Research Article

Explicit Formulas Involving $q$-Euler Numbers and Polynomials

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We deal with $q$-Euler numbers and $q$-Bernoulli numbers. We derive some interesting relations for $q$-Euler numbers and polynomials by using their generating function and derivative operator. Also, we derive relations between the $q$-Euler numbers and $q$-Bernoulli numbers via the $p$-adic $q$-integral in the $p$-adic integer ring.

1. Preliminaries

Imagine that $p$ is a fixed odd prime number. Throughout this paper we use the following notations, where $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The $p$-adic absolute value is defined by

$$|p|_p = \frac{1}{p}. \tag{1.1}$$

In this paper, we will assume that $|q - 1|_p < 1$ as an indeterminate. $[x]_q$ is a $q$-extension of $x$, which is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}. \tag{1.2}$$

We note that $\lim_{q \to 1} [x]_q = x$ (see [1–12]).
We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \), if the difference quotient
\[
F_f(x, y) = \frac{f(x) - f(y)}{x - y},
\]
has a limit \( f'(a) \) as \( (x, y) \to (a, a) \) and denote this by \( f \in \text{UD}(\mathbb{Z}_p) \).

Let \( \text{UD}(\mathbb{Z}_p) \) be the set of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in \text{UD}(\mathbb{Z}_p) \), let us start with the expression
\[
\frac{1}{[p^N]} \sum_{0 \leq \xi < p^N} f(\xi) q^\xi = \sum_{0 \leq \xi < p^N} f(\xi) \mu_q(\xi + p^N \mathbb{Z}_p),
\]
which represents \( p \)-adic \( q \)-analogue of Riemann sums for \( f \). The integral of \( f \) on \( \mathbb{Z}_p \) will be defined as the limit \( \lim_{N \to \infty} \) of these sums, when it exists. The \( p \)-adic \( q \)-integral of function \( f \in \text{UD}(\mathbb{Z}_p) \) is defined by Kim
\[
I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_q(\xi) = \lim_{N \to \infty} \frac{1}{[p^N]} \sum_{\xi = 0}^{p^N - 1} f(\xi) q^\xi.
\]

The bosonic integral is considered as a bosonic limit \( q \to 1 \), \( I_1(f) = \lim_{q \to 1} I_q(f) \). Similarly, the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) is introduced by Kim as follows:
\[
I_{-q}(f) = \lim_{q \to -q} I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_{-q}(\xi)
\]
(for more details, see [9–12]).

In [6], the \( q \)-Euler polynomials with weight 0 are introduced as
\[
\tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x + y)^n d\mu_{-q}(y).
\]

From (1.7), we have
\[
\tilde{E}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^l \tilde{E}_{n-l,q},
\]
where \( \tilde{E}_{n,q}(0) = \tilde{E}_{n,q} \) are called \( q \)-Euler numbers with weight 0. Then, \( q \)-Euler numbers are defined as
\[
q(\tilde{E}_q + 1)^n + \tilde{E}_{n,q} = \begin{cases} [2]^n_q & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}
\]
where the usual convention about replacing \( (\tilde{E}_q)^n \) by \( \tilde{E}_{n,q} \) is used.
Similarly, the \( q \)-Bernoulli polynomials and numbers with weight 0 are defined, respectively, as
\[
\tilde{B}_{n,q}(x) = \lim_{n \to \infty} \frac{1}{[p^n]_q} \sum_{y=0}^{p^n-1} (x + y)^n q^y
\]
\[
= \int_{\mathbb{Z}_p} (x + y)^n d\mu_q(y), \tag{1.10}
\]
\[
\tilde{B}_{n,q} = \int_{\mathbb{Z}_p} y^n d\mu_q(y)
\]
(for more information, see [4]).

We, by using the Kim et al. method in [2], will investigate some interesting identities on the \( q \)-Euler numbers and polynomials arising from their generating function and derivative operator. Consequently, we derive some properties on the \( q \)-Euler numbers and polynomials and \( q \)-Bernoulli numbers and polynomials by using \( q \)-Volkenborn integral and fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \).

2. On the \( q \)-Euler Numbers and Polynomials

Let us consider Kim’s \( q \)-Euler polynomials as follows:
\[
F^q_x = F^q_x(t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!}. \tag{2.1}
\]

Here \( x \) is a fixed parameter. Thus, by expression of (2.1), we can readily see the following:
\[
qe^t F^q_x + F^q_x = [2]_q e^{xt}. \tag{2.2}
\]

Last from equality, taking derivative operator \( D \) as \( D = d/dt \) on the both sides of (2.2). Then, we easily see that
\[
qe^t (D + I)^k F^q_x + D^k F^q_x = [2]_q x^k e^{xt}, \tag{2.3}
\]
where \( k \in \mathbb{N}^{*} \) and \( I \) is identity operator. By multiplying \( e^{-t} \) on both sides of (2.3), we get
\[
q(D + I)^k F^q_x + e^{-t} D^k F^q_x = [2]_q x^k e^{(x-1)t}. \tag{2.4}
\]

Let us take derivative operator \( D^m (m \in \mathbb{N}) \) on both sides of (2.4). Then we get
\[
qe^t D^m (D + I)^k F^q_x + D^m (D - I)^m F^q_x = [2]_q x^k (x - 1)^m e^{xt}. \tag{2.5}
\]
Let $G[0]$ (not $G(0)$) be the constant term in a Laurent series of $G(t)$. Then, from (2.5), we get
\[
\sum_{j=0}^{k} \binom{k}{j} (q e^{t} D^{k+m-j} F^{q}(t)) [0] + \sum_{j=0}^{m} \binom{m}{j} (-1)^{j} (D^{k+m-j} F^{q}(t)) [0] = [2]_{q} x^{k} (x - 1)^{m}. \tag{2.6}
\]

By (2.1), we see
\[
(D^{N} F^{q}(t)) [0] = \tilde{E}_{N,q}(x), \quad (e^{t} D^{N} F^{q}(t)) [0] = \tilde{E}_{N,q}(x). \tag{2.7}
\]

By expressions of (2.6) and (2.7), we see that
\[
\max\{k,m\} \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^{j} \binom{m}{j} \right] \tilde{E}_{k+m-j,q}(x) = [2]_{q} x^{k} (x - 1)^{m}. \tag{2.8}
\]

From (2.1), we note that
\[
\frac{d}{dx} \tilde{E}_{n,q}(x) = n \sum_{l=0}^{n-1} \binom{n-1}{l} \tilde{E}_{l,q} x^{n-1-l} = n \tilde{E}_{n-1,q}(x). \tag{2.9}
\]

By (2.9), we easily see
\[
\int_{0}^{1} \tilde{E}_{n,q}(x) dx = \frac{\tilde{E}_{n+1,q}(1) - \tilde{E}_{n+1,q}}{n+1} = - \frac{[2]_{q} (-1)^{n+1}}{n+1} \tilde{E}_{n+1,q}. \tag{2.10}
\]

Now, let us consider definition of integral from 0 to 1 in (2.8), then we have
\[
- \frac{[2]_{q} (-1)^{m} B(k+1,m+1)}{[2]_{q} (-1)^{m}} = [2]_{q} (-1)^{m} B(k+1,m+1)
\]
\[
= [2]_{q} (-1)^{m} \frac{\Gamma(k+1)\Gamma(m+1)}{\Gamma(k+m+2)}, \tag{2.11}
\]

where $B(m, n)$ is beta function which is defined by
\[
B(m, n) = \int_{0}^{1} x^{m-1} (1 - x)^{n-1} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m > 0, \ n > 0. \tag{2.12}
\]

As a result, we obtain the following theorem.
Theorem 2.1. For \( n \in \mathbb{N} \), one has

\[
\sum_{j=1}^{\max\{k,m\}} \left[ q\left(\begin{array}{c} k \\ j \end{array}\right) + (-1)^j \left(\begin{array}{c} m \\ j \end{array}\right) \right] \frac{\tilde{E}_{k+m-j+1,q}}{k + m - j + 1} = q \frac{(-1)^{m+1}}{(k + m + 1) (k + m)} - [2]_q \frac{\tilde{E}_{k+m+1,q}}{k + m + 1}.
\]  

(2.13)

Substituting \( m = k + 1 \) into Theorem 2.1, we readily get

\[
\sum_{j=1}^{k+1} \left[ q\left(\begin{array}{c} k \\ j \end{array}\right) + (-1)^j \left(\begin{array}{c} k + 1 \\ j \end{array}\right) \right] \frac{\tilde{E}_{2k+2-j,q}}{2k + 2 - j} = q \frac{(-1)^k}{(2k + 2) (2^{k+1})} - [2]_q \frac{\tilde{E}_{2k+2,q}}{2k + 2}.
\]  

(2.14)

By (2.1), it follows that

\[
\sum_{j=0}^{\max\{k,m\}} (k + m - j) \left[ q\left(\begin{array}{c} k \\ j \end{array}\right) + (-1)^j \left(\begin{array}{c} m \\ j \end{array}\right) \right] \tilde{E}_{k+m-j-1,q}(x) = [2]_q x^{k-1} (x - 1)^{m-1} ((k + m)x - k).
\]  

(2.15)

Let \( m = k \) in (2.1), we see that

\[
\sum_{j=0}^{k} \left[ q\left(\begin{array}{c} k \\ j \end{array}\right) + (-1)^j \left(\begin{array}{c} k \\ j \end{array}\right) \right] \tilde{E}_{2k-j,q}(x) = [2]_q x^k (x - 1)^k.
\]  

(2.16)

Last from equality, we discover the following:

\[
[2]_q \sum_{j=0}^{[k/2]} \left(\begin{array}{c} k \\ 2j \end{array}\right) \tilde{E}_{2k-2j,q}(x) + (q - 1) \sum_{j=0}^{[k/2]} \left(\begin{array}{c} k \\ 2j + 1 \end{array}\right) \tilde{E}_{2k-2j-1,q}(x) = [2]_q x^k (x - 1)^k.
\]  

(2.17)

Here \([\cdot]\) is Gauss’ symbol. Then, taking integral from 0 to 1 in both sides of last equality, we get

\[
- [2]_q \prod_{j=0}^{[k/2]} \left(\begin{array}{c} k \\ 2j \end{array}\right) \frac{\tilde{E}_{2k-2j+1,q}}{2k - 2j + 1} + [2]_q \prod_{j=0}^{[k/2]} \left(\begin{array}{c} k \\ 2j + 1 \end{array}\right) \frac{\tilde{E}_{2k-2j,q}}{2k - 2j}
\]

\[
= [2]_q (-1)^k B(k + 1, k + 1) \]  

(2.18)

\[
= \frac{[2]_q (-1)^k}{(2k + 1) \left(\frac{2^k}{k}\right)}.
\]

Consequently, we derive the following theorem.
Theorem 2.2. The following identity

\[
[2]^q \sum_{j=0}^{[k/2]} \binom{k}{2j} \frac{\tilde{E}_{2k-2j+1,q}}{2k-2j+1} + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \frac{\tilde{E}_{2k-2j,q}}{2k-2j} = \frac{q(-1)^{k+1}}{(2k+1)(2k)}
\]

(2.19)

is true.

In view of (2.1) and (2.17), we discover the following applications:

\[
\begin{align*}
&= \sum_{j=0}^{k+1} \left[ q\binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \tilde{E}_{2k+1-j,q}(x) \\
&= [2]^q \tilde{E}_{2k+1,q}(x) + \sum_{j=1}^{[k+1]/2} \left[ q\binom{k}{2j} + \binom{k}{2j+1} + \binom{k}{2j} \right] \tilde{E}_{2k-2j,q}(x) \\
&\quad + \sum_{j=0}^{[k+1]/2} \left[ q\binom{k}{2j+1} - \binom{k}{2j+1} - \binom{k}{2j} \right] \tilde{E}_{2k-2j,q}(x) \\
&= \sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j+1}(x) \\
&\quad + [2]^q \sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \tilde{E}_{2k+1-2j,q}(x) \\
&\quad + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j+1}(x).
\end{align*}
\]

(2.20)

By expressions (2.17) and (2.20), we have the following theorem.

Theorem 2.3. For \( k \in \mathbb{N} \), one has

\[
[2]^q \sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \tilde{E}_{2k+1-2j,q}(x) \\
+ (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j,q}(x) + \frac{1}{1+q} \tilde{E}_{2k-2j+1}(x)
\]

(2.21)

\[= x^k(x-1)^k \left[ [2]^q x - q \right].\]
3. *p*-adic Integral on $\mathbb{Z}_p$ Associated with Kim’s $q$-Euler Polynomials

In this section, we consider Kim’s $q$-Euler polynomials by means of $p$-adic $q$-integral on $\mathbb{Z}_p$. Now we start with the following assertion.

Let $m, k \in \mathbb{N}$. Then by (2.8),

$$I_1 = [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_q(x)$$

$$= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_q(x)$$

$$= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}. \quad (3.1)$$

On the other hand, in right hand side of (2.8),

$$I_2 = \max_{k,m} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x)$$

$$= \max_{k,m} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q}. \quad (3.2)$$

Equating $I_1$ and $I_2$, we get the following theorem.

**Theorem 3.1.** For $m, k \in \mathbb{N}$, one has

$$\sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q}$$

$$= [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}. \quad (3.3)$$

Let us take fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ in left hand side of (2.21), we get

$$I_3 = \int_{\mathbb{Z}_p} x^k (x-1)^k \left[ [2]_q x - q \right] d\mu_q(x)$$

$$= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{l} d\mu_q(x)$$

$$= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l,q} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l,q} \quad (3.4)$$
In other words, we consider right hand side of (2.21) as follows:

\[ I_4 = [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \bar{E}_{2k+1-2j-l,q} \int_{Z_p} x^l d\mu_{-q}(x) \]
\[ + \frac{[k/2]}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \bar{E}_{2k+1-2j-l,q} \int_{Z_p} x^l d\mu_{-q}(x) \]
\[ + \frac{[k/2]}{2j+1} \left[ \binom{q-1}{2k-2j} \bar{E}_{2k-2j-1,q} \int_{Z_p} x^l d\mu_{-q}(x) \right] \]
\[ = [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \bar{E}_{2k+1-2j-l,q} \bar{E}_{l,q} \]
\[ + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \binom{2k-2j+1}{l} \bar{E}_{2k+1-2j-l,q} \bar{E}_{l,q} \]
\[ + \frac{[k/2]}{2j+1} \left[ \binom{q-1}{2k-2j} \bar{E}_{2k-2j-1,q} \bar{E}_{l,q} \right]. \]

(3.5)

Equating \( I_3 \) and \( I_4 \), we get the following theorem.

**Theorem 3.2.** For \( k \in \mathbb{N} \), one has

\[
\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left\{ [2]_q \bar{E}_{k+1-q} - q \bar{E}_{k+1,q} \right\}
\[ = [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \bar{E}_{2k+1-2j-l,q} \bar{E}_{l,q} \]
\[ + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \bar{E}_{2k+1-2j-l,q} \bar{E}_{l,q} \]
\[ + \frac{[k/2]}{2j+1} \left\{ \binom{q-1}{2k-2j} \bar{E}_{2k-2j-1,q} \bar{E}_{l,q} \right\}. \]

(3.6)
Now, we consider (2.8) and (2.1) by means of \( q \)-Volkenborn integral. Then, by (2.8), we see

\[
[2]_q \int_{\mathbb{Z}_p} x^k(x-1)^m d\mu_q(x) = [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_q(x) = [2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{B}_{l+k,q}.
\]  

(3.7)

On the other hand,

\[
\sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x)
\]

\[
= \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{B}_{l,q}.
\]  

(3.8)

Therefore, we get the following theorem.

**Theorem 3.3.** For \( m, k \in \mathbb{N} \), one has

\[
[2]_q \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \tilde{B}_{l+k,q}
\]

\[
= \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{B}_{l,q}.
\]  

(3.9)

By using fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) in left hand side of (2.21), we get

\[
I_5 = [2]_q \int_{\mathbb{Z}_p} x^k(x-1)^k ([2] x - q) d\mu_q(x)
\]

\[
= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_q(x) - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x)
\]

\[
= [2]_q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l+1,q} - q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l,q}.
\]  

(3.10)
Also, we consider right hand side of (2.21) as follows:

\[
I_6 = [2]^k q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \left( \frac{2k - 2j + 1}{l} \right) E_{2k+1-2j-l,q} \int_{Z_p} x^l d\mu_q(x) \\
+ \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \left( \frac{2k - 2j + 1}{l} \right) \bar{E}_{2k+1-2j-l,q} \int_{Z_p} x^l d\mu_q(x) \\
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[ \left( q - 1 \right) \sum_{j=0}^{2k-2j} \left( \frac{2k - 2j}{l} \right) \bar{E}_{2k-2j-l,q} \int_{Z_p} x^l d\mu_q(x) \right] \\
+ \frac{q - 1}{1 + q} \sum_{j=0}^{2k-2j+1} \left( \frac{2k - 2j + 1}{l} \right) \bar{E}_{2k-2j-l+1,q} \int_{Z_p} x^l d\mu_q(x)
\]

Equating \( I_5 \) and \( I_6 \), we get the following corollary.

**Corollary 3.4.** For \( k \in \mathbb{N} \), one gets

\[
\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left\{ [2]^q \bar{B}_{k+l+1,q} - q\bar{B}_{k+l,q} \right\} \\
= [2]^k q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \left( \frac{2k - 2j + 1}{l} \right) E_{2k+1-2j-l,q} \bar{B}_{l,q} \\
+ \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \left( \frac{2k - 2j + 1}{l} \right) \bar{E}_{2k+1-2j-l,q} \bar{B}_{l,q} \\
+ \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \left( q - 1 \right) \sum_{j=0}^{2k-2j} \left( \frac{2k - 2j}{l} \right) \bar{E}_{2k-2j-l,q} \bar{B}_{l,q} \right\} \\
+ \frac{q - 1}{1 + q} \sum_{j=0}^{2k-2j+1} \left( \frac{2k - 2j + 1}{l} \right) \bar{E}_{2k-2j-l+1,q} \bar{B}_{l,q}.
\]
Abstract and Applied Analysis

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References


