x) = \sum_{\ell=0}^n \binom{n}{\ell} x^{n-\ell} \int_{\mathbb{Z}_p} q^y y^\ell d\mu_{-1}(y) = \sum_{\ell=0}^n \binom{n}{\ell} x^{n-\ell} E_{\ell,q}, \quad (1.7)$$

where $\binom{n}{\ell} = n!/\ell!(n-\ell)! = n(n-1) \cdots (n-\ell+1)/\ell!$ (see [1, 2, 4–13]). Cohen introduced many interesting and valuable identities related to Euler and Bernoulli numbers and polynomials in his book (see [14]). In [13], Ryoo has introduced the q -Euler numbers and polynomials with weight α , and Simsek et al. have studied q -Euler numbers and polynomials, and they introduced many interesting identities and properties (see [3, 15, 16]). In this paper, we consider the q -Euler numbers and polynomials with weight $\alpha = 1$. By applying the fermionic p -adic integral on $\mathbb{Z}_p$, we derive many not only new but also some interesting identities on the q -extension of Euler numbers and polynomials. In particular, we consider that Theorems 2.5, 2.6, 2.7, and 2.9 are important identities because these identities are closely related to Frobenius-Euler numbers and polynomials. As is well known, Frobenius-Euler numbers and polynomials are important to study p -adic l -functions in the number theory and mathematical physics related to fermionic distributions. In [17], Bayad and Kim have studied some interesting identities and properties on the q -Euler numbers and polynomials associated with Bernstein polynomials. Recently, several authors have studied some properties of q -Euler numbers and polynomials (see [1–19]). The purpose of this paper

is to give some interesting new identities for the q -Euler numbers and polynomials by using the fermionic p -adic integral on $\mathbb{Z}_p$ and (1.7).

2. Some Identities on q -Euler Polynomials

From (1.4), we note that

$$\int_{\mathbb{Z}_p} e^{(x+y+z)t} q^z d\mu_{-1}(z) = e^{(x+y)t} \frac{2}{qe^t + 1} = \sum_{n=0}^{\infty} E_{n,q}(x+y) \frac{t^n}{n!}. \quad (2.1)$$

Thus, by (1.4) and (2.1), we get

$$\begin{aligned} E_{n,q}(x+y) &= \int_{\mathbb{Z}_p} (x+y+z)^n q^z d\mu_{-1}(z) \\ &= \sum_{\ell=0}^n \binom{n}{\ell} y^{n-\ell} \int_{\mathbb{Z}_p} (x+z)^\ell q^z d\mu_{-1}(z) \\ &= \sum_{\ell=0}^n \binom{n}{\ell} y^{n-\ell} E_{\ell,q}(x), \quad \text{where } n \in \mathbb{Z}_+. \end{aligned} \quad (2.2)$$

By (2.2), we get

$$\begin{aligned} E_{n,q}(x+y) &= \sum_{j=0}^n \binom{n}{j} y^{n-j} E_{j,q}(x) \\ &= \frac{2}{[2]_q} y^n + \sum_{j=1}^n \frac{n}{j} \binom{n-1}{j-1} y^{n-j} E_{j,q}(x). \end{aligned} \quad (2.3)$$

From (2.3), we can derive the following equation (2.4):

$$\sum_{j=0}^{n-1} \binom{n-1}{j} y^{n-1-j} \frac{E_{j+1,q}(x)}{j+1} = \frac{E_{n,q}(x+y) - \left(\frac{2}{[2]_q}\right) y^n}{n}. \quad (2.4)$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{Z}_+$, one has

$$\sum_{j=0}^n \binom{n}{j} y^{n-j} \frac{E_{j+1,q}(x)}{j+1} = \frac{1}{n+1} \left(E_{n+1,q}(x+y) - \frac{2}{[2]_q} y^{n+1} \right). \quad (2.5)$$

Let us replace y by $-y$ in Theorem 2.1. Then we get

$$\sum_{j=0}^n \binom{n}{j} (-1)^{n-j} y^{n-j} \frac{E_{j+1,q}(x)}{j+1} = \frac{1}{n+1} \left(E_{n+1,q}(x-y) - \frac{(-1)^{n+1} 2}{[2]_q} y^{n+1} \right). \quad (2.6)$$

Thus, we have

$$\sum_{j=0}^n \binom{n}{j} (-1)^j y^{n-j} \frac{E_{j+1,q}(x)}{j+1} = \frac{1}{n+1} \left((-1)^n E_{n+1,q}(x-y) + \frac{2}{[2]_q} y^{n+1} \right). \quad (2.7)$$

Therefore, by Theorem 2.1 and (2.7), we obtain the following corollary.

Corollary 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{j=1}^{[n/2]} \binom{n}{2j} y^{n-2j} \frac{E_{2j+1,q}(x)}{2j+1} = \frac{E_{n+1,q}(x+y) + (-1)^n E_{n+1,q}(x-y)}{2n+2}. \quad (2.8)$$

From (2.2), we have

$$\sum_{j=1}^n \frac{1}{j} \binom{n-1}{j-1} y^{n-j} (-1)^j E_{j,q}(x) = \frac{(-1)^n E_{n,q}(x-y) - y^n (2/[2]_q)}{n}. \quad (2.9)$$

Therefore, by (2.3) and (2.9), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$, one has*

$$\sum_{j=1}^{[(n+1)/2]} \binom{n}{2j-1} y^{n+1-2j} \frac{E_{2j,q}(x)}{j} = \frac{E_{n+1,q}(x+y) + (-1)^{n+1} E_{n+1,q}(x-y) - (4/[2]_q) y^{n+1}}{4n+4}. \quad (2.10)$$

Letting $y = 1$ in Theorem 2.1, we see that

$$\begin{aligned} q \sum_{j=0}^n \binom{n}{j} \frac{E_{j+1,q}(x)}{j+1} &= \frac{q E_{n+1,q}(x+1) - 2q/[2]_q}{n+1}, \\ q E_{n+1,q}(x+1) &= \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} (E_q + 1)^\ell x^{n+1-\ell} \\ &= (2 - E_{0,q}) x^{n+1} - \sum_{\ell=1}^{n+1} \binom{n+1}{\ell} E_{\ell,q} x^{n+1-\ell} \end{aligned}$$

$$\begin{aligned}
&= 2x^{n+1} - \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} E_{\ell,q} x^{n+1-\ell} \\
&= 2x^{n+1} - E_{n+1,q}(x).
\end{aligned} \tag{2.11}$$

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.4. For $n \in \mathbb{Z}_+$, one has

$$\sum_{j=0}^n \binom{n}{j} \frac{qE_{j+1,q}(x)}{j+1} = -\frac{E_{n+1,q}}{n+1} + \frac{2x^{n+1}}{n+1} - \frac{2}{[2]_q} \frac{1}{n+1}. \tag{2.12}$$

Replacing y by 1 and n by $2n$ in Corollary 2.2, we have

$$\begin{aligned}
&\sum_{j=0}^n \binom{2n}{2j} \frac{E_{2j+1,q}(x)}{2j+1} \\
&= \frac{E_{2n+1,q}(x+1) + E_{2n+1,q}(x-1)}{4n+2} \\
&= \frac{(1/q)(qE_{2n+1,q}(x+1) + E_{2n+1,q}(x)) + (qE_{2n+1,q}(x) + E_{2n+1,q}(x-1))}{4n+2} \\
&\quad - \frac{E_{2n+1,q}(x)}{q(4n+2)} - q \frac{E_{2n+1,q}(x)}{4n+2} \\
&= \frac{2x^{2n+1}}{q(4n+2)} + \frac{2(x-1)^{2n+1}}{4n+2} - \frac{E_{2n+1,q}(x)}{q(4n+2)} - q \frac{E_{2n+1,q}(x)}{4n+2}.
\end{aligned} \tag{2.13}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.5. For $n \in \mathbb{Z}_+$, one has

$$\sum_{j=0}^n \binom{2n}{2j} \frac{E_{2j+1,q}(x)}{2j+1} = \frac{x^{2n+1}}{q(2n+1)} + \frac{(x-1)^{2n+1}}{2n+1} - \frac{E_{2n+1,q}(x)}{q(4n+2)} - q \frac{E_{2n+1,q}(x)}{4n+2}. \tag{2.14}$$

Replacing y by 1 and n by $2n$ in Theorem 2.3, we have

$$\begin{aligned}
&\sum_{j=1}^n \binom{2n}{2j-1} \frac{E_{2j,q}(x)}{j} \\
&= \frac{E_{2n+1,q}(x+1) - E_{2n+1,q}(x-1)}{8n+4} - \frac{1}{(2n+1)[2]_q}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2x^{2n+1}}{q(8n+4)} - \frac{2(x-1)^{2n+1}}{8n+4} - \frac{1}{(2n+1)[2]_q} - \frac{E_{2n+1,q}(x)}{q(8n+4)} + \frac{qE_{2n+1,q}(x)}{8n+4} \\
&= \frac{x^{2n+1}}{2q(2n+1)} - \frac{(x-1)^{2n+1}}{2(2n+1)} - \frac{1}{(2n+1)[2]_q} - \frac{E_{2n+1,q}(x)}{q(8n+4)} + \frac{qE_{2n+1,q}(x)}{8n+4}.
\end{aligned} \tag{2.15}$$

Therefore, by (2.15), we obtain the following theorem.

Theorem 2.6. For $n \in \mathbb{Z}_+$, one has

$$\sum_{\ell=1}^n \binom{2n}{2j-1} \frac{E_{2j,q}(x)}{j} = \frac{x^{2n+1}}{2q(2n+1)} - \frac{2(x-1)^{2n+1}}{2(2n+1)} - \frac{1}{(2n+1)[2]_q} - \frac{E_{2n+1,q}(x)}{q(8n+4)} + q \frac{E_{2n+1,q}(x)}{8n+4}. \tag{2.16}$$

Replacing y by $1/2$ and n by $2n$ in Theorem 2.3, we get

$$\begin{aligned}
&\sum_{j=1}^n \binom{2n}{2j-1} \left(\frac{1}{2}\right)^{2n+1-2j} \frac{E_{2j}(x)}{j} \\
&= \frac{E_{2n+1,q}(x+1/2) - E_{2n+1}(x-1/2) - (4/[2]_q)(1/2)^{2n+1}}{8n+4}.
\end{aligned} \tag{2.17}$$

Thus, by (2.17), we get

$$\sum_{j=1}^n \binom{2n}{2j-1} 2^{2j} \frac{E_{2j}(x)}{j} = \frac{2^{2n}(E_{2n+1,q}(x+1/2) - E_{2n+1}(x-1/2))}{4n+2} - \frac{1}{(2n+1)[2]_q}. \tag{2.18}$$

Note that

$$\begin{aligned}
qE_{2n+1,q}\left(x + \frac{1}{2}\right) &= qE_{2n+1,q}\left(x - \frac{1}{2} + 1\right) \\
&= q \sum_{\ell=0}^{2n+1} \binom{2n+1}{\ell} \left(x - \frac{1}{2}\right)^{2n+1-\ell} (E_q + 1)^\ell \\
&= \left(x - \frac{1}{2}\right)^{2n+1} (2 - E_{0,q}) - \sum_{\ell=1}^{2n+1} \binom{2n+1}{\ell} \left(x - \frac{1}{2}\right)^{2n+1-\ell} E_{\ell,q} \\
&= 2\left(x - \frac{1}{2}\right)^{2n+1} - E_{2n+1,q}\left(x - \frac{1}{2}\right).
\end{aligned} \tag{2.19}$$

Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 2.7. For $n \in \mathbb{N}$, one has

$$\sum_{j=1}^n \binom{2n}{2j-1} 2^{2j} \frac{E_{2j}(x)}{j} = \frac{2^{2n}(x-1/2)^{2n+1}}{2n+1} - \frac{[2]_q 2^n}{4n+2} E_{2n+1,q} \left(x - \frac{1}{2} \right) - \frac{1}{(2n+1)[2]_q}. \quad (2.20)$$

Replacing y by 1 and n by $2n+1$ in Corollary 2.2, we see that

$$\begin{aligned} & \sum_{j=0}^n \binom{2n+1}{2j} \frac{E_{2j+1,q}(x)}{2j+1} \\ &= \frac{E_{2n+2,q}(x+1) - E_{2n+2,q}(x-1)}{4n+4} \\ &= \left(\frac{(1/q)(qE_{2n+2,q}(x+1) + E_{2n+2,q}(x))}{4n+4} \right) - \left(\frac{qE_{2n+2,q}(x) + E_{2n+2,q}(x-1)}{4n+4} \right) \\ &\quad - \left(\frac{E_{2n+2,q}(x)}{q(4n+4)} - q \frac{E_{2n+2,q}(x)}{4n+4} \right) \\ &= \frac{2x^{2n+2}}{q(4n+4)} - \frac{2(x-1)^{2n+2}}{4n+4} - \left(\frac{E_{2n+2,q}(x)}{q(4n+4)} - q \frac{E_{2n+2,q}(x)}{4n+4} \right). \end{aligned} \quad (2.21)$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 2.8. For $n \in \mathbb{Z}_+$, one has

$$\sum_{j=0}^n \binom{2n+1}{2j} \frac{E_{2j+1,q}(x)}{2j+1} = \frac{x^{2n+2}}{q(2n+2)} - \frac{(x-1)^{2n+2}}{2n+2} - (1-q) \frac{[2]_q E_{2n+2,q}(x)}{q(4n+4)}. \quad (2.22)$$

Replacing n by $2n+1$ and y by 1 in Theorem 2.3, we get

$$\begin{aligned} & \sum_{j=1}^{n+1} \binom{2n+1}{2j-1} \frac{E_{2j,q}(x)}{j} \\ &= \frac{E_{2n+2,q}(x+1) + E_{2n+2,q}(x-1)}{8n+8} - \frac{1}{(2n+2)[2]_q} \\ &= \frac{qE_{2n+2,q}(x+1) + E_{2n+2,q}(x)}{q(8n+8)} + \frac{qE_{2n+2,q}(x) + E_{2n+2,q}(x)}{8n+8} \\ &\quad - \frac{E_{2n+2,q}(x)}{q(8n+8)} - \frac{qE_{2n+2,q}(x)}{8n+8} - \frac{1}{(2n+2)[2]_q} \\ &= \frac{2x^{2n+2}}{q(8n+8)} + \frac{q2(x-1)^{2n+2}}{8n+8} - \frac{E_{2n+2,q}(x)}{q(8n+8)} - q \frac{E_{2n+2,q}(x)}{8n+8} - \frac{1}{(2n+2)[2]_q}. \end{aligned} \quad (2.23)$$

Therefore, by (2.23), we obtain the following theorem.

Theorem 2.9. For $n \in \mathbb{Z}_+$, one has

$$\sum_{j=1}^{n+1} \binom{2n+1}{2j-1} \frac{E_{2j,q}(x)}{j} = \frac{x^{2n+2}}{q(4n+4)} + q \frac{(x-1)^{2n+2}}{4n+4} - (1+q^2) \frac{E_{2n+2,q}(x)}{8n+8} - \frac{1}{(2n+2)[2]_q}. \quad (2.24)$$

Acknowledgments

This paper was supported by the research grant of Kwangwoon University in 2012. The authors would like to express their sincere gratitude to referees for their valuable comments and suggestions.

References

- [1] T. Kim, "Some identities on the q -Euler polynomials of higher order and q -Stirling numbers by the fermionic p -adic integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 4, pp. 484–491, 2009.
- [2] H. Ozden, I. N. Cangul, and Y. Simsek, "Multivariate interpolation functions of higher-order q -Euler numbers and their applications," *Abstract and Applied Analysis*, vol. 2008, Article ID 390857, 16 pages, 2008.
- [3] Y. Simsek, "Complete sum of products of (h, q) -extension of Euler polynomials and numbers," *Journal of Difference Equations and Applications*, vol. 16, no. 11, pp. 1331–1348, 2010.
- [4] H. Ozden, I. N. Cangul, and Y. Simsek, "Remarks on q -Bernoulli numbers associated with Daehee numbers," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 1, pp. 41–48, 2009.
- [5] J. Choi, T. Kim, and Y. H. Kim, "A note on the extended q -Bernoulli numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, no. 4, pp. 351–354, 2011.
- [6] M. Can, M. Cenkci, V. Kurt, and Y. Simsek, "Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l -functions," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 135–160, 2009.
- [7] I. N. Cangul, V. Kurt, H. Ozden, and Y. Simsek, "On the higher-order w - q -Genocchi numbers," *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 1, pp. 39–57, 2009.
- [8] T. Kim, "On the multiple q -Genocchi and Euler numbers," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 481–486, 2008.
- [9] T. Kim, "New approach to q -Euler, Genocchi numbers and their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 105–112, 2009.
- [10] T. Kim, "New approach to q -Euler, Genocchi numbers and their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 105–112, 2008.
- [11] Y.-H. Kim, W. Kim, and C. S. Ryoo, "On the twisted q -Euler zeta function associated with twisted q -Euler numbers," *Proceedings of the Jangjeon Mathematical Society*, vol. 12, no. 1, pp. 93–100, 2009.
- [12] C. S. Ryoo, "A note on the Frobenius-Euler polynomials," *Proceedings of the Jangjeon Mathematical Society*, vol. 14, no. 4, pp. 495–501, 2011.
- [13] C. S. Ryoo, "A note on the weighted q -Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics Journal*, vol. 21, no. 1, pp. 47–54, 2011.
- [14] H. Cohen, *Number Theory: Analytic and Modern Tools*, vol. II of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2007.
- [15] Y. Simsek, "Special functions related to Dedekind-type DC-sums and their applications," *Russian Journal of Mathematical Physics*, vol. 17, no. 4, pp. 495–508, 2010.
- [16] H. Ozden, I. N. Cangul, and Y. Simsek, "On the behavior of two variable twisted p -adic Euler q - l -functions," *Nonlinear Analysis*, vol. 71, no. 12, pp. e942–e951, 2009.
- [17] A. Bayad and T. Kim, "Identities involving values of Bernstein, q -Bernoulli, and q -Euler polynomials," *Russian Journal of Mathematical Physics*, vol. 18, no. 2, pp. 133–143, 2011.

- [18] H. Y. Lee, N. S. Jung, and C. S. Ryoo, "A note on the q -Euler numbers and polynomials with weak weight α ," *Journal of Applied Mathematics*, vol. 2011, Article ID 497409, 14 pages, 2011.
- [19] S. Araci, D. Erdal, and J. J. Seo, "A study on the fermionic p -adic q -integral representation on $\mathbb{Z}_p$ associated with weighted q -Bernstein and q -Genocchi polynomials," *Abstract and Applied Analysis*, vol. 2011, Article ID 649248, 10 pages, 2011.