Research Article

Nearly Quadratic Mappings over p -Adic Fields

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We establish some stability results over p -adic fields for the generalized quadratic functional equation $\sum_{k=2}^n \sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^n f(\sum_{i=1}^n, i \neq i_1, \dots, i_{n-k+1}) x_i - \sum_{r=1}^{n-k+1} x_{i_r}) + f(\sum_{i=1}^n x_i) = 2^{n-1} \sum_{i=1}^n f(x_i)$, where $n \in \mathbb{N}$ and $n \geq 2$.

1. Introduction and Preliminaries

In 1899, Hensel [1] discovered the p -adic numbers as a number of theoretical analogue of power series in complex analysis. Fix a prime number p . For any nonzero rational number x , there exists a unique integer n_x such that $x = (a/b)p^{n_x}$, where a and b are integers not divisible by p . Then, p -adic absolute value $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, and it is called the p -adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x} a_k p^k$, where $|a_k| \leq p - 1$ are integers (see, e.g., [2, 3]). Note that if $p > 2$, then $|2^n|_p = 1$ for each integer n .

During the last three decades, p -adic numbers have gained the interest of physicists for their research, in particular, in problems coming from quantum physics, p -adic strings, and superstrings [4, 5]. A key property of p -adic numbers is that they do not satisfy the Archimedean axiom: For $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$.

Let $\mathbb{K}$ denote a field and function (valuation absolute) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$. A non-Archimedean valuation is a function $|\cdot|$ that satisfies the strong triangle inequality; namely, $|x + y| \leq \max\{|x|, |y|\} \leq |x| + |y|$ for all $x, y \in \mathbb{K}$. The associated field $\mathbb{K}$ is referred to as a non-Archimedean field. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $|\cdot|$ is nontrivial, that is, there is a $z \in \mathbb{K}$ such that $|z| \neq 0, 1$.

Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a non-Archimedean norm if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. Then, $(X, \|\cdot\|)$ is called a non-Archimedean space. In any such a space, a sequence $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy if and only if $\{x_{n+1} - x_n\}_{n \in \mathbb{N}}$ converges to zero. By a complete non-Archimedean space, we mean one in which every Cauchy sequence is convergent.

The study of stability problems for functional equations is related to a question of Ulam [6] concerning the stability of group homomorphisms, which was affirmatively answered for Banach spaces by Hyers [7]. Subsequently, the result of Hyers was generalized by Aoki [8] for additive mappings and by Rassias [9] for linear mappings by considering an unbounded Cauchy difference. The paper by Rassias has provided a lot of influences in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Rassias [10] considered the Cauchy difference controlled by a product of different powers of norm. The above results have been generalized by Forti [11] and Găvruta [12] who permitted the Cauchy difference to become arbitrary unbounded (see also [13–22]). Arriola and Beyer [23] investigated stability of approximate additive functions $f : \mathbb{Q}_p \rightarrow \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \rightarrow \mathbb{R}$ is a continuous function for which there exists a fixed ε such that $|f(x + y) - f(x) - f(y)| \leq \varepsilon$ for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive function $T : \mathbb{Q}_p \rightarrow \mathbb{R}$ such that $|f(x) - T(x)| \leq \varepsilon$ for all $x \in \mathbb{Q}_p$. For more details about the results concerning such problems, the reader is referred to [24–45].

Recently, Khodaei and Rassias [46] introduced the generalized additive functional equation

$$\sum_{k=2}^n \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f \left(\sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r} x_{i_r} \right) + f \left(\sum_{i=1}^n a_i x_i \right) = 2^{n-1} a_1 f(x_1) \quad (1.1)$$

and proved the generalized Hyers-Ulam stability of the above functional equation. The functional equation

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2) \quad (1.2)$$

is related to symmetric biadditive function and is called a quadratic functional equation [47, 48]. Every solution of the quadratic equation (1.2) is said to be a quadratic function.

Now, we introduce the generalized quadratic functional equation in n -variables as follows:

$$\sum_{k=2}^n \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f \left(\sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) + f \left(\sum_{i=1}^n x_i \right) = 2^{n-1} \sum_{i=1}^n f(x_i), \quad (1.3)$$

where $n \geq 2$. Moreover, we investigate the generalized Hyers-Ulam stability of functional equation (1.3) over the p -adic field $\mathbb{Q}_p$.

As a special case, if $n = 2$ in (1.3), then we have the functional equation (1.2). Also, if $n = 3$ in (1.3), we obtain

$$\sum_{i_1=2}^2 \sum_{i_2=i_1+1}^3 f\left(\sum_{i=1, i \neq i_1, i_2}^3 x_i - \sum_{r=1}^2 x_{i_r}\right) + \sum_{i_1=2}^3 f\left(\sum_{i=1, i \neq i_1}^3 x_i - x_{i_1}\right) + f\left(\sum_{i=1}^3 x_i\right) = 2^2 \sum_{i=1}^3 f(x_i), \quad (1.4)$$

that is,

$$f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) = 4f(x_1) + 4f(x_2) + 4f(x_3). \quad (1.5)$$

2. Stability of Quadratic Functional Equation (1.3) over p -Adic Fields

We will use the following lemma.

Lemma 2.1. *Let X and Y be real vector spaces. A function $f : X \rightarrow Y$ satisfies the functional equation (1.3) if and only if the function f is quadratic.*

Proof. Let f satisfy the functional equation (1.3). Setting $x_i = 0$ ($i = 1, \dots, n$) in (1.3), we have

$$\sum_{k=2}^n \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f(0) + f(0) = 2^{n-1} \sum_{i=1}^n f(0), \quad (2.1)$$

that is,

$$\sum_{i_1=2}^2 \sum_{i_2=i_1+1}^3 \cdots \sum_{i_{n-1}=i_{n-2}+1}^n f(0) + \sum_{i_1=2}^3 \sum_{i_2=i_1+1}^4 \cdots \sum_{i_{n-2}=i_{n-3}+1}^n f(0) + \cdots + \sum_{i_1=2}^n f(0) + f(0) = 2^{n-1} \sum_{i=1}^n f(0), \quad (2.2)$$

or

$$\left(\binom{n-1}{n-1} + \binom{n-1}{n-2} + \cdots + \binom{n-1}{1} + 1 \right) f(0) = 2^{n-1} \sum_{i=1}^n f(0), \quad (2.3)$$

but $1 + \sum_{j=1}^{n-1} \binom{n-1}{j} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$, and also $n > j \geq 1$ so $2^{n-1}(n-1)f(0) = 0$.

Putting $x_i = 0$ ($i = 2, \dots, n-1$) in (1.3) and then using $f(0) = 0$, we get

$$\begin{aligned} & f(x_1 - x_n) + \left(\binom{n-2}{1} f(x_1 - x_n) + \binom{n-2}{n-2} f(x_1 + x_n) \right) + \cdots \\ & + \left(\binom{n-2}{n-3} f(x_1 - x_n) + \binom{n-2}{2} f(x_1 + x_n) \right) \end{aligned}$$

$$\begin{aligned}
& + \left(\binom{n-2}{n-2} f(x_1 - x_n) + \binom{n-2}{1} f(x_1 + x_n) \right) + f(x_1 + x_n) \\
& = 2^{n-1} f(x_1) + 2^{n-1} f(x_n),
\end{aligned} \tag{2.4}$$

that is,

$$\left(1 + \sum_{j=1}^{n-2} \binom{n-2}{j} \right) (f(x_1 + x_n) + f(x_1 - x_n)) = 2^{n-1} f(x_1) + 2^{n-1} f(x_n), \tag{2.5}$$

for all $x_1, x_n \in X$, this shows that f satisfies the functional equation (1.2). So the function f is quadratic.

Conversely, suppose that f is quadratic, thus f satisfies the functional equation (1.2). Hence, we have $f(0) = 0$ and f is even.

We are going to prove our assumption by induction on $n \geq 2$. It holds on $n = 2$. Assume that it holds on the case where $n = t$; that is, we have

$$\sum_{k=2}^t \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{t-k+1}=i_{t-k}+1}^t \right) f \left(\sum_{i=1, i \neq i_1, \dots, i_{t-k+1}}^t x_i - \sum_{r=1}^{t-k+1} x_{i_r} \right) + f \left(\sum_{i=1}^t x_i \right) = 2^{t-1} \sum_{i=1}^t f(x_i) \tag{2.6}$$

for all $x_1, \dots, x_t \in X$. It follows from (1.2) that

$$f \left(\sum_{i=1}^t x_i + x_{t+1} \right) + f \left(\sum_{i=1}^t x_i - x_{t+1} \right) = 2f \left(\sum_{i=1}^t x_i \right) + 2f(x_{t+1}) \tag{2.7}$$

for all $x_1, \dots, x_{t+1} \in X$. Replacing x_t by $-x_t$ in (2.7), we obtain

$$f \left(\sum_{i=1}^{t-1} x_i - x_t + x_{t+1} \right) + f \left(\sum_{i=1}^{t-1} x_i - x_t - x_{t+1} \right) = 2f \left(\sum_{i=1}^{t-1} x_i - x_t \right) + 2f(x_{t+1}) \tag{2.8}$$

for all $x_1, \dots, x_{t+1} \in X$. Adding (2.7) to (2.8), we have

$$\begin{aligned}
& f \left(\sum_{i=1}^{t-1} x_i - x_t - x_{t+1} \right) + f \left(\sum_{i=1}^{t-1} x_i - x_t + x_{t+1} \right) + f \left(\sum_{i=1}^{t-1} x_i + x_t - x_{t+1} \right) + f \left(\sum_{i=1}^{t-1} x_i + x_t + x_{t+1} \right) \\
& = 2 \left[f \left(\sum_{i=1}^{t-1} x_i - x_t \right) + f \left(\sum_{i=1}^{t-1} x_i + x_t \right) \right] + 4f(x_{t+1})
\end{aligned} \tag{2.9}$$

for all $x_1, \dots, x_{t+1} \in X$. Replacing x_{t-1} by $-x_{t-1}$ in (2.9), we get

$$\begin{aligned} & f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t - x_{t+1}\right) \\ & + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t + x_{t+1}\right) = 2\left[f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t\right)\right] + 4f(x_{t+1}) \end{aligned} \quad (2.10)$$

for all $x_1, \dots, x_{t+1} \in X$. Adding (2.9) to (2.10), one gets

$$\begin{aligned} & f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t - x_{t+1}\right) \\ & + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t + x_{t+1}\right) \\ & + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} + x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t+1} x_i\right) \\ & = 2\left[f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t\right) + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t\right) \right. \\ & \quad \left. + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} + x_t\right)\right] + 8f(x_{t+1}) \end{aligned} \quad (2.11)$$

for all $x_1, \dots, x_{t+1} \in X$. By using the above method, for x_{t-2} until x_2 , we infer that

$$\begin{aligned} & \sum_{k=2}^{t+1} \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{t-k+2}=i_{t-k+1}+1}^{t+1} \right) f\left(\sum_{i=1, i \neq i_1, \dots, i_{t-k+2}}^{t+1} x_i - \sum_{r=1}^{t-k+2} x_{i_r}\right) + f\left(\sum_{i=1}^{t+1} x_i\right) \\ & = 2 \left[\sum_{k=2}^t \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{t-k+1}=i_{t-k}+1}^t \right) f\left(\sum_{i=1, i \neq i_1, \dots, i_{t-k+1}}^t x_i - \sum_{r=1}^{t-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^t x_i\right) \right] + 2^t f(x_{t+1}) \end{aligned} \quad (2.12)$$

for all $x_1, \dots, x_{t+1} \in X$. Now, by the case $n = t$, we lead to

$$\begin{aligned} & \sum_{k=2}^{t+1} \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{t-k+2}=i_{t-k+1}+1}^{t+1} \right) f\left(\sum_{i=1, i \neq i_1, \dots, i_{t-k+2}}^{t+1} x_i - \sum_{r=1}^{t-k+2} x_{i_r}\right) + f\left(\sum_{i=1}^{t+1} x_i\right) \\ & = 2 \left[2^{t-1} \sum_{i=1}^t f(x_i) \right] + 2^t f(x_{t+1}) \end{aligned} \quad (2.13)$$

for all $x_1, \dots, x_{t+1} \in X$, so (1.3) holds for $n = t + 1$. This completes the proof of the lemma. $\square$

Corollary 2.2. *A function $f : X \rightarrow Y$ satisfies the functional equation (1.3) if and only if there exists a symmetric biadditive function $B_1 : X \times X \rightarrow Y$ such that $f(x) = B_1(x, x)$ for all $x \in X$.*

Now, we investigate the stability of the functional equation (1.3) from a Banach space B into p -adic field $\mathbb{Q}_p$. For convenience, we define the difference operator D_f for a given function f :

$$\begin{aligned} D_f(x_1, \dots, x_n) := & \sum_{k=2}^n \left(\sum_{i_1=2}^k \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^n \right) f \left(\sum_{i=1, i \neq i_1, \dots, i_{n-k+1}}^n x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) \\ & + f \left(\sum_{i=1}^n x_i \right) - 2^{n-1} \sum_{i=1}^n f(x_i). \end{aligned} \quad (2.14)$$

Theorem 2.3. *Let B be a Banach space and let $\varepsilon > 0$, λ be real numbers. Suppose that a function $f : \mathbb{Q}_p \rightarrow B$ with $f(0) = 0$ satisfies the inequality*

$$\|D_f(x_1, \dots, x_n)\| \leq \varepsilon \sum_{i=1}^n |x_i|_p^\lambda \quad (2.15)$$

for all $x_1, \dots, x_n \in \mathbb{Q}_p$. Then there exists a unique quadratic function $Q : \mathbb{Q}_p \rightarrow B$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{\varepsilon}{2^{n-1} - 2^{n-\lambda-3}} |x|_p^\lambda, & p = 2, \lambda > -2; \\ \frac{\varepsilon}{3 \cdot 2^{n-3}} |x|_p^\lambda, & p > 2; \end{cases} \quad (2.16)$$

for all nonzero $x \in \mathbb{Q}_p$.

Proof. Letting $x_1 = x_2 = x \neq 0$ and $x_i = 0$ ($i = 3, \dots, n$) in (2.15), we obtain

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{\varepsilon}{2^{n-1}} |x|_p^\lambda \quad (2.17)$$

for all $x \in \mathbb{Q}_p$. Hence,

$$\left\| \frac{1}{2^{2l}} f(2^l x) - \frac{1}{2^{2m}} f(2^m x) \right\| \leq \frac{\varepsilon}{2^{n-1}} \sum_{j=l}^{m-1} \frac{|2|_p^{\lambda j}}{2^{2j}} |x|_p^\lambda \quad (2.18)$$

for all nonnegative integers m and l with $m > l$ and for all $x \in \mathbb{Q}_p$. It follows from (2.18) that the sequence $\{(1/2^{2m})f(2^m x)\}$ is a Cauchy sequence for all $x \in \mathbb{Q}_p$. Since B is complete, the sequence $\{(1/2^{2m})f(2^m x)\}$ converges. Therefore, one can define the function $Q : \mathbb{Q}_p \rightarrow B$ by

$$Q(x) := \lim_{m \rightarrow \infty} \frac{1}{2^{2m}} f(2^m x) \quad (2.19)$$

for all $x \in \mathbb{Q}_p$. It follows from (2.15) and (2.19) that

$$\|D_Q(x_1, \dots, x_n)\| = \lim_{m \rightarrow \infty} \frac{1}{2^{2m}} \|D_f(2^m x_1, \dots, 2^m x_n)\| \leq \lim_{m \rightarrow \infty} \frac{|2|_p^{\lambda m}}{2^{2m}} \sum_{i=1}^n \varepsilon |x_i|_p^\lambda = 0 \quad (2.20)$$

for all $x_1, \dots, x_n \in \mathbb{Q}_p$. So $D_Q(x_1, \dots, x_n) = 0$. By Lemma 2.1, the function $Q : \mathbb{Q}_p \rightarrow B$ is quadratic.

Taking the limit $m \rightarrow \infty$ in (2.18) with $l = 0$, we find that the function Q is quadratic function satisfying the inequality (2.16) near the approximate function $f : \mathbb{Q}_p \rightarrow B$ of (1.3).

To prove the aforementioned uniqueness, we assume now that there is another additive function $Q' : \mathbb{Q}_p \rightarrow B$ which satisfies (1.3) and the inequality (2.16). So

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \frac{1}{2^{2m}} \|Q(2^m x) - Q'(2^m x)\| \\ &\leq \frac{1}{2^{2m}} (\|Q(2^m x) - f(2^m x)\| + \|f(2^m x) - Q'(2^m x)\|) \\ &\leq \begin{cases} \frac{\varepsilon}{2^{2m+\lambda m} (2^{n-2} - 2^{n-\lambda-4})} |x|_p^\lambda, & p = 2, \lambda > -2; \\ \frac{\varepsilon}{3 \cdot 2^{2m+n-4}} |x|_p^\lambda, & p > 2; \end{cases} \end{aligned} \quad (2.21)$$

which tends to zero as $m \rightarrow \infty$ for all nonzero $x \in \mathbb{Q}_p$. This proves the uniqueness of Q , completing the proof of uniqueness. $\square$

The following example shows that the above result is not valid over p -adic fields.

Example 2.4. Let $p > 2$ be a prime number and define $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ by $f(x) = x^2 - 2x$. Since $|2^n|_p = 1$,

$$|D_f(x_1, \dots, x_n)|_p = \left| 2^n \sum_{i=2}^n x_i \right|_p = \left| \sum_{i=2}^n x_i \right|_p \leq \sum_{i=1}^n |x_i|_p \quad (2.22)$$

for all $x_1, \dots, x_n \in \mathbb{Q}_p$. Hence, the conditions of Theorem 2.3 for $\varepsilon = 1$ and $\lambda = 1$ hold. However for each $n \in \mathbb{N}$, we have

$$\left| \frac{1}{2^{2(m+1)}} f(2^{m+1}x) - \frac{1}{2^{2m}} f(2^m x) \right|_p = \frac{|x|_p}{|2^m|_p} = |x|_p \quad (2.23)$$

for all $x \in \mathbb{Q}_p$. Hence $\{(1/2^{2m})f(2^m x)\}$ is not convergent for all nonzero $x \in \mathbb{Q}_p$.

In the next result, which can be compared with Theorem 2.3, we will show that the stability of the functional equation (1.3) in non-Archimedean spaces over p -adic fields.

Theorem 2.5. Let $\ell \in \{-1, 1\}$ be fixed. Let $\mathcal{V}$ be a non-Archimedean space and $\mathcal{W}$ be a complete non-Archimedean space over $\mathbb{Q}_p$, where $p > 2$ is a prime number. Suppose that a function $f : \mathcal{V} \rightarrow \mathcal{W}$ satisfies the inequality

$$\|D_f(x_1, \dots, x_n)\|_{\mathcal{W}} \leq \begin{cases} \varepsilon \sum_{i=1}^n \|x_i\|_{\mathcal{V}}^{\lambda}, & \lambda\ell > 2\ell; \\ \varepsilon \sum_{i=2}^n \|x_1\|_{\mathcal{V}}^{\lambda_1} \|x_i\|_{\mathcal{V}}^{\lambda_i}, & (\lambda_1 + \lambda_i)\ell > 2\ell; \\ \varepsilon \max\{\|x_i\|_{\mathcal{V}}^{\lambda}; 1 \leq i \leq n\}, & \lambda\ell > 2\ell; \end{cases} \quad (2.24)$$

for all $x_1, \dots, x_n \in \mathcal{V}$, where $\varepsilon, \lambda_1, \dots, \lambda_n$ and λ are nonnegative real numbers. Then, the limit

$$Q(x) := \lim_{m \rightarrow \infty} \frac{1}{p^{2\ell m}} f(p^{\ell m} x) \quad (2.25)$$

exists for all $x \in \mathcal{V}$ and $Q : \mathcal{V} \rightarrow \mathcal{W}$ is a unique quadratic function satisfying

$$\|f(x) - Q(x)\|_{\mathcal{W}} \leq \begin{cases} 2p^{1+\ell+(1-\ell)\lambda/2} \varepsilon \|x\|_{\mathcal{V}}^{\lambda}, \\ p^{1+\ell+((1-\ell)(\lambda_1+\lambda_2)/2)} \varepsilon \|x\|_{\mathcal{V}}^{\lambda_1+\lambda_2}, \\ p^{1+\ell+(1-\ell)\lambda/2} \varepsilon \|x\|_{\mathcal{V}}^{\lambda}, \end{cases} \quad (2.26)$$

for all $x \in \mathcal{V}$.

Proof. By (2.24),

$$\|D_f(x_1, \dots, x_n)\|_{\mathcal{W}} \leq \varepsilon \sum_{i=1}^n \|x_i\|_{\mathcal{V}}^{\lambda} \quad (2.27)$$

for all $x_1, \dots, x_n \in \mathcal{V}$, where $\lambda\ell > 2\ell$. Putting $x_i = 0$ ($i = 1, \dots, n$) in (2.27) to obtain $f(0) = 0$, setting $x_i = 0$ ($i = 3, \dots, n$) in (2.27), we obtain

$$\left\| 2^{n-2} f(x_1 + x_2) + 2^{n-2} f(x_1 - x_2) - 2^{n-1} f(x_1) - 2^{n-1} f(x_2) \right\|_{\mathcal{W}} \leq \varepsilon (\|x_1\|_{\mathcal{V}}^{\lambda} + \|x_2\|_{\mathcal{V}}^{\lambda}) \quad (2.28)$$

for all $x_1, x_2 \in \mathcal{V}$. So

$$\|f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) - 2f(x_2)\|_{\mathcal{W}} \leq \varepsilon (\|x_1\|_{\mathcal{V}}^{\lambda} + \|x_2\|_{\mathcal{V}}^{\lambda}) \quad (2.29)$$

for all $x_1, x_2 \in \mathcal{V}$. Letting $x_1 = x_2 = x$ in (2.29), we have

$$\|f(2x) - 4f(x)\|_{\mathcal{W}} \leq 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda} \quad (2.30)$$

for all $x \in \mathcal{U}$. By induction on j , we will show that for each $j \geq 2$,

$$\left\| f(jx) - j^2 f(x) \right\|_{\mathcal{W}} \leq 2\varepsilon \|x\|_{\mathcal{U}}^{\lambda} \quad (2.31)$$

for all $x \in \mathcal{U}$. It holds on $j = 2$; see (2.30). Let (2.31) hold for $j = 2, \dots, k$. Replacing x_1 and x_2 by kx and x in (2.29), respectively, we get

$$\left\| f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x) \right\|_{\mathcal{W}} \leq \varepsilon \left(1 + |k|_p^{\lambda} \right) \|x\|_{\mathcal{U}}^{\lambda} \quad (2.32)$$

for all $x \in \mathcal{U}$. It follows from (2.32) and our induction hypothesis that

$$\begin{aligned} \left\| f((k+1)x) - (k+1)^2 f(x) \right\|_{\mathcal{W}} &= \left\| f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x) \right. \\ &\quad \left. - f((k-1)x) + (k-1)^2 f(x) - 2(f(kx) - k^2 f(x)) \right\|_{\mathcal{W}} \\ &\leq \max \left\{ 2\varepsilon \|x\|_{\mathcal{U}}^{\lambda}, \varepsilon \left(1 + |k|_p^{\lambda} \right) \|x\|_{\mathcal{U}}^{\lambda} \right\} = 2\varepsilon \|x\|_{\mathcal{U}}^{\lambda} \end{aligned} \quad (2.33)$$

for all $x \in \mathcal{U}$. This proves (2.31) for each $j \geq 2$. In particular,

$$\left\| f(px) - p^2 f(x) \right\|_{\mathcal{W}} \leq 2\varepsilon \|x\|_{\mathcal{U}}^{\lambda} \quad (2.34)$$

for all $x \in \mathcal{U}$. So

$$\begin{aligned} \left\| f(x) - \frac{1}{p^2} f(px) \right\|_{\mathcal{W}} &\leq 2p^2 \varepsilon \|x\|_{\mathcal{U}}^{\lambda}, \\ \left\| f(x) - p^2 f\left(\frac{x}{p}\right) \right\|_{\mathcal{W}} &\leq 2p^{\lambda} \varepsilon \|x\|_{\mathcal{U}}^{\lambda} \end{aligned} \quad (2.35)$$

for all $x \in \mathcal{U}$. Hence,

$$\left\| \frac{1}{p^{2\ell j}} f(p^{\ell j} x) - \frac{1}{p^{2\ell(j+1)}} f(p^{\ell(j+1)} x) \right\|_{\mathcal{W}} \leq \frac{2p^{2\ell j + (1-\ell)\lambda/2 + 1 + \ell}}{p^{\lambda\ell j}} \varepsilon \|x\|_{\mathcal{U}}^{\lambda} \quad (2.36)$$

for all $x \in \mathcal{U}$. Since the right side of the above inequality tends to zero as $j \rightarrow \infty$, $\{(1/p^{2\ell m})f(p^{\ell m}x)\}$ is a Cauchy sequence in complete non-Archimedean space $\mathcal{W}$, thus it

converges to some function $Q(x) = \lim_{m \rightarrow \infty} (1/p^{2\ell m})f(p^{\ell m}x)$ for all $x \in \mathcal{U}$. Using (2.35) and induction, one can show that for any $m \in \mathbb{N}$, we have

$$\begin{aligned} \left\| f(x) - \frac{1}{p^{2\ell m}} f(p^{\ell m}x) \right\|_{\mathcal{W}} &= \left\| \sum_{j=0}^{m-1} \frac{1}{p^{2\ell j}} f(p^{\ell j}x) - \frac{1}{p^{2\ell(j+1)}} f(p^{\ell(j+1)}x) \right\|_{\mathcal{W}} \\ &\leq \max \left\{ \left\| \frac{1}{p^{2\ell j}} f(p^{\ell j}x) - \frac{1}{p^{2\ell(j+1)}} f(p^{\ell(j+1)}x) \right\|_{\mathcal{W}} ; 0 \leq j < m \right\} \\ &\leq \max \left\{ 2p^{1+\ell+(1-\ell)\lambda/2+\ell j(2-\lambda)} \varepsilon \|x\|_{\mathcal{U}}^{\lambda} ; 0 \leq j < m \right\} \end{aligned} \quad (2.37)$$

for all $x \in \mathcal{U}$. Letting $m \rightarrow \infty$ in this inequality, we see that

$$\|f(x) - Q(x)\|_{\mathcal{W}} \leq 2p^{1+\ell+(1-\ell)\lambda/2} \varepsilon \|x\|_{\mathcal{U}}^{\lambda} \quad (2.38)$$

for all $x \in \mathcal{U}$. Moreover,

$$\|D_Q(x_1, \dots, x_n)\|_{\mathcal{W}} = \lim_{m \rightarrow \infty} \left\| \frac{1}{p^{2\ell m}} D_f(p^{\ell m}x_1, \dots, p^{\ell m}x_n) \right\|_{\mathcal{W}} \leq \lim_{m \rightarrow \infty} \frac{p^{2\ell m}}{p^{\lambda\ell m}} \sum_{i=1}^n \varepsilon \|x_i\|_{\mathcal{U}}^{\lambda} = 0 \quad (2.39)$$

for all $x_1, \dots, x_n \in \mathcal{U}$. So $D_Q(x_1, \dots, x_n) = 0$. By Lemma 2.1, the function $Q : \mathcal{U} \rightarrow \mathcal{W}$ is quadratic.

Now, let $Q' : \mathcal{U} \rightarrow \mathcal{W}$ be another quadratic function satisfying (1.3) and (2.38). So

$$\begin{aligned} \|Q(x) - Q'(x)\|_{\mathcal{W}} &\leq p^{2\ell m} \max \left\{ \|Q(p^{\ell m}x) - f(p^{\ell m}x)\|_{\mathcal{W}'}, \|f(p^{\ell m}x) - Q'(p^{\ell m}x)\|_{\mathcal{W}'} \right\} \\ &\leq \frac{2p^{2\ell m+(1-\ell)\lambda/2+1+\ell}}{p^{\lambda\ell m}} \varepsilon \|x\|_{\mathcal{U}}^{\lambda}, \end{aligned} \quad (2.40)$$

which tends to zero as $m \rightarrow \infty$ for all $x \in \mathcal{U}$. This proves the uniqueness of Q .

The rest of the proof is similar to the above proof, hence it is omitted. $\square$

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