Research Article
Weighted Approximation for Jackson-Matsuoka Polynomials on the Sphere

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We consider the best approximation by Jackson-Matsuoka polynomials in the weighted $L^p$ space on the unit sphere of $\mathbb{R}^d$. Using the relation between $K$-functionals and modulus of smoothness on the sphere, we obtain the direct and inverse estimate of approximation by these polynomials for the $h$-spherical harmonics.

1. Introduction and Notations

Let $S := S^{d-1} = \{x : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^d$ ($d \geq 3$), $d \in \mathbb{N}$, where $\|x\|$ denotes the usual Euclidean norm, $\mathbb{R}$ the set of real numbers. For a nonzero vector $v \in \mathbb{R}^d$, let $\sigma_v$ denote the reflection with respect to the hyperplane perpendicular to $v$, $x \sigma_v := x - 2(\langle x, v \rangle / \|v\|^2)v$, $x \in \mathbb{R}^d$, where $\langle x, v \rangle$ denote the usual Euclidean inner product. Let $G$ be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system $\mathbb{R}_+$, normalized so that $\langle v, v \rangle = 2$ for all $v \in \mathbb{R}_+$. Then $G$ is a subgroup of the orthogonal group generated by the reflections $\{\sigma_v : v \in \mathbb{R}_+\}$. Let $\kappa$ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on $\mathbb{R}_+$ with the property that $\kappa_v = \kappa_u$ whenever $\sigma_u$ is conjugate to $\sigma_v$ in $G$, then $v \mapsto \kappa_v$ is a $G$-invariant function. We consider the weighted best $L^p$ approximation with respect to the measure $h^2_\kappa d\omega$ on $S$, where $h^2_\kappa$ is defined by

$$h^2_\kappa = \prod_{v \in \mathbb{R}_+} |\langle x, v \rangle|^\kappa_v, \quad x \in \mathbb{R}^d, \quad (1.1)$$

d$\omega$ is the surface (Lebesgue) measure on $S$. The function $h_\kappa$ is a positive homogeneous function of degree $\gamma_\kappa := \sum_{v \in \mathbb{R}_+} \kappa_v$, and it is invariant under the reflection group. We denote
by $a_κ$ the normalization constant of $h_κ$, $a_κ^{-1} = \int_S h_κ^2(y)\,dω$ and denote by $L_p(h_κ^2)$, $1 \leq p \leq \infty$, the space of functions defined on $S$ with the finite norm

$$\|f\|_κ^p := \left(a_κ \int_S |f(y)|^p h_κ^2(y)\,dω(y)\right)^{1/p}, \quad 1 \leq p < \infty,$$

(1.2)

and for $p = \infty$ we assume that $L_∞$ is replaced by $C(S)$ the space of continuous functions on $S$ with the usual uniform norm $\|f\|_∞$.

$Δ_h$ denote the $h$-Laplacian. $Δ_{h,0}$ is the Laplace-Beltrami operator on the sphere. $P_n^d$ denote the subspace of homogeneous polynomials of degree $n$ in $d$ variables. The $h$-harmonics are defined as the homogeneous polynomials satisfying the equation $Δ_h P = 0$, $P \in P_n^d$. Furthermore, let $σ_n^d(h_κ^2)$ denote the space of $h$-spherical harmonics of degree $n$ in $d$ variables. The spherical $h$-harmonics are the restriction of $h$-harmonics on the unit sphere. It is well known that spherical $h$-harmonics are eigenfunctions of $Δ_{h,0}$; that is,

$$Δ_{h,0} Y(x) = -n(n + 2λ)Y(x), \quad x \in S, \quad Y \in σ_n^d(h_κ^2).$$

(1.3)

The standard Hilbert space theory shows that $L_2(h_κ^2) = \sum_{n=0}^\infty \oplus σ_n^d(h_κ^2)$. That is, with each $f \in L_2(h_κ^2)$ we can associate its $h$-harmonic expansion

$$f(x) = \sum_{n=0}^\infty Y_n(h_κ^2; f, x), \quad x \in S,$$

(1.4)

in $L_2(h_κ^2)$ norm. For the surface measure ($κ = 0$), such a series is called the Laplace series (see [1]). The orthogonal projection $Y_n(h_κ^2) : L_2(h_κ^2) \rightarrow σ_n^d(h_κ^2)$ takes the form

$$Y_n(h_κ^2; f, x) := \int_Σ f(y) P_n(h_κ^2; x, y) h_κ^2(y)\,dω(y),$$

(1.5)

where $P_n(h_κ^2; x, y)$ is the reproducing kernel of the space of $h$-harmonics $σ_n^d(h_κ^2)$, which is given by (see [2])

$$P_n(h_κ^2; x, y) = \frac{n + 1}{λ} V_κ \left[ C_n^λ(Δ, y) \right](x).$$

(1.6)

$C_n^λ$ is the ultraspherical polynomial of degree $n, λ := γ_κ + (d - 2)/2$, $γ_κ = \sum_{\nu \in \mathbb{N}}, κ_ν$, and the intertwining operator $V_κ$ is a linear operator uniquely determined by

$$V_κ P_n \subset P_n, \quad V_κ 1 = 1, \quad ∂_i V_κ = V_κ ∂_i, \quad 1 \leq i \leq d.$$
The spherical means are denoted by

$$T_{\theta}(f) = \frac{1}{|S^{d-2}|} \int |\sin \theta|^{d-2} f(y) d\omega(y),$$

where $|S^{d-2}| = \int_{S^{d-2}} d\omega = 2\pi^{(d-1)/2} / \Gamma((d - 1)/2)$.

The spherical means associated with $h_\kappa$, in which $T_{\theta}^\kappa(f)$ is defined by

$$c_1 \int_0^\pi T_{\theta}^\kappa(f, x) g(\cos \theta)(\sin \theta)^{2\kappa} d\theta = a_\kappa \int_\mathbb{R} f(y) V_\kappa g((x, y)) h_\kappa^2(y) d\omega(y),$$

where $g$ is any function $[-1, 1] \mapsto \mathbb{R}$ such that the integral in the right-hand side is finite,

$c_1 = \int_1^0 (1 - t^2)^{(\lambda - 1)/2} dt = \Gamma(\lambda + 1/2)\sqrt{\pi}/\Gamma(\lambda + 1)$. $T_{\theta}^\kappa(f)$ is a proper extension of $T_{\theta}(f)$, since $T_{\theta}(f)$ satisfies $T_{\theta}^\kappa(f)$ when $\kappa = 0$ and $V_\kappa = id$, and the properties of $T_{\theta}^\kappa$ are well known (see [2]). In particular, the function $T_{\theta}^\kappa f(x)$ has the expansion

$$T_{\theta}^\kappa(f) \sim \sum_{n=0}^\infty \frac{C_n^1(\cos \theta)}{C_n^1(1)} \mathcal{Y}_n(h_\kappa^2; f) := \sum_{n=0}^\infty \frac{Q_n^1(\cos \theta)}{C_n^1(1)} \mathcal{Y}_n(h_\kappa^2; f).$$

Simultaneously, they lead to the following definition of an analog of the modulus of smoothness.

**Definition 1.1** (see [2]). For $f \in L_p(h_\kappa^2)$, $1 \leq p < \infty$, or $f \in C(\mathbb{S})$, the modulus of smoothness on the sphere is given by

$$\omega(f; t)_{\kappa,p} := \sup_{0 < \theta \leq \pi} \left\| f - T_{\theta}^\kappa(f) \right\|_{\kappa,p}.$$

The $K$-functional of the sphere is given by

$$K(f; t^2)_{\kappa,p} = \inf_{g \in W_p(h_\kappa^2)} \left\{ \left\| f - g \right\|_{\kappa,p} + t^2 \left\| \Delta_{h,0} g \right\|_{\kappa,p} \right\}.$$

where $W_p(h_\kappa^2) := \{ f : f \in L_p(h_\kappa^2), -k(2 + 2\lambda)P_k(h_\kappa^2; f) = P_k(h_\kappa^2; g) \text{ for some } g \in L_p(h_\kappa^2) \}$, $0 < t < t_0$, $t_0$ is a positive constant.

In [2], Xu proved the weak equivalence relation

$$C^{-1} \omega(f; t)_{\kappa,p} \leq K(f; t^2)_{\kappa,p} \leq C \omega(f; t)_{\kappa,p}.$$

Throughout this paper, $C$ denotes a positive constant independent on $n$ and $f$ and $C(a)$ denotes a positive constant dependent on $a$, which may be different according to the circumstances.
Based on the classical Jackson-Matsuoka kernel (see [3]), we define a new kernel

\[ M_{n,j,i,s}(\theta) := \frac{1}{\Omega_{n,j,i,s}} \left( \frac{\sin^2 n\theta / 2}{\sin^2 \theta / 2} \right)^{2s}, \quad n = 1, 2, \ldots, \theta \in \mathbb{R}, \quad (1.14) \]

where \( j, i, s \in \mathbb{N}, \Omega_{n,j,i,s} \) is a constant chosen such that \( c_1 \int_0^\pi M_{n,j,i,s}(\theta) \sin^{2s} \theta d\theta = 1 \). It is known that \( M_{n,j,i,s}(\theta) \) is an even nonnegative operator. In particular, it is an even nonnegative trigonometric polynomial of degree at most \( 2s(nj+2j-2i) \) for \( j > i \) and the Jackson polynomial for \( j = i \). Using \( M_{n,j,i,s}(\theta) \) we consider the spherical convolution

\[ J_{n,j,i,s}(f; x) := (f * M_{n,j,i,s})(x) := c_1 \int_0^\pi T^s_\theta(f; x) M_{n,j,i,s}(\theta)(\sin^{2s} \theta) d\theta. \quad (1.15) \]

It is called the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel. In particular, \((f_0 * M_{n,j,i,s})(x) = 1 \) for \( f_0(x) = 1 \). The classical Jackson-Matsuoka polynomials in the classical \( L_p \) space have been studied by many authors (see [3, 4]).

The purpose of this paper is to consider approximation by \( h \)-harmonic polynomials, which in the \( L_p \) metric can be viewed as weighted approximation, in which the measure \( d\omega \) on the sphere is replaced by \( h^p_\omega d\omega \). It is well known that the situation can be quite different from that of ordinary harmonics; the weighted approximation is not a simple extension. Since the orthogonal group acts transitively on the sphere \( S \), much of the results for the ordinary harmonics can be proved by considering just one point; the reflection groups do not act transitively on the sphere.

In this paper, we consider weighted approximation of the Jackson-Matsuoka polynomials on the sphere. With the help of the relation between \( K \)-functionals and modulus of smoothness of sphere and the properties of the spherical means, we obtain the direct and inverse estimate for the best approximation by Jackson-Matsuoka polynomials in the weighted \( L_p \) space on the unit sphere of \( \mathbb{R}^d \). We only consider best weighted approximation by Jackson-Matsuoka polynomials, and for the other polynomials on the unit sphere of \( \mathbb{R}^d \), the methods and the results are similar.

### 2. Auxiliary Lemmas

We need the following lemmas.

**Lemma 2.1.** Let \( \Omega_{n,j,i,s} = \int_0^\pi ((\sin^2 n\theta / 2) / (\sin^2 \theta / 2))^{2s} \sin^{2s} \theta d\theta \). Then, the weak equivalence

\[ \Omega_{n,j,i,s} \asymp n^{4si-2j-1} \quad (2.1) \]

holds true for \( 4si > 2\lambda + 1, \ j \geq i \), where the weak equivalence relation \( A(n) \asymp B(n) \) means that \( A(n) \ll B(n) \) and \( B(n) \ll A(n) \), and relation \( A_n \ll B_n \) means that there is a positive constant \( C \) independent on \( n \) such that \( A(n) \leq CB(n) \) holds.

The proof is similar to that of Lemma 2.2 and we omit it.
Lemma 2.2. For $4is > r + 2\lambda + 1, j \geq i, r \in \mathbb{R}$, there is a constant $C(\lambda, j, i, s)$ such that

$$
\int_0^\pi \theta^r M_{n,j,i,s}(\theta) \sin^{2\lambda} \theta \, d\theta \leq C(\lambda, j, i, s) n^{-r}.
$$

(2.2)

Proof. Since $\theta/\pi \leq \sin(\theta/2) \leq \theta/2$ and $\sin \theta \leq \theta$ hold for $0 \leq \theta \leq \pi$, by $\Omega_{n,j,i,s} \sim n^{4is-2\lambda-1}$, we have

$$
\int_0^\pi \theta^r M_{n,j,i,s}(\theta) \sin^{2\lambda} \theta \, d\theta \leq C(\lambda, j, i, s) n^{-4is+2\lambda+1} \int_0^\pi \theta^r \left( \frac{\sin^{2\lambda} n\theta/2}{\sin^{2\lambda} \theta/2} \right)^{2s} \sin^{2\lambda} \theta \, d\theta
\leq C(\lambda, j, i, s) n^{-4is+2\lambda+1} n^{4is-2\lambda-1} \int_0^{n\pi/2} t^{r+2\lambda} \left( \frac{\sin^{2\lambda} t}{t^{2\lambda}} \right)^{2s} \, dt
\leq C(\lambda, j, i, s) n^{-r} \left( \int_0^{\pi/2} t^{r+2\lambda} \left( \frac{\sin^{2\lambda} t}{t^{2\lambda}} \right)^{2s} \, dt + \int_{\pi/2}^\infty t^{r+2\lambda} \left( \frac{\sin^{2\lambda} t}{t^{2\lambda}} \right)^{2s} \, dt \right)
\leq C(\lambda, j, i, s) C_2 n^4 \leq C(\lambda, j, i, s) n^4,
$$

(2.3)

where

$$
C_2 = \int_0^{\pi/2} t^4 \left( \frac{\sin^{2\lambda} t}{t^{2\lambda}} \right)^{2s} \, dt + \int_{\pi/2}^\infty t^4 \left( \frac{\sin^{2\lambda} t}{t^{2\lambda}} \right)^{2s} \, dt, \quad 4is > r + 2\lambda + 1, j \geq i.
$$

(2.4)

Lemma 2.2 has been proved.

Lemma 2.3 (see [2]). For $0 \leq \theta \leq \pi$, one has

$$
T_0^x(g;x) - g(x) = \int_0^\theta \sin^{-2\lambda} t \, dt \int_0^t T_0^x(\Delta_{h,0} g) \sin^{2\lambda} u \, du
\leq \int_0^\theta \sin^{-2\lambda} t \Phi(t) B_t(\Delta_{h,0} g, x) \, dt,
$$

(2.5)

where

$$
B_t(\Delta_{h,0} g, x) = \frac{1}{\Phi(t)} \int_0^t T_0^x(\Delta_{h,0} g) \sin^{2\lambda} u \, du,
$$

(2.6)

and $\Phi(t) = c^{-1}_\lambda \int_0^t \sin^{2\lambda} u \, du$. 

Lemma 2.4. Let \( g, \Delta_{h,0}g, \Delta_{h,0}^2g \in L^p(\mathbb{H}^2_2), 1 \leq p \leq \infty, J_{n,j,i,s}(f;x) \) be the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel, \( 4is > 2\lambda + 5, j \geq i \). Then, there is a constant \( C(\lambda, j, i, s) \) such that

\[
\|J_{n,j,i,s}g - g - \alpha(n)\Delta_{h,0}g\|_{p} \leq C(\lambda, j, i, s)n^{-\frac{4}{p}}\|\Delta_{h,0}^2g\|_{p},
\]

where \( \alpha(n) \approx n^{-2} \).

Proof. By Lemma 2.3, we have

\[
J_{n,j,i,s}(g;x) - g(x) = c_1 \int_0^\pi M_{n,j,i,s}(\theta) (T_\theta^0 (g;x) - g(x)) \sin^2 \theta \, d\theta
\]

\[
= c_1 \int_0^\pi M_{n,j,i,s}(\theta) \sin^2 \theta \, d\theta \int_0^\theta \frac{\Phi(t)}{\sin^2 t} B_t(\Delta_{h,0}g, x) \, dt
\]

\[
+ c_1 \Delta_{h,0}g(x) \int_0^\pi M_{n,j,i,s}(\theta) \sin^2 \theta \, d\theta \int_0^\theta \frac{\Phi(t)}{\sin^2 t} \left( B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x) \right) \, dt
\]

\[
= \Delta_{h,0}g(x) \int_0^\pi M_{n,j,i,s}(\theta) \sin^2 \theta \, d\theta \int_0^\theta \frac{dt}{\sin^2 t} \int_0^t \sin^2 u \, du
\]

\[
+ \int_0^\pi M_{n,j,i,s}(\theta) \sin^2 \theta \, d\theta \int_0^\theta \frac{dt}{\sin^2 t} \int_0^t \sin^2 u \, du \left( B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x) \right) \, du
\]

\[
:= \alpha(n)\Delta_{h,0}g(x) + \int_0^\pi M_{n,j,i,s}(\theta) \sin^2 \theta \Psi_\theta(g, x) \, d\theta,
\]

where

\[
\alpha(n) := \int_0^\pi M_{n,j,i,s}(\theta) \sin^2 \theta \, d\theta \int_0^\theta \frac{dt}{\sin^2 t} \int_0^t \sin^2 u \, du,
\]

\[
\Psi_\theta(g, x) := \int_0^\theta \frac{dt}{\sin^2 t} \int_0^t \sin^2 u \left( B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x) \right) \, du
\]

By Lemma 2.1, we have

\[
\alpha(n) = \int_0^\pi M_{n,j,i,s}(\theta) \sin^2 \theta \, d\theta \int_0^\theta \frac{dt}{\sin^2 t} \int_0^t \sin^2 u \, du
\]

\[
\times \int_0^\pi M_{n,j,i,s}(\theta) \sin^2 \theta \, d\theta \int_0^\theta \frac{t \sin^2 \xi}{\sin^2 t} \, dt
\]

\[
\times \int_0^\pi \theta^2 M_{n,j,i,s}(\theta) \sin^2 \theta \, d\theta \propto n^{-2}, \quad (0 < \xi < t).
\]
We now estimate, using Lemma 2.3 again, the expression \( B_t(\Delta h, 0 \cdot g, x) = \Delta h, 0 \cdot g, x \), and obtain
\[
\| \Psi_\theta (g) \|_{\kappa,p} \leq C (\lambda, j, i, s) \theta^4 \| \Delta^2 h, 0 \cdot g \|_{\kappa,p}. \tag{2.11}
\]

By Lemma 2.2 and Hölder-Minkowski inequality shows that
\[
\left\| \int_0^\pi M_{n,j,i,s}(\theta) \sin^{2\lambda} \Psi_\theta (g, x) d\theta \right\|_{\kappa,p} \leq C (\lambda, j, i, s) \| \Delta^2 h, 0 \cdot g \|_{\kappa,p} \int_0^\pi \theta^4 M_{n,j,i,s}(\theta) \sin^{2\lambda} d\theta
\leq C (\lambda, j, i, s) n^{-4} \| \Delta^2 h, 0 \cdot g \|_{\kappa,p}. \tag{2.12}
\]

Consequently, by (2.8), (2.10), and (2.12) we complete the proof of this lemma.

**Lemma 2.5.** For \( t \geq 0 \), there is a constant \( C \) such that
\[
\omega(f; t\delta)_{\kappa,p} \leq C \max\{1, t^2\} \omega(f; \delta)_{\kappa,p}. \tag{2.13}
\]

**Proof.** By the equivalence relation between the modulus of smoothness and \( K \)-functional, and the definition of \( K(f; t^2)_{\kappa,p} \), we have
\[
\omega(f; t\delta)_{\kappa,p} \leq CK(f; (t\delta)^2)_{\kappa,p} \leq C \left( \| f - g \|_{\kappa,p} + t^2 \delta^2 \| \Delta h, 0 \cdot g \|_{\kappa,p} \right)
\leq C \max\{1, t^2\} \left( \| f - g \|_{\kappa,p} + \delta^2 \| \Delta h, 0 \cdot g \|_{\kappa,p} \right) \tag{2.14}
\leq C \max\{1, t^2\} K(f; \delta^2)_{\kappa,p} \leq C \max\{1, t^2\} \omega(f; \delta)_{\kappa,p}.
\]

Lemma 2.5 has been proved.

**3. Main Results**

Our main results are the following.

**Theorem 3.1.** Suppose that \( f \in L_p(h^2_x) \), \( 1 \leq p \leq \infty \), \( J_{n,j,i,s}(f; x) \) is the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel, \( 4is > 2\lambda + 5 \), \( j \geq i \). Then
\[
\| J_{n,j,i,s}(f) - f \|_{\kappa,p} \leq \omega(f; n^{-1})_{\kappa,p}. \tag{3.1}
\]
Proof. First we prove \( \| J_{n; j, i, s}(f) - f \|_{k, p} \ll \omega(f; n^{-1})_{k, p} \). Since \( (f_0 \ast M_{n; j, i, s})(x) = 1 \) for \( f_0(x) = 1 \), therefore, we have that

\[
\| J_{n; j, i, s}(f) - f \|_{k, p} = \left\| \int_0^\pi M_{n; j, i, s}(\theta)(f(x) - T_\theta^*(f; x)) \sin^2 \theta \, d\theta \right\|_{k, p} \\
\leq \int_0^\pi \| f - T_\theta^*(f) \|_{k, p} M_{n; j, i, s}(\theta) \sin^2 \theta \, d\theta.
\]

(3.2)

Splitting the integral over \([0, \pi]\) into two integrals over \([0, 1/n]\) and \([1/n, \pi]\), respectively, and using the definition of \( \omega(f; t)_{k, p} \), we conclude that

\[
\| f - T_\theta^*(f) \|_{k, p} \leq \omega(f; n^{-1})_{k, p} + \int_{1/n}^\pi \omega(f; \theta)_{k, p} M_{n; j, i, s}(\theta) \sin^2 \theta \, d\theta.
\]

(3.3)

From Lemma 2.5 it follows that, for \( \theta \geq n^{-1} \),

\[
\omega(f; \theta)_{k, p} = \omega\left(f; \frac{\theta}{n}\right)_{k, p} \leq C \max\{1, n^2 \theta^2\} \omega(f; \theta)_{k, p} \leq C n^2 \theta^2 \omega(f; \theta)_{k, p}.
\]

(3.4)

Therefore, it follows that

\[
\| J_{n; j, i, s}(f) - f \|_{k, p} \leq \omega(f; \theta)_{k, p} \left(1 + C n^2 \int_{1/n}^\pi \theta^2 M_{n; j, i, s}(\theta) \sin^2 \theta \, d\theta\right).
\]

(3.5)

From Lemma 2.2, we get

\[
\| J_{n; j, i, s}(f) - f \|_{k, p} \leq C(\lambda, j, i, s) \omega(f; n^{-1})_{k, p}.
\]

(3.6)

Next we prove \( \omega(f; n^{-1})_{k, p} \ll \| J_{n; j, i, s}(f) - f \|_{k, p} \). Let \( m \) be a fixed positive integer Denote by

\[
J_{m; n; j, i, s}^m(f) := \sum_{k=0}^m \left( \int_0^\pi M_{n; j, i, s}(\theta) Q_k^i(\cos \theta) \sin^2 \theta \, d\theta \right)^m Y_k(h^2; f).
\]

(3.7)

By orthogonality of the orthogonal projector \( Y_k \), we have that

\[
J_{m; n; j, i, s}^m(f) = \sum_{k=0}^m \left( \int_0^\pi M_{n; j, i, s}(\theta) Q_k^i(\cos \theta) \sin^2 \theta \, d\theta \right)^m
\times Y_k\left(h^2; \sum_{i=0}^m \left( \int_0^\pi M_{n; j, i, s}(\theta) Q_k^i(\cos \theta) \sin^2 \theta \, d\theta \right)^i Y_{\nu}\left(h^2; f\right)\right)
\]

(3.8)

\[
= J_{n; j, i, s}^m\left(J_{m; n; j, i, s}(f)\right).
\]
Letting \( g = J_{m,i,s}^n(f) \), by (3.8) we get

\[
\|f - g\|_{\kappa,p} = \|f - J_{m,i,s}^n(f)\|_{\kappa,p} \\
\leq \sum_{k=1}^{m} \|J_{m,i,s}^{k-1}(f) - J_{m,i,s}^k(f)\|_{\kappa,p} \\
\leq C(\lambda, j, i, s) \sum_{k=1}^{m} \|J_{m,i,s}^{k-1}(f) - J_{m,i,s}^k(f)\|_{\kappa,p} \\
\leq C(\lambda, j, i, s) m \|f - J_{m,i,s}(f)\|_{\kappa,p}
\]

(3.9)

where \( J_{m,i,s}(f) = f \).

On the other hand,

\[
\|\Delta_{h,0} J_{m,i,s}^m(f)\|_{\kappa,p} \leq \sum_{k=0}^{m} k(k + 2\lambda) \left( \int_0^{\pi} M_{n,j,s}(\theta) Q_k^1(\cos \theta) |\sin^{2\lambda} \theta d\theta \right) \sum_{k=0}^{m} \left( \int_0^{\pi} M_{n,j,s}(\theta) \theta^{-\lambda} \sin^{2\lambda} \theta d\theta \right) \sum_{k=0}^{m} Y_k \left( h_{2\lambda}^2; f \right).
\]

(3.10)

Note that [5]

\[
|Q_k^1(\cos \theta)| = \left| \frac{C_k^1(\cos \theta)}{C_k^1(1)} \right| \leq C \min\left\{ (k\theta)^{-1}, 1 \right\}. 
\]

(3.11)

For \( k\theta \geq 1 \), from (2.2) it follows that

\[
\|\Delta_{h,0} J_{m,i,s}^m(f)\|_{\kappa,p} \leq C(\lambda, j, i, s) \sum_{k=0}^{m} k(k + 2\lambda) k^{-\frac{m\lambda}{2}} \left( \int_0^{\pi} M_{n,j,s}(\theta) \theta^{-\lambda} \sin^{2\lambda} \theta d\theta \right) \sum_{k=0}^{m} Y_k \left( h_{2\lambda}^2; f \right) \leq C(\lambda, j, i, s) m \|f\|_{\kappa,p}.
\]

(3.12)
holds for $m > 3/\lambda$. For $k\theta < 1$, by (2.2), we get

$$
\|\Delta_{h,0} J_{n,j,i,s}^m(f)\|_{k,p} \\
\leq \left\| \sum_{k=0}^{m} \left( \int_0^{\pi} M_{n,j,i,s}(\theta) \theta^{-2/m} (\theta^2 k(k+2\lambda))^{1/m} |Q_1^{1}(\cos \theta)| \sin^2 \theta d\theta \right)^m \right\|_{k,p} \\
\leq C(\lambda, j, i, s) \left\| \sum_{k=0}^{m} \left( \int_0^{\pi} M_{n,j,i,s}(\theta) \theta^{-2/m} (k\theta^2)^{2/m} \sin^2 \theta d\theta \right)^m \right\|_{k,p} \\
\leq C(\lambda, j, i, s) \left\| \sum_{k=0}^{m} \left( \int_0^{\pi} M_{n,j,i,s}(\theta) \theta^{-2/m} \sin^2 \theta d\theta \right)^m \right\|_{k,p} \\
\leq C(\lambda, j, i, s)n^2 \left\| \sum_{k=0}^{\infty} Y_k \left( h_{k,i}^2; f \right) \right\|_{k,p} \leq Cn^2 \|f\|_{k,p}.
$$

Consequently, the inequality

$$
\|\Delta_{h,0} J_{n,j,i,s}^m(f)\|_{k,p} \leq C(\lambda, j, i, s)n^2 \|f\|_{k,p}
$$

holds uniformly for $m > 3/\lambda$. Without loss of generality, we may assume $m_1 > 3/\lambda$, $m > m_1 + 3/\lambda$. Using Lemma 2.4 and (3.8), we have

$$
\alpha(n) \|\Delta_{h,0} J_{n,j,i,s}^m(f)\|_{k,p} = \|\alpha(n) \Delta_{h,0} J_{n,j,i,s}^m(f)\|_{k,p} \\
\leq \|J_{n,j,i,s}(f) - f\|_{k,p} + C(\lambda, j, i, s)n^2 \|\Delta_{h,0} J_{n,j,i,s}^m(f)\|_{k,p} \\
\leq m \|J_{n,j,i,s}(f) - f\|_{k,p} + C(\lambda, j, i, s)n^2 \|\Delta_{h,0} J_{n,j,i,s}^{m-m_1}(f)\|_{k,p} \\
\leq m \|J_{n,j,i,s}(f) - f\|_{k,p} \\
+ C(\lambda, j, i, s) \left( n^{-2} \|\Delta_{h,0} J_{n,j,i,s}^m(f)\|_{k,p} + n^{-2} \|J_{n,j,i,s}^m(f) - J_{n,j,i,s}^{m-m_1}(f)\|_{k,p} \right) \\
\leq m \|J_{n,j,i,s}(f) - f\|_{k,p} \\
+ C(\lambda, j, i, s) \left( \|J_{n,j,i,s}(f) - f\|_{k,p} + \|J_{n,j,i,s}^m(f)\|_{k,p} \right) \\
\leq C(\lambda, j, i, s) \left( \|J_{n,j,i,s}(f) - f\|_{k,p} + \|f\|_{k,p} \right).
$$
Consequently, \( n^{-2}\|\Delta_{n,0} J_{n,j,i,s}^m(f)\|_{\kappa,p} \leq C(\lambda, j, i, s)\|f - J_{n,j,i,s}(f)\|_{\kappa,p} \) by the definition of \( K(f; t^2)_{\kappa,p} \) and (1.13) shows that

\[
\omega(f; n^{-1})_{\kappa,p} \leq CK(f; n^{-2})_{\kappa,p} \\
\leq C\left(\|f - J_{n,j,i,s}^m(f)\|_{\kappa,p} + n^{-2}\|\Delta_{n,0} J_{n,j,i,s}^m(f)\|_{\kappa,p}\right) \\
\leq C(\lambda, j, i, s)\|f - J_{n,j,i,s}(f)\|_{\kappa,p} \\
\leq C(\lambda, j, i, s)\|f - J_{n,j,i,s}(f)\|_{\kappa,p'}
\]

that is, \( \omega(f; n^{-1})_{\kappa,p} \ll \|f - J_{n,j,i,s}(f)\|_{\kappa,p'} \).

The proof is completed. \( \square \)

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References
