Research Article

Nearly Quadratic Mappings over $p$-Adic Fields

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Received 30 October 2011; Revised 20 November 2011; Accepted 21 November 2011

Abstract and Applied Analysis

We establish some stability results over $p$-adic fields for the generalized quadratic functional equation

$$\sum_{k=0}^{n} \sum_{i=1}^{k} \sum_{\ell_{i+1}=1}^{k+1} \cdots \sum_{\ell_{n-k}=1}^{k} x_i = f(\sum_{i=1}^{n-k} f_i) + f(\sum_{j=1}^{n} x_j) = 2^{n-1} \sum_{i=1}^{n} f(x_i),$$

where $n \in \mathbb{N}$ and $n \geq 2$.

1. Introduction and Preliminaries

In 1899, Hensel [1] discovered the $p$-adic numbers as a number of theoretical analogue of power series in complex analysis. Fix a prime number $p$. For any nonzero rational number $x$, there exists a unique integer $n_x$ such that $x = (a/b)p^{n_x}$, where $a$ and $b$ are integers not divisible by $p$. Then, $p$-adic absolute value $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $Q$. The completion of $Q$ with respect to the metric $d(x, y) = |x - y|_p$, is denoted by $Q_p$, and it is called the $p$-adic number field. In fact, $Q_p$ is the set of all formal series $x = \sum_{k=0}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$ are integers (see, e.g., [2, 3]). Note that if $p > 2$, then $|2^n|_p = 1$ for each integer $n$.

During the last three decades, $p$-adic numbers have gained the interest of physicists for their research, in particular, in problems coming from quantum physics, $p$-adic strings, and superstrings [4, 5]. A key property of $p$-adic numbers is that they do not satisfy the Archimedean axiom: For $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$.

Let $K$ denote a field and function (valuation absolute) $| \cdot |$ from $K$ into $[0, \infty)$. A non-Archimedean valuation is a function $| \cdot |$ that satisfies the strong triangle inequality; namely, $|x + y| \leq \max(|x|, |y|) \leq |x| + |y|$ for all $x, y \in K$. The associated field $K$ is referred to as a non-Archimedean field. Clearly, $|1| = |1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $| \cdot |$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $| \cdot |$ is nontrivial, that is, there is a $z \in K$ such that $|z| \neq 0, 1$.\n
Let $X$ be a linear space over a field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a non-Archimedean norm if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. Then, $(X, \|\cdot\|)$ is called a non-Archimedean space. In any such a space, a sequence $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy if and only if $\{x_{n+1} - x_n\}_{n \in \mathbb{N}}$ converges to zero. By a complete non-Archimedean space, we mean one in which every Cauchy sequence is convergent.

The study of stability problems for functional equations is related to a question of Ulam [6] concerning the stability of group homomorphisms, which was affirmatively answered for Banach spaces by Hyers [7]. Subsequently, the result of Hyers was generalized by Aoki [8] for additive mappings and by Rassias [9] for linear mappings by considering an unbounded Cauchy difference. The paper by Rassias has provided a lot of influences in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Rassias [10] considered the Cauchy difference controlled by a product of different powers of norm. The above results have been generalized by Forti [11] and Găvruţa [12] who permitted the Cauchy difference to become arbitrary unbounded (see also [13–22]). Arriola and Beyer [23] investigated stability of approximate additive functions $f : \mathbb{Q}_p \rightarrow \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \rightarrow \mathbb{R}$ is a continuous function for which there exists a fixed $\varepsilon$ such that $|f(x + y) - f(x) - f(y)| \leq \varepsilon$ for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive function $T : \mathbb{Q}_p \rightarrow \mathbb{R}$ such that $|f(x) - T(x)| \leq \varepsilon$ for all $x \in \mathbb{Q}_p$. For more details about the results concerning such problems, the reader is referred to [24–45].

Recently, Khodaei and Rassias [46] introduced the generalized additive functional equation

$$
\sum_{k=2}^{n} \left( \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_n-k+1=1}^{n} \right) f \left( \sum_{i=1}^{n} a_{i_1} x_{i_1} - \sum_{r=1}^{n-k+1} a_{i_r} x_{i_r} \right) + f \left( \sum_{i=1}^{n} a_i x_i \right) = 2^{n-1} a_1 f(x_1)
$$

(1.1)

and proved the generalized Hyers-Ulam stability of the above functional equation. The functional equation

$$
f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2)
$$

(1.2)

is related to symmetric biadditive function and is called a quadratic functional equation [47, 48]. Every solution of the quadratic equation (1.2) is said to be a quadratic function.

Now, we introduce the generalized quadratic functional equation in $n$-variables as follows:

$$
\sum_{k=2}^{n} \left( \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_n-k+1=1}^{n} \right) f \left( \sum_{i=1}^{n} x_{i_1} - \sum_{r=1}^{n-k+1} x_{i_r} \right) + f \left( \sum_{i=1}^{n} x_i \right) = 2^{n-1} \sum_{i=1}^{n} f(x_i)
$$

(1.3)

where $n \geq 2$. Moreover, we investigate the generalized Hyers-Ulam stability of functional equation (1.3) over the $p$-adic field $\mathbb{Q}_p$. 

\[ \sum_{k=2}^{n} \left( \sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_n-k+1=1}^{n} \right) f \left( \sum_{i=1}^{n} x_{i_1} - \sum_{r=1}^{n-k+1} x_{i_r} \right) + f \left( \sum_{i=1}^{n} x_i \right) = 2^{n-1} \sum_{i=1}^{n} f(x_i) \]
As a special case, if \( n = 2 \) in (1.3), then we have the functional equation (1.2). Also, if \( n = 3 \) in (1.3), we obtain

\[
\sum_{i=2}^{2} \sum_{i=1+1}^{3} f \left( \sum_{i=1, i \neq i_1, i_2}^{3} x_j - \sum_{r=1}^{2} x_{i_2} \right) + \sum_{i=2}^{3} f \left( \sum_{i=1, i \neq i_1}^{3} x_i - x_{i_1} \right) + f \left( \sum_{i=1}^{3} x_i \right) = 2^2 \sum_{i=1}^{3} f(x_i),
\]

that is,

\[
f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) = 4f(x_1) + 4f(x_2) + 4f(x_3).
\]

2. Stability of Quadratic Functional Equation (1.3) over \( p \)-Adic Fields

We will use the following lemma.

**Lemma 2.1.** Let \( X \) and \( Y \) be real vector spaces. A function \( f : X \to Y \) satisfies the functional equation (1.3) if and only if the function \( f \) is quadratic.

**Proof.** Let \( f \) satisfy the functional equation (1.3). Setting \( x_i = 0 \) \( (i = 1, \ldots, n) \) in (1.3), we have

\[
\sum_{k=2}^{n} \left( \sum_{i=2}^{k} \sum_{i=1+1}^{k+1} \cdots \sum_{i=k-n+1}^{n} \right) f(0) + f(0) = 2^{n-1} \sum_{i=1}^{n} f(0),
\]

that is,

\[
\sum_{i=2}^{2} \sum_{i=1+1}^{3} f(0) + \sum_{i=2}^{3} \sum_{i=1+1}^{4} \sum_{i=2+1}^{n} f(0) + \cdots + \sum_{i=2}^{n} f(0) + f(0) = 2^{n-1} \sum_{i=1}^{n} f(0),
\]

or

\[
\left( \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1} + 1 \right) f(0) = 2^{n-1} \sum_{i=1}^{n} f(0),
\]

but \( 1 + \sum_{j=1}^{n-j} \binom{n-j}{j} = \sum_{j=0}^{n-j} \binom{n-j}{j} = 2^{n-j} \), and also \( n > j \geq 1 \) so \( 2^{n-1}(n-1)f(0) = 0 \).

Putting \( x_i = 0 \) \( (i = 2, \ldots, n-1) \) in (1.3) and then using \( f(0) = 0 \), we get

\[
f(x_1 - x_n) + \left( \binom{n-2}{1} f(x_1 - x_n) + \binom{n-2}{2} f(x_1 + x_n) \right) + \cdots + \left( \binom{n-2}{n-3} f(x_1 - x_n) + \binom{n-2}{n-2} f(x_1 + x_n) \right)
\]
that it holds on the case where \( n \) is even.

Hence, we have

\[
\begin{align*}
2n^{-1}f(x_1) + 2n^{-1}f(x_n),
\end{align*}
\]

that is,

\[
\left(1 + \sum_{j=1}^{n-2} \binom{n-2}{j}\right) (f(x_1 + x_n) + f(x_1 - x_n)) = 2n^{-1}f(x_1) + 2n^{-1}f(x_n),
\]

(2.5)

for all \( x_1, x_n \in X \), this shows that \( f \) satisfies the functional equation (1.2). So the function \( f \) is quadratic.

Conversely, suppose that \( f \) is quadratic, thus \( f \) satisfies the functional equation (1.2). Hence, we have \( f(0) = 0 \) and \( f \) is even.

We are going to prove our assumption by induction on \( n \geq 2 \). It holds on \( n = 2 \). Assume that it holds on the case where \( n = t \); that is, we have

\[
\sum_{k=2}^{t} \left( \sum_{i=2}^{k} \sum_{j=1}^{k-1} \cdots \sum_{l_{k-1}=i+1}^{k-1} \right) f\left( x_i - \sum_{r=1}^{t_i-1} x_i - \sum_{r=1}^{t} x_i \right) + f\left( \sum_{i=1}^{t} x_i \right) = 2^{t-1} \sum_{i=1}^{t} f(x_i)
\]

(2.6)

for all \( x_1, \ldots, x_t \in X \). It follows from (1.2) that

\[
f\left( \sum_{i=1}^{t} x_i + x_{i+1} \right) + f\left( \sum_{i=1}^{t} x_i - x_{i+1} \right) = 2f\left( \sum_{i=1}^{t} x_i \right) + 2f(x_{i+1})
\]

(2.7)

for all \( x_1, \ldots, x_{t+1} \in X \). Replacing \( x_i \) by \(-x_i\) in (2.7), we obtain

\[
f\left( \sum_{i=1}^{t-1} x_i - x_i + x_{i+1} \right) + f\left( \sum_{i=1}^{t-1} x_i - x_i - x_{i+1} \right) = 2f\left( \sum_{i=1}^{t-1} x_i - x_i \right) + 2f(x_{i+1})
\]

(2.8)

for all \( x_1, \ldots, x_{t+1} \in X \). Adding (2.7) to (2.8), we have

\[
f\left( \sum_{i=1}^{t-1} x_i - x_i - x_{i+1} \right) + f\left( \sum_{i=1}^{t-1} x_i - x_i + x_{i+1} \right) + f\left( \sum_{i=1}^{t-1} x_i + x_i - x_{i+1} \right) + f\left( \sum_{i=1}^{t-1} x_i + x_i + x_{i+1} \right)
\]

\[
= 2 \left[ f\left( \sum_{i=1}^{t-1} x_i - x_i \right) + f\left( \sum_{i=1}^{t-1} x_i + x_i \right) \right] + 4f(x_{i+1})
\]

(2.9)
for all $x_1, \ldots, x_{t+1} \in X$. Replacing $x_t-1$ by $-x_{t-1}$ in (2.9), we get

$$f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t - x_{t+1}\right)$$

$$+ f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t + x_{t+1}\right) = 2\left[f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t\right)\right] + 4f(x_{t+1})$$

(2.10)

for all $x_1, \ldots, x_{t+1} \in X$. Adding (2.9) to (2.10), one gets

$$f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t - x_{t+1}\right)$$

$$+ f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} + x_t - x_{t+1}\right)$$

$$+ f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} + x_t + x_{t+1}\right) = 2\left[f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t\right) + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t\right)\right]$$

$$+ f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} + x_t\right) + 8f(x_{t+1})$$

(2.11)

for all $x_1, \ldots, x_{t+1} \in X$. By using the above method, for $x_{t-2}$ until $x_2$, we infer that

$$\sum_{k=2}^{t+1} \left(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{k-2}=i_{k-1}+1}^{k+1} \right) f\left(\sum_{i=1}^{t+1} x_i - \sum_{r=1}^{t-k+2} x_{i_r}\right) + f\left(\sum_{i=1}^{t} x_i\right)$$

$$= 2\left[\sum_{k=2}^{t} \left(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{k-2}=i_{k-1}+1}^{k+1} \right) f\left(\sum_{i=1}^{t} x_i - \sum_{r=1}^{t-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{t} x_i\right)\right] + 2^k f(x_{t+1})$$

(2.12)

for all $x_1, \ldots, x_{t+1} \in X$. Now, by the case $n = t$, we lead to

$$\sum_{k=2}^{t+1} \left(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{k-2}=i_{k-1}+1}^{k+1} \right) f\left(\sum_{i=1}^{t+1} x_i - \sum_{r=1}^{t-2} x_{i_r}\right) + f\left(\sum_{i=1}^{t+1} x_i\right)$$

$$= 2^{t-1} \sum_{i=1}^{t} f(x_i) + 2^t f(x_{t+1})$$

(2.13)

for all $x_1, \ldots, x_{t+1} \in X$, so (1.3) holds for $n = t + 1$. This completes the proof of the lemma. □
Corollary 2.2. A function \( f : X \to Y \) satisfies the functional equation (1.3) if and only if there exists a symmetric biadditive function \( B_1 : X \times X \to Y \) such that \( f(x) = B_1(x, x) \) for all \( x \in X \).

Now, we investigate the stability of the functional equation (1.3) from a Banach space \( B \) into \( p \)-adic field \( \mathbb{Q}_p \). For convenience, we define the difference operator \( D_f \) for a given function \( f \):

\[
D_f(x_1, \ldots, x_n) = \sum_{k=2}^{n} \left( \sum_{i_1=1}^{k-1} \cdots \sum_{i_{n-k+1}=1}^{k} \right) f\left( \sum_{i=1}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r} \right) + f\left( \sum_{i=1}^{n} x_i \right) - 2^{n-1} \sum_{i=1}^{n} f(x_i). \tag{2.14}
\]

Theorem 2.3. Let \( B \) be a Banach space and let \( \varepsilon > 0, \lambda \) be real numbers. Suppose that a function \( f : \mathbb{Q}_p \to B \) with \( f(0) = 0 \) satisfies the inequality

\[
\|D_f(x_1, \ldots, x_n)\| \leq \varepsilon \sum_{i=1}^{n} |x_i|_p^\lambda \tag{2.15}
\]

for all \( x_1, \ldots, x_n \in \mathbb{Q}_p \). Then there exists a unique quadratic function \( Q : \mathbb{Q}_p \to B \) such that

\[
\|f(x) - Q(x)\| \leq \begin{cases} 
\varepsilon & p = 2, \lambda > -2; \\
\frac{\varepsilon}{32^{n-1} |x|^\lambda_2} & p > 2; \\
\frac{\varepsilon}{2^{n-1} |x|^\lambda_2} & 1 \leq p < 2.
\end{cases} \tag{2.16}
\]

for all nonzero \( x \in \mathbb{Q}_p \).

Proof. Letting \( x_1 = x_2 = x \neq 0 \) and \( x_i = 0 \ (i = 3, \ldots, n) \) in (2.15), we obtain

\[
\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{\varepsilon}{2^{n-1} |x|^\lambda_2} \tag{2.17}
\]

for all \( x \in \mathbb{Q}_p \). Hence,

\[
\left\| \frac{1}{2^m} f(2^mx) - \frac{1}{2^{m+l}} f(2^{m+l}x) \right\| \leq \frac{\varepsilon}{2^{n-1} \sum_{j=1}^{m} |x|^\lambda_2} \sum_{j=1}^{m} |x|^\lambda_2 \tag{2.18}
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and for all \( x \in \mathbb{Q}_p \). It follows from (2.18) that the sequence \( \{(1/2^m)f(2^mx)\} \) is a Cauchy sequence for all \( x \in \mathbb{Q}_p \). Since \( B \) is complete, the sequence \( \{(1/2^m)f(2^mx)\} \) converges. Therefore, one can define the function \( Q : \mathbb{Q}_p \to B \) by

\[
Q(x) := \lim_{m \to \infty} \frac{1}{2^m} f(2^mx) \tag{2.19}
\]
for all \( x \in \mathbb{Q}_p \). It follows from (2.15) and (2.19) that

\[
\|D_Q(x_1, \ldots, x_n)\| = \lim_{m \to \infty} \frac{1}{2^m} \|D_f(2^m x_1, \ldots, 2^m x_n)\| \leq \lim_{m \to \infty} \frac{|2|^{l_m}}{2^{m+1}} \sum_{i=1}^{n} \epsilon |x_i|_p = 0 \quad (2.20)
\]

for all \( x_1, \ldots, x_n \in \mathbb{Q}_p \). So \( D_Q(x_1, \ldots, x_n) = 0 \). By Lemma 2.1, the function \( Q : \mathbb{Q}_p \to B \) is quadratic.

Taking the limit \( m \to \infty \) in (2.18) with \( l = 0 \), we find that the function \( Q \) is quadratic function satisfying the inequality (2.16) near the approximate function \( f : \mathbb{Q}_p \to B \) of (1.3).

To prove the aforementioned uniqueness, we assume now that there is another additive function \( Q' : \mathbb{Q}_p \to B \) which satisfies (1.3) and the inequality (2.16). So

\[
\|Q(x) - Q'(x)\| = \frac{1}{2^m} \|Q(2^m x) - Q'(2^m x)\|
\]

\[
\leq \frac{1}{2^m} \left( \|Q(2^m x) - f(2^m x)\| + \|f(2^m x) - Q'(2^m x)\| \right)
\]

\[
\leq \begin{cases} 
\frac{\epsilon}{2^{m+1}} |x|_p & p = 2, \lambda > -2; \\
\frac{\epsilon}{3.2^{2m+n-4}} |x|_p & p > 2;
\end{cases}
\]

which tends to zero as \( m \to \infty \) for all nonzero \( x \in \mathbb{Q}_p \). This proves the uniqueness of \( Q \), completing the proof of uniqueness.

The following example shows that the above result is not valid over \( p \)-adic fields.

**Example 2.4.** Let \( p > 2 \) be a prime number and define \( f : \mathbb{Q}_p \to \mathbb{Q}_p \) by \( f(x) = x^2 - 2x \). Since \(|2^n|_p = 1\),

\[
|D_f(x_1, \ldots, x_n)|_p = \left| \sum_{i=2}^{n} x_i \right|_p = \left| \sum_{i=2}^{n} x_i \right|_p = \left| \sum_{i=1}^{n} x_i \right|_p \quad (2.22)
\]

for all \( x_1, \ldots, x_n \in \mathbb{Q}_p \). Hence, the conditions of Theorem 2.3 for \( \epsilon = 1 \) and \( \lambda = 1 \) hold. However for each \( n \in \mathbb{N} \), we have

\[
\left| \frac{1}{2^{2(m+1)}} f(2^{m+1} x) - \frac{1}{2^m} f(2^m x) \right|_p = \left| \frac{|x|_p}{2^{m+1}} = |x|_p \right|
\]

for all \( x \in \mathbb{Q}_p \). Hence \( \{(1/2^m) f(2^m x)\} \) is not convergent for all nonzero \( x \in \mathbb{Q}_p \).

In the next result, which can be compared with Theorem 2.3, we will show that the stability of the functional equation (1.3) in non-Archimedean spaces over \( p \)-adic fields.
Theorem 2.5. Let \( \ell \in \{-1, 1\} \) be fixed. Let \( U \) be a non-Archimedean space and \( W \) be a complete non-Archimedean space over \( \mathbb{Q}_p \), where \( p > 2 \) is a prime number. Suppose that a function \( f : U \to W \) satisfies the inequality

\[
\| Df(x_1, \ldots, x_n) \|_W \leq \begin{cases} 
\varepsilon \sum_{i=1}^{n} \| x_i \|_U^{1/2}, & \lambda \ell > 2 \ell; \\
\varepsilon \sum_{i=2}^{n} \| x_1 \|_U^{1/2} \| x_i \|_U^{1/4}, & (\lambda_1 + \lambda_i) \ell > 2 \ell; \\
\varepsilon \max \left\{ \| x_i \|_U^{1/4}, 1 \leq i \leq n \right\}, & \lambda \ell > 2 \ell;
\end{cases}
\]

(2.24)

for all \( x_1, \ldots, x_n \in U \), where \( \varepsilon, \lambda_1, \ldots, \lambda_n \) and \( \lambda \) are nonnegative real numbers. Then, the limit

\[
Q(x) := \lim_{m \to \infty} \frac{1}{p^{2 \ell m}} f \left( p^{2 \ell m} x \right)
\]

exists for all \( x \in U \) and \( Q : U \to W \) is a unique quadratic function satisfying

\[
\| f(x) - Q(x) \|_W \leq \begin{cases} 
2p^{1+\ell+(1-\ell)\lambda/2} \varepsilon \| x \|_U^{1/4}, & \\
p^{1+\ell+(1-\ell)\lambda/2} \varepsilon \| x \|_U^{1/4}, & \lambda \ell > 2 \ell.
\end{cases}
\]

(2.26)

for all \( x \in U \).

Proof. By (2.24),

\[
\| Df(x_1, \ldots, x_n) \|_W \leq \varepsilon \sum_{i=1}^{n} \| x_i \|_U^{1/2}
\]

(2.27)

for all \( x_1, \ldots, x_n \in U \), where \( \lambda \ell > 2 \ell \). Putting \( x_i = 0 \) (\( i = 1, \ldots, n \)) in (2.27) to obtain \( f(0) = 0 \), setting \( x_i = 0 \) (\( i = 3, \ldots, n \)) in (2.27), we obtain

\[
\left\| 2^{n-2} f(x_1 + x_2) + 2^{n-2} f(x_1 - x_2) - 2^{n-1} f(x_1) - 2^{n-1} f(x_2) \right\|_W \leq \varepsilon \left( \| x_1 \|_U + \| x_2 \|_U^{1/2} \right)
\]

(2.28)

for all \( x_1, x_2 \in U \). So

\[
\| f(x_1 + x_2) + f(x_1 - x_2) - 2 f(x_1) - 2 f(x_2) \|_W \leq \varepsilon \left( \| x_1 \|_U + \| x_2 \|_U^{1/2} \right)
\]

(2.29)

for all \( x_1, x_2 \in U \). Letting \( x_1 = x_2 = x \) in (2.29), we have

\[
\| f(2x) - 4 f(x) \|_W \leq 2 \varepsilon \| x \|_U^{1/2}
\]

(2.30)
for all \( x \in U \). By induction on \( j \), we will show that for each \( j \geq 2 \),

\[
\left\| f(jx) - j^2 f(x) \right\| \leq 2\epsilon \|x\|_U^4
\]  

(2.31)

for all \( x \in U \). It holds on \( j = 2 \); see (2.30). Let (2.31) hold for \( j = 2, \ldots, k \). Replacing \( x_1 \) and \( x_2 \) by \( kx \) and \( x \) in (2.29), respectively, we get

\[
\left\| f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x) \right\| \leq \epsilon \left( 1 + |k|_p^4 \right) \|x\|_U^4
\]  

(2.32)

for all \( x \in U \). It follows from (2.32) and our induction hypothesis that

\[
\left\| f((k+1)x) - (k+1)^2 f(x) \right\| = \left\| f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x) \right. \\
- f((k-1)x) + (k-1)^2 f(x) - 2f(kx) - 2f(x) \left\| \right\|_U^4
\]

\[
\leq \max \left\{ 2\epsilon \|x\|_U^4, \epsilon \left( 1 + |k|_p^4 \right) \|x\|_U^4 \right\} = 2\epsilon \|x\|_U^4
\]  

(2.33)

for all \( x \in U \). This proves (2.31) for each \( j \geq 2 \). In particular,

\[
\left\| f(px) - p^2 f(x) \right\| \leq 2\epsilon \|x\|_U^4
\]  

(2.34)

for all \( x \in U \). So

\[
\left\| f(x) - \frac{1}{p^2} f(px) \right\| \leq 2p^2 \epsilon \|x\|_U^4,
\]

\[
\left\| f(x) - p^2 f\left( \frac{x}{p} \right) \right\| \leq 2p^3 \epsilon \|x\|_U^4
\]  

(2.35)

for all \( x \in U \). Hence,

\[
\left\| \frac{1}{p^{2\ell}} f\left( p^{\ell} x \right) - \frac{1}{p^{2\ell(j+1)}} f\left( p^{\ell(j+1)} x \right) \right\| \leq \frac{2p^{2\ell(j+1)-1} \epsilon}{p^{2\ell}} \|x\|_U^4
\]  

(2.36)

for all \( x \in U \). Since the right side of the above inequality tends to zero as \( j \to \infty \), \( \left\{ (1/p^{2\ell m}) f\left( p^{\ell m} x \right) \right\} \) is a Cauchy sequence in complete non-Archimedean space \( \mathcal{K} \), thus it
converges to some function $Q(x) = \lim_{m \to \infty} (1/p^{2^m}) f(p^{\ell m} x)$ for all $x \in \mathcal{U}$. Using (2.35) and

\[
\left\| f(x) - \frac{1}{p^{2^m}} f(p^{\ell m} x) \right\|_{\mathcal{W}} = \left\| \sum_{j=0}^{m-1} \frac{1}{p^{2^j}} f(p^{\ell j} x) - \frac{1}{p^{2^j}} f(p^{\ell (j+1)} x) \right\|_{\mathcal{W}} \\
\leq \max \left\{ \left\| \frac{1}{p^{2^j}} f(p^{\ell j} x) - \frac{1}{p^{2^j}} f(p^{\ell (j+1)} x) \right\|_{\mathcal{W}} ; 0 \leq j < m \right\} \\
\leq \max \left\{ 2p^{1+\ell+(1-\ell)/2+\ell}(2-\ell)\epsilon \|x\|_{\mathcal{V}}^1 ; 0 \leq j < m \right\}
\]

(2.37)

for all $x \in \mathcal{U}$. Letting $m \to \infty$ in this inequality, we see that

\[
\left\| f(x) - Q(x) \right\|_{\mathcal{W}} \leq 2p^{1+\ell+(1-\ell)/2} \epsilon \|x\|_{\mathcal{V}}^1
\]

(2.38)

for all $x \in \mathcal{U}$. Moreover,

\[
\left\| D_Q(x_1, \ldots, x_n) \right\|_{\mathcal{W}} = \lim_{m \to \infty} \left\| \frac{1}{p^{2^m}} D_f(p^{\ell m} x_1, \ldots, p^{\ell m} x_n) \right\|_{\mathcal{W}} \leq \lim_{m \to \infty} \frac{p^{2^m}}{p^{3\ell m}} \sum_{i=1}^{n} \|x_i\|_{\mathcal{V}}^1 = 0
\]

(2.39)

for all $x_1, \ldots, x_n \in \mathcal{U}$. So $D_Q(x_1, \ldots, x_n) = 0$. By Lemma 2.1, the function $Q : \mathcal{U} \to \mathcal{W}$ is quadratic.

Now, let $Q' : \mathcal{U} \to \mathcal{W}$ be another quadratic function satisfying (1.3) and (2.38). So

\[
\left\| Q(x) - Q'(x) \right\|_{\mathcal{W}} \leq p^{2^m} \max \left\{ \left\| Q(p^{\ell m} x) - f(p^{\ell m} x) \right\|_{\mathcal{W}} , \left\| f(p^{\ell m} x) - Q'(p^{\ell m} x) \right\|_{\mathcal{W}} \right\}
\]

\[
\leq \frac{2p^{2^m+(1-\ell)/2+1+\ell}}{p^{3\ell m}} \epsilon \|x\|_{\mathcal{V}}^1
\]

which tends to zero as $m \to \infty$ for all $x \in \mathcal{U}$. This proves the uniqueness of $Q$.

The rest of the proof is similar to the above proof, hence it is omitted. □

**Acknowledgments**

The third author of this work was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant number: 2011-0005197).

**References**

Abstract and Applied Analysis


