Research Article

Weighted Approximation for Jackson-Matsuoka Polynomials on the Sphere

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We consider the best approximation by Jackson-Matsuoka polynomials in the weighted L_p space on the unit sphere of $\mathbb{R}^d$. Using the relation between K -functionals and modulus of smoothness on the sphere, we obtain the direct and inverse estimate of approximation by these polynomials for the h -spherical harmonics.

1. Introduction and Notations

Let $\mathbb{S} := \mathbb{S}^{d-1} = \{x : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^d$ ($d \geq 3$), $d \in \mathbb{N}$, where $\|x\|$ denotes the usual Euclidean norm, $\mathbb{R}$ the set of real numbers. For a nonzero vector $v \in \mathbb{R}^d$, let σ_v denote the reflection with respect to the hyperplane perpendicular to v , $x\sigma_v := x - 2(\langle x, v \rangle / \|v\|^2)v$, $x \in \mathbb{R}^d$, where $\langle x, v \rangle$ denote the usual Euclidean inner product. Let G be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system $\mathbb{R}_+$, normalized so that $\langle v, v \rangle = 2$ for all $v \in \mathbb{R}_+$. Then G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_v : v \in \mathbb{R}_+\}$. Let κ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on $\mathbb{R}_+$ with the property that $\kappa_u = \kappa_v$ whenever σ_u is conjugate to σ_v in G , then $v \mapsto \kappa_v$ is a G -invariant function. We consider the weighted best L_p approximation with respect to the measure $h_\kappa^2 d\omega$ on $\mathbb{S}$, where h_κ^2 is defined by

$$h_\kappa = \prod_{v \in \mathbb{R}_+} |\langle x, v \rangle|^{\kappa_v}, \quad x \in \mathbb{R}^d, \quad (1.1)$$

$d\omega$ is the surface (Lebesgue) measure on $\mathbb{S}$. The function h_κ is a positive homogeneous function of degree $\gamma_\kappa := \sum_{v \in \mathbb{R}_+} \kappa_v$, and it is invariant under the reflection group. We denote

by a_κ the normalization constant of h_κ , $a_\kappa^{-1} = \int_{\mathbb{S}} h_\kappa^2(y) d\omega(y)$ and denote by $L_p(h_\kappa^2)$, $1 \leq p \leq \infty$, the space of functions defined on $\mathbb{S}$ with the finite norm

$$\|f\|_{\kappa,p} := \left(a_\kappa \int_{\mathbb{S}} |f(y)|^p h_\kappa^2(y) d\omega(y) \right)^{1/p}, \quad 1 \leq p < \infty, \quad (1.2)$$

and for $p = \infty$ we assume that L_∞ is replaced by $C(\mathbb{S})$ the space of continuous functions on $\mathbb{S}$ with the usual uniform norm $\|f\|_\infty$.

Δ_h denote the h -Laplacian. $\Delta_{h,0}$ is the Laplace-Beltrami operator on the sphere. $\mathcal{P}_n^d$ denote the subspace of homogeneous polynomials of degree n in d variables. The h -harmonics are defined as the homogeneous polynomials satisfying the equation $\Delta_h P = 0$, $P \in \mathcal{P}_n^d$. Furthermore, let $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of h -spherical harmonics of degree n in d variables. The spherical h -harmonics are the restriction of h -harmonics on the unit sphere. It is well known that spherical h -harmonics are eigenfunctions of $\Delta_{h,0}$; that is,

$$\Delta_{h,0} Y(x) = -n(n+2\lambda)Y(x), \quad x \in \mathbb{S}, \quad Y \in \mathcal{H}_n^d(h_\kappa^2). \quad (1.3)$$

The standard Hilbert space theory shows that $L_2(h_\kappa^2) = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n^d(h_\kappa^2)$. That is, with each $f \in L_2(h_\kappa^2)$ we can associate its h -harmonic expansion

$$f(x) = \sum_{n=0}^{\infty} Y_n(h_\kappa^2; f, x), \quad x \in \mathbb{S}, \quad (1.4)$$

in $L_2(h_\kappa^2)$ norm. For the surface measure ($\kappa = 0$), such a series is called the Laplace series (see [1]). The orthogonal projection $Y_n(h_\kappa^2) : L_2(h_\kappa^2) \rightarrow \mathcal{H}_n^d(h_\kappa^2)$ takes the form

$$Y_n(h_\kappa^2; f, x) := \int_{\mathbb{S}} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y) d\omega(y), \quad (1.5)$$

where $P_n(h_\kappa^2; x, y)$ is the reproducing kernel of the space of h -harmonics $\mathcal{H}_n^d(h_\kappa^2)$, which is given by (see [2])

$$P_n(h_\kappa^2; x, y) = \frac{n+\lambda}{\lambda} V_\kappa \left[C_n^\lambda(\langle \cdot, y \rangle) \right] (x). \quad (1.6)$$

C_n^λ is the ultraspherical polynomial of degree n , $\lambda := \gamma_\kappa + (d-2)/2$, $\gamma_\kappa = \sum_{v \in \mathbb{R}_+} \kappa_v$, and the intertwining operator V_κ is a linear operator uniquely determined by

$$V_\kappa \mathcal{P}_n \subset \mathcal{P}_n, \quad V_\kappa 1 = 1, \quad \mathfrak{D}_i V_\kappa = V_\kappa \partial_i, \quad 1 \leq i \leq d. \quad (1.7)$$

The spherical means are denoted by

$$T_\theta(f) = \frac{1}{|\mathbb{S}^{d-2}|(\sin \theta)^{d-2}} \int_{\langle x, y \rangle = \cos \theta} f(y) d\omega(y), \quad (1.8)$$

where $|\mathbb{S}^{d-2}| = \int_{\mathbb{S}^{d-2}} d\omega = 2\pi^{(d-1)/2} / \Gamma((d-1)/2)$.

The spherical means associated with $h_\kappa^2 d\omega$, in which $T_\theta^\kappa(f)$ is defined by

$$c_\lambda \int_0^\pi T_\theta^\kappa(f, x) g(\cos \theta) (\sin \theta)^{2\lambda} d\theta = a_\kappa \int_{\mathbb{S}} f(y) V_\kappa g(\langle x, y \rangle) h_\kappa^2(y) d\omega(y), \quad (1.9)$$

where g is any function $[-1, 1] \mapsto \mathbb{R}$ such that the integral in the right-hand side is finite, $c_\lambda^{-1} = \int_{-1}^1 (1-t^2)^{\lambda-1/2} dt = \Gamma(\lambda+1/2)\sqrt{\pi}/\Gamma(\lambda+1)$. $T_\theta^\kappa(f)$ is a proper extension of $T_\theta(f)$, since $T_\theta(f)$ satisfies $T_\theta^\kappa(f)$ when $\kappa = 0$ and $V_\kappa = id$, and the properties of T_θ^κ are well known (see [2]). In particular, the function $T_\theta^\kappa f(x)$ has the expansion

$$T_\theta^\kappa(f) \sim \sum_{n=0}^{\infty} \frac{C_n^\lambda(\cos \theta)}{C_n^\lambda(1)} Y_n(h_\kappa^2; f) := \sum_{n=0}^{\infty} Q_n^\lambda(\cos \theta) Y_n(h_\kappa^2; f). \quad (1.10)$$

Simultaneously, they lead to the following definition of an analog of the modulus of smoothness.

Definition 1.1 (see [2]). For $f \in L_p(h_\kappa^2)$, $1 \leq p < \infty$, or $f \in C(\mathbb{S})$, the modulus of smoothness on the sphere is given by

$$\omega(f; t)_{\kappa, p} := \sup_{0 < \theta \leq t} \|f - T_\theta^\kappa(f)\|_{\kappa, p}. \quad (1.11)$$

The K -functional of the sphere is given by

$$K(f; t^2)_{\kappa, p} = \inf_{g \in W_p(h_\kappa^2)} \left\{ \|f - g\|_{\kappa, p} + t^2 \|\Delta_{h, 0} g\|_{\kappa, p} \right\}, \quad (1.12)$$

where $W_p(h_\kappa^2) := \{f : f \in L_p(h_\kappa^2), -k(k+2\lambda)P_k(h_\kappa^2; f) = P_k(h_\kappa^2; g) \text{ for some } g \in L_p(h_\kappa^2)\}$, $0 < t < t_0$, t_0 is a positive constant.

In [2], Xu proved the weak equivalence relation

$$C^{-1} \omega(f; t)_{\kappa, p} \leq K(f; t^2)_{\kappa, p} \leq C \omega(f; t)_{\kappa, p}. \quad (1.13)$$

Throughout this paper, C denotes a positive constant independent on n and f and $C(a)$ denotes a positive constant dependent on a , which may be different according to the circumstances.

Based on the classical Jackson-Matsuoka kernel (see [3]), we define a new kernel

$$M_{n;j,i,s}(\theta) := \frac{1}{\Omega_{n;j,i,s}} \left(\frac{\sin^{2j} n\theta/2}{\sin^{2i} \theta/2} \right)^{2s}, \quad n = 1, 2, \dots, \theta \in \mathbb{R}, \quad (1.14)$$

where $j, i, s \in \mathbb{N}$, $\Omega_{n;j,i,s}$ is a constant chosen such that $c_\lambda \int_0^\pi M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta d\theta = 1$. It is known that $M_{n;j,i,s}(\theta)$ is an even nonnegative operator. In particular, it is an even nonnegative trigonometric polynomial of degree at most $2s(nj+2j-2i)$ for $j > i$ and the Jackson polynomial for $j = i$. Using $M_{n;j,i,s}(\theta)$ we consider the spherical convolution

$$J_{n;j,i,s}(f; x) := (f * M_{n;j,i,s})(x) := c_\lambda \int_0^\pi T_\theta^\kappa(f; x) M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta d\theta. \quad (1.15)$$

It is called the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel. In particular, $(f_0 * M_{n;j,i,s})(x) = 1$ for $f_0(x) = 1$. The classical Jackson-Matsuoka polynomials in the classical L_p space have been studied by many authors (see [3, 4]).

The purpose of this paper is to consider approximation by h -harmonic polynomials, which in the L_p metric can be viewed as weighted approximation, in which the measure $d\omega$ on the sphere is replaced by $h_\kappa^2 d\omega$. It is well known that the situation can be quite different from that of ordinary harmonics; the weighted approximation is not a simple extension. Since the orthogonal group acts transitively on the sphere $\mathbb{S}$, much of the results for the ordinary harmonics can be proved by considering just one point; the reflection groups do not act transitively on the sphere.

In this paper, we consider weighted approximation of the Jackson-Matsuoka polynomials on the sphere. With the help of the relation between K -functionals and modulus of smoothness of sphere and the properties of the spherical means, we obtain the direct and inverse estimate for the best approximation by Jackson-Matsuoka polynomials in the weighted L_p space on the unit sphere of $\mathbb{R}^d$. We only consider best weighted approximation by Jackson-Matsuoka polynomials, and for the other polynomials on the unit sphere of $\mathbb{R}^d$, the methods and the results are similar.

2. Auxiliary Lemmas

We need the following lemmas.

Lemma 2.1. Let $\Omega_{n;j,i,s} = \int_0^\pi ((\sin^{2j} n\theta/2) / (\sin^{2i} \theta/2))^{2s} \sin^{2\lambda} \theta d\theta$. Then, the weak equivalence

$$\Omega_{n;j,i,s} \asymp n^{4is-2\lambda-1} \quad (2.1)$$

holds true for $4si > 2\lambda + 1$, $j \geq i$, where the weak equivalence relation $A(n) \asymp B(n)$ means that $A(n) \ll B(n)$ and $B(n) \ll A(n)$, and relation $A_n \ll B_n$ means that there is a positive constant C independent on n such that $A(n) \leq CB(n)$ holds.

The proof is similar to that of Lemma 2.2 and we omit it.

Lemma 2.2. For $4is > r + 2\lambda + 1$, $j \geq i$, $r \in \mathbb{R}$, there is a constant $C(\lambda, j, i, s)$ such that

$$\int_0^\pi \theta^r M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta d\theta \leq C(\lambda, j, i, s) n^{-r}. \quad (2.2)$$

Proof. Since $\theta/\pi \leq \sin(\theta/2) \leq \theta/2$ and $\sin \theta \leq \theta$ hold for $0 \leq \theta \leq \pi$, by $\Omega_{n;j,i,s} \asymp n^{4is-2\lambda-1}$, we have

$$\begin{aligned} \int_0^\pi \theta^r M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta d\theta &\leq C(\lambda, j, i, s) n^{-4is+2\lambda+1} \int_0^\pi \theta^r \left(\frac{\sin^{2j} n\theta/2}{\sin^{2i} \theta/2} \right)^{2s} \sin^{2\lambda} \theta d\theta \\ &\leq C(\lambda, j, i, s) n^{-4is+2\lambda+1} n^{4is-r-2\lambda-1} \int_0^{n\pi/2} t^{r+2\lambda} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt \\ &\leq C(\lambda, j, i, s) n^{-r} \left(\int_0^{\pi/2} t^{r+2\lambda} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt + \int_{\pi/2}^\infty t^{r+2\lambda} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt \right) \\ &\leq C(\lambda, j, i, s) C_2 n^\lambda \leq C(\lambda, j, i, s) n^\lambda, \end{aligned} \quad (2.3)$$

where

$$C_2 = \int_0^{\pi/2} t^\lambda \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt + \int_{\pi/2}^\infty t^\lambda \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt, \quad 4is > r + 2\lambda + 1, \quad j \geq i. \quad (2.4)$$

Lemma 2.2 has been proved. $\square$

Lemma 2.3 (see [2]). For $0 \leq \theta \leq \pi$, one has

$$\begin{aligned} T_\theta^\kappa(g; x) - g(x) &= \int_0^\theta \sin^{-2\lambda} t dt \int_0^t T_u^\kappa(\Delta_{h,0} g) \sin^{2\lambda} u du \\ &= \int_0^\theta \sin^{-2\lambda} t \Phi(t) B_t(\Delta_{h,0} g, x) dt, \end{aligned} \quad (2.5)$$

where

$$B_t(\Delta_{h,0} g, x) = \frac{1}{\Phi(t)} \int_0^t T_u^\kappa(\Delta_{h,0} g) \sin^{2\lambda} u du, \quad (2.6)$$

and $\Phi(t) = c_\lambda^{-1} \int_0^t \sin^{2\lambda} u du$.

Lemma 2.4. Let $g, \Delta_{h,0}g, \Delta_{h,0}^2g \in L_p(h_\kappa^2)$, $1 \leq p \leq \infty$, $J_{n;j,i,s}(f; x)$ be the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel, $4is > 2\lambda + 5$, $j \geq i$. Then, there is a constant $C(\lambda, j, i, s)$ such that

$$\|J_{n;j,i,s}g - g - \alpha(n)\Delta_{h,0}g\|_{\kappa,p} \leq C(\lambda, j, i, s)n^{-4}\|\Delta_{h,0}^2g\|_{\kappa,p}, \quad (2.7)$$

where $\alpha(n) \asymp n^{-2}$.

Proof. By Lemma 2.3, we have

$$\begin{aligned} J_{n;j,i,s}(g; x) - g(x) &= c_\lambda \int_0^\pi M_{n;j,i,s}(\theta)(T_\theta^\kappa(g; x) - g(x))\sin^{2\lambda}\theta d\theta \\ &= c_\lambda \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \frac{\Phi(t)}{\sin^{2\lambda}t} B_t(\Delta_{h,0}g, x) dt \\ &= c_\lambda \Delta_{h,0}g(x) \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \frac{\Phi(t)}{\sin^{2\lambda}t} dt \\ &\quad + c_\lambda \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \frac{\Phi(t)}{\sin^{2\lambda}t} (B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x)) dt \\ &= \Delta_{h,0}g(x) \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u du \\ &\quad + \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u (B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x)) du \\ &:= \alpha(n)\Delta_{h,0}g(x) + \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \Psi_\theta(g, x) d\theta, \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} \alpha(n) &:= \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u du, \\ \Psi_\theta(g, x) &:= \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u (B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x)) du. \end{aligned} \quad (2.9)$$

By Lemma 2.1, we have

$$\begin{aligned} \alpha(n) &= \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u du \\ &\asymp \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \frac{t \sin^{2\lambda}\xi}{\sin^{2\lambda}t} dt \\ &\asymp \int_0^\pi \theta^2 M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \asymp n^{-2}, \quad (0 < \xi < t). \end{aligned} \quad (2.10)$$

We now estimate, using Lemma 2.3 again, the expression $B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x)$, and obtain

$$\|\Psi_\theta(g)\|_{\kappa,p} \leq C(\lambda, j, i, s) \theta^4 \left\| \Delta_{h,0}^2 g \right\|_{\kappa,p}. \quad (2.11)$$

By Lemma 2.2 and Hölder-Minkowski inequality shows that

$$\begin{aligned} \left\| \int_0^\pi M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta \Psi_\theta(g, x) d\theta \right\|_{\kappa,p} &\leq C(\lambda, j, i, s) \left\| \Delta_{h,0}^2 g \right\|_{\kappa,p} \int_0^\pi \theta^4 M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta d\theta \\ &\leq C(\lambda, j, i, s) n^{-4} \left\| \Delta_{h,0}^2 g \right\|_{\kappa,p}. \end{aligned} \quad (2.12)$$

Consequently, by (2.8), (2.10), and (2.12) we complete the proof of this lemma. $\square$

Lemma 2.5. *For $t \geq 0$, there is a constant C such that*

$$\omega(f; t\delta)_{\kappa,p} \leq C \max\{1, t^2\} \omega(f; \delta)_{\kappa,p}. \quad (2.13)$$

Proof. By the equivalence relation between the modulus of smoothness and K -functional, and the definition of $K(f; t^2)_{\kappa,p}$, we have

$$\begin{aligned} \omega(f; t\delta)_{\kappa,p} &\leq CK(f; (t\delta)^2)_{\kappa,p} \leq C(\|f - g\|_{\kappa,p} + t^2 \delta^2 \|\Delta_{h,0}g\|_{\kappa,p}) \\ &\leq C \max\{1, t^2\} (\|f - g\|_{\kappa,p} + \delta^2 \|\Delta_{h,0}g\|_{\kappa,p}) \\ &\leq C \max\{1, t^2\} K(f; \delta^2)_{\kappa,p} \leq C \max\{1, t^2\} \omega(f; \delta)_{\kappa,p}. \end{aligned} \quad (2.14)$$

Lemma 2.5 has been proved. $\square$

3. Main Results

Our main results are the following.

Theorem 3.1. *Suppose that $f \in L_p(h_\kappa^2)$, $1 \leq p \leq \infty$, $J_{n;j,i,s}(f; x)$ is the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel, $4is > 2\lambda + 5$, $j \geq i$. Then*

$$\|J_{n;j,i,s}(f) - f\|_{\kappa,p} \asymp \omega(f; n^{-1})_{\kappa,p}. \quad (3.1)$$

Proof. First we prove $\|J_{n;j,i,s}(f) - f\|_{\kappa,p} \ll \omega(f; n^{-1})_{\kappa,p}$. Since $(f_0 * M_{n;j,i,s})(x) = 1$ for $f_0(x) = 1$, therefore, we have that

$$\begin{aligned} \|J_{n;j,i,s}(f) - f\|_{\kappa,p} &= \left\| \int_0^\pi M_{n;j,i,s}(\theta) (f(x) - T_\theta^\kappa(f; x)) \sin^{2\lambda} \theta d\theta \right\|_{\kappa,p} \\ &\leq \int_0^\pi \|f - T_\theta^\kappa(f)\|_{\kappa,p} M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta d\theta. \end{aligned} \quad (3.2)$$

Splitting the integral over $[0, \pi]$ into two integrals over $[0, 1/n]$ and $[1/n, \pi]$, respectively, and using the definition of $\omega(f; t)_{\kappa,p}$, we conclude that

$$\|f - T_\theta^\kappa(f)\|_{\kappa,p} \leq \omega\left(f; n^{-1}\right)_{\kappa,p} + \int_{1/n}^\pi \omega(f; \theta)_{\kappa,p} M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta d\theta. \quad (3.3)$$

From Lemma 2.5 it follows that, for $\theta \geq n^{-1}$,

$$\omega(f; \theta)_{\kappa,p} = \omega\left(f; n \frac{\theta}{n}\right)_{\kappa,p} \leq C \max\{1, n^2 \theta^2\} \omega(f; \theta)_{\kappa,p} \leq C n^2 \theta^2 \omega(f; \theta)_{\kappa,p}. \quad (3.4)$$

Therefore, it follows that

$$\|J_{n;j,i,s}(f) - f\|_{\kappa,p} \leq \omega(f; \theta)_{\kappa,p} \left(1 + C n^2 \int_{1/n}^\pi \theta^2 M_{n;j,i,s}(\theta) \sin^{2\lambda} \theta d\theta\right). \quad (3.5)$$

From Lemma 2.2, we get

$$\|J_{n;j,i,s}(f) - f\|_{\kappa,p} \leq C(\lambda, j, i, s) \omega(f; n^{-1})_{\kappa,p}. \quad (3.6)$$

Next we prove $\omega(f; n^{-1})_{\kappa,p} \ll \|J_{n;j,i,s}(f) - f\|_{\kappa,p}$. Let m be a fixed positive integer. Denote by

$$J_{n;j,i,s}^m(f) := \sum_{k=0}^m \left(\int_0^\pi M_{n;j,i,s}(\theta) Q_k^\lambda(\cos \theta) \sin^{2\lambda} \theta d\theta \right)^m Y_k(h_\kappa^2; f). \quad (3.7)$$

By orthogonality of the orthogonal projector Y_k , we have that

$$\begin{aligned} J^{m+l}(f) &= \sum_{k=0}^m \left(\int_0^\pi M_{n;j,i,s}(\theta) Q_k^\lambda(\cos \theta) \sin^{2\lambda} \theta d\theta \right)^m \\ &\quad \times Y_k \left(h_\kappa^2; \sum_{v=0}^m \left(\int_0^\pi M_{n;j,i,s}(\theta) Q_v^\lambda(\cos \theta) \sin^{2\lambda} \theta d\theta \right)^l Y_v(h_\kappa^2; f) \right) \\ &= J_{n;j,i,s}^m(J_{n;j,i,s}^l(f)). \end{aligned} \quad (3.8)$$

Letting $g = J_{n;j,i,s}^m(f)$, by (3.8) we get

$$\begin{aligned}
 \|f - g\|_{\kappa,p} &= \|f - J_{n;j,i,s}^m(f)\|_{\kappa,p} \\
 &\leq \sum_{k=1}^m \|J_{n;j,i,s}^{k-1}(f) - J_{n;j,i,s}^k(f)\|_{\kappa,p} \\
 &\leq C(\lambda, j, i, s) \sum_{k=1}^m \|J_{n;j,i,s}^{k-1}((f) - J_{n;j,i,s}(f))\|_{\kappa,p} \\
 &\leq C(\lambda, j, i, s) m \|f - J_{n;j,i,s}(f)\|_{\kappa,p}
 \end{aligned} \tag{3.9}$$

where $J_{n;j,i,s}^0(f) = f$.

On the other hand,

$$\|\Delta_{h,0} J_{n;j,i,s}^m(f)\|_{\kappa,p} \leq \sum_{k=0}^m k(k+2\lambda) \left(\int_0^\pi M_{n;j,i,s}(\theta) |Q_k^\lambda(\cos \theta)| \sin^{2\lambda} \theta d\theta \right)^m Y_k(h_\kappa^2; f). \tag{3.10}$$

Note that [5]

$$|Q_k^\lambda(\cos \theta)| \equiv \left| \frac{C_k^\lambda(\cos \theta)}{C_k^\lambda(1)} \right| \leq C \min\{(k\theta)^{-1}, 1\}. \tag{3.11}$$

For $k\theta \geq 1$, from (2.2) it follows that

$$\begin{aligned}
 \|\Delta_{h,0} J_{n;j,i,s}^m(f)\|_{\kappa,p} &\leq C(\lambda, j, i, s) \left\| \sum_{k=0}^m k(k+2\lambda) k^{-m\lambda} \left(\int_0^\pi M_{n;j,i,s}(\theta) \theta^{-\lambda} \sin^{2\lambda} \theta d\theta \right)^m Y_k(h_\kappa^2; f) \right\|_{\kappa,p} \\
 &\leq C(\lambda, j, i, s) n^{m\lambda} \|f\|_{\kappa,p} \sum_{k=0}^\infty k^{2-m\lambda} \leq C(\lambda, j, i, s) n^{m\lambda} \|f\|_{\kappa,p}.
 \end{aligned} \tag{3.12}$$

holds for $m > 3/\lambda$. For $k\theta < 1$, by (2.2), we get

$$\begin{aligned}
& \left\| \Delta_{h,0} J_{n;j,i,s}^m(f) \right\|_{\kappa,p} \\
& \leq \left\| \sum_{k=0}^m \left(\int_0^\pi M_{n;j,i,s}(\theta) \theta^{-2/m} (\theta^2 k(k+2\lambda))^{1/m} |Q_k^\lambda(\cos \theta)| \sin^{2\lambda} \theta d\theta \right)^m Y_k(h_\kappa^2; f) \right\|_{\kappa,p} \\
& \leq C(\lambda, j, i, s) \left\| \sum_{k=0}^m \left(\int_0^\pi M_{n;j,i,s}(\theta) \theta^{-2/m} ((k\theta)^2)^{2/m} \sin^{2\lambda} \theta d\theta \right)^m Y_k(h_\kappa^2; f) \right\|_{\kappa,p} \\
& \leq C(\lambda, j, i, s) \left\| \sum_{k=0}^m \left(\int_0^\pi M_{n;j,i,s}(\theta) \theta^{-2/m} \sin^{2\lambda} \theta d\theta \right)^m Y_k(h_\kappa^2; f) \right\|_{\kappa,p} \\
& \leq C(\lambda, j, i, s) n^2 \left\| \sum_{k=0}^\infty Y_k(h_\kappa^2; f) \right\|_{\kappa,p} \leq C n^2 \|f\|_{\kappa,p}.
\end{aligned} \tag{3.13}$$

Consequently, the inequality

$$\left\| \Delta_{h,0} J_{n;j,i,s}^m(f) \right\|_{\kappa,p} \leq C(\lambda, j, i, s) n^2 \|f\|_{\kappa,p} \tag{3.14}$$

holds uniformly for $m > 3/\lambda$. Without loss of generality, we may assume $m_1 > 3/\lambda$, $m > m_1 + 3/\lambda$. Using Lemma 2.4 and (3.8), we have

$$\begin{aligned}
\alpha(n) \left\| \Delta_{h,0} J_{n;j,i,s}^m(f) \right\|_{\kappa,p} &= \left\| \alpha(n) \Delta_{h,0} J_{n;j,i,s}^m(f) \right\|_{\kappa,p} \\
&\leq \left\| J_{n;j,i,s}^m(f) - f \right\|_{\kappa,p} + C(\lambda, j, i, s) n^{-4} \left\| \Delta_{h,0}^2 J_{n;j,i,s}^m(f) \right\|_{\kappa,p} \\
&\leq m \left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} + C(\lambda, j, i, s) n^{-2} \left\| \Delta_{h,0}^2 J_{n;j,i,s}^{m-m_1}(f) \right\|_{\kappa,p} \\
&\leq m \left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} \\
&\quad + C(\lambda, j, i, s) \left(n^{-2} \left\| \Delta_{h,0} J_{n;j,i,s}^m(f) \right\|_{\kappa,p} + n^{-2} \left\| J_{n;j,i,s}^m(f) - J_{n;j,i,s}^{m-m_1}(f) \right\|_{\kappa,p} \right) \\
&\leq m \left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} \\
&\quad + C(\lambda, j, i, s) \left(n^{-2} \left\| \Delta_{h,0} J_{n;j,i,s}^m(f) \right\|_{\kappa,p} + \left\| J_{n;j,i,s}^{m_1}(f) - f \right\|_{\kappa,p} \right) \\
&\leq C(\lambda, j, i, s) \left(\left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} + n^{-2} \left\| \Delta_{h,0} J_{n;j,i,s}^m(f) \right\|_{\kappa,p} \right) \\
&\leq C(\lambda, j, i, s) \left(\left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} + \|f\|_{\kappa,p} \right).
\end{aligned} \tag{3.15}$$

Consequently, $n^{-2} \|\Delta_{h,0} J_{n;j,i,s}^m(f)\|_{\kappa,p} \leq C(\lambda, j, i, s) \|f - J_{n;j,i,s}(f)\|_{\kappa,p'}$, by the definition of $K(f; t^2)_{\kappa,p}$ and (1.13) shows that

$$\begin{aligned} \omega(f; n^{-1})_{\kappa,p} &\leq CK(f; n^{-2})_{\kappa,p} \\ &\leq C \left(\|f - J_{n;j,i,s}^m(f)\|_{\kappa,p} + n^{-2} \|\Delta_{h,0} J_{n;j,i,s}^m(f)\|_{\kappa,p} \right) \\ &\leq C(\lambda, j, i, s) \|f - J_{n;j,i,s}(f)\|_{\kappa,p'} \end{aligned} \quad (3.16)$$

that is, $\omega(f; n^{-1})_{\kappa,p} \ll \|f - J_{n;j,i,s}(f)\|_{\kappa,p}$.

The proof is completed. $\square$

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