Research Article

General Univalence Criterion Associated with the \( n \)th Derivative

Oqlah Al-Refai\(^1\) and Maslina Darus\(^2\)

\(^1\) Department of Mathematics, Faculty of Science and Information Technology, Zarqa University, Zarqa 13132, Jordan
\(^2\) School of Mathematical Sciences, Universiti Kebangsaan Malaysia, Selangor, 43600 Bangi, Malaysia

Correspondence should be addressed to Maslina Darus, maslina@ukm.my

Received 23 February 2012; Revised 2 May 2012; Accepted 10 May 2012

Academic Editor: Saminathan Ponnusamy

Copyright \( \odot \) 2012 O. Al-Refai and M. Darus. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

For normalized analytic functions \( f(z) \) with \( f(z) \neq 0 \) for \( 0 < |z| < 1 \), we introduce a univalence criterion defined by sharp inequality associated with the \( n \)th derivative of \( z/f(z) \), where \( n \in \{3, 4, 5, \ldots\} \).

1. Introduction

Let \( \mathcal{A} \) denote the class of functions of the following form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are normalized analytic in the open unit disk \( U := \{ z : |z| < 1 \} \).

In [1], Aksentev proved that the condition

\[
\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1 \quad (1.2)
\]

or equivalently \( \text{Re}(f^2(z)/z^2 f'(z)) \geq 1/2 \), for \( z \in \mathbb{U} \), is sufficient for \( f(z) \in \mathcal{A} \) to be univalent in \( \mathbb{U} \). By virtue of the aforementioned result of Aksentev, the class of functions defined by (1.2) was extensively studied by Obradović and Ponnusamy [2, 3], Ozaki and Nunokawa [4],
Let us prove the following theorem.

**Theorem 2.1.** Let \( f(z) \in \mathcal{A} \) with \( f(z) \neq 0 \) for \( 0 < |z| < 1 \) and let \( g(z) \in \mathcal{A} \) be bounded in \( \mathbb{U} \) and satisfy

\[
m = \inf \left\{ \left| \frac{g(z_1) - g(z_2)}{z_1 - z_2} \right| : z_1, z_2 \in \mathbb{U} \right\} > 0.
\]  

(2.1)

For any \( n \in \{3, 4, \ldots\} \), if

\[
\left| \frac{d^n}{dz^n} \left( \frac{z}{f(z)} - \frac{z}{g(z)} \right) \right| \leq K \quad (z \in \mathbb{U}),
\]

(2.2)
where
\[ K = \frac{n!}{n-1} \left( \frac{m}{M^2} - \sum_{k=2}^{n-1} \frac{k-1}{k!} |\alpha_k| \right), \quad \alpha_k = \frac{d^k}{dz^k} \left( \frac{z - g(z)}{f(z)} \right) \bigg|_{z=0}, \tag{2.3} \]
and \( M = \sup\{|g(z)| : z \in U\} \), then \( f(z) \) is univalent in \( U \).

**Proof.** If we put
\[ h(z) = \frac{d^n}{dz^n} \left( \frac{z - g(z)}{f(z)} \right), \tag{2.4} \]
then the function \( h \) is analytic in \( U \) and, by integration from 0 to \( z \), we get
\[ \frac{d^{n-1}}{dz^{n-1}} \left( \frac{z - g(z)}{f(z)} \right) = \alpha_{n-1} + \int_0^z h(u_1)du_1. \tag{2.5} \]
Integrating both sides of the previous equation \((n-1)\)-times from 0 to \( z \) gives
\[ \frac{z}{f(z)} - \frac{z}{g(z)} = \sum_{k=1}^{n-1} \frac{\alpha_k}{k!} z^k + \int_0^z \int_0^{u_n} \int_0^{u_{n-1}} \cdots \int_0^{u_3} \int_0^{u_2} h(u_1)du_1. \tag{2.6} \]
Thus, we have
\[ f(z) = \frac{g(z)}{1 + g(z) \sum_{k=1}^{n-1} (\alpha_k / k!) z^{k-1} + g(z) (\psi(z)/z)}, \tag{2.7} \]
where
\[ \psi(z) = \int_0^z \int_0^{u_n} \int_0^{u_{n-1}} \cdots \int_0^{u_3} \int_0^{u_2} h(u_1)du_1. \tag{2.8} \]
Next, for \( n = 3 \), we have
\[ z^2 \left( \frac{\psi(z)}{z} \right)' = \int_0^z u \psi''(u)du = \int_0^z u \psi' \int_0^u h(u_1)du_1, \tag{2.9} \]
and for \( n = 4 \),

\[
z^2 \left( \frac{\psi(z)}{z} \right)' = \int_0^z u \psi''(u) \, du = \int_0^z u \, du \int_0^u \int_0^{u_2} h(u_1) \, du_1.
\]

(2.10)

In general, for \( n \in \{3, 4, \ldots\} \),

\[
z^2 \left( \frac{\psi(z)}{z} \right)' = \int_0^z u \psi''(u) \, du
\]

\[
= \int_0^z u \, du \int_0^u \int_0^{u_{n-2}} \int_0^{u_{n-3}} \cdots \int_0^{u_2} h(u_1) \, du_1
\]

\[
= \int_0^1 z^2 \, dt \int_0^{z_2} \int_0^{u_{n-2}} \int_0^{u_{n-3}} \cdots \int_0^{u_2} h(u_1) \, du_1 \quad \text{(by setting} \ u = zt) \]

\[
= \int_0^1 z^3 \, dt \int_0^1 \int_0^{u_{n-2}} \int_0^{u_{n-3}} \cdots \int_0^{u_2} h(u_1) \, du_1 \quad \text{(by setting} \ u_{n-2} = zts_1) \]

\[
= \int_0^1 z^4 \, dt \int_0^1 \int_0^{u_{n-2}} \int_0^{u_{n-3}} \cdots \int_0^{u_2} h(u_1) \, du_1 \quad \text{(by setting} \ u_{n-3} = zts_1s_2) \]

\[
= \int_0^1 z^n \, dt \int_0^1 \int_0^{u_{n-2}} \int_0^{u_{n-3}} \cdots \int_0^{u_2} h(u_1) \, du_1 \quad \text{(by setting} \ u_1 = zts_1s_2 \cdots s_{n-2}) \]

\[
\int_0^1 h(zts_1 \cdots s_{n-2}) \, ds_{n-2} \quad \text{(by setting} \ u_1 = zts_1s_2 \cdots s_{n-2}) \]

(2.11)

therefore

\[
\left| \left( \frac{\psi(z)}{z} \right)' \right| \leq \frac{|z|^{n-2}}{n} \cdot \frac{1}{n-2} \cdot \frac{1}{n-3} \cdots \frac{1}{2} \int_0^1 |h(zts_1s_2 \cdots s_{n-2})| \, ds_{n-2} \leq \frac{n-1}{n!} K,
\]

(2.12)

and so

\[
\left| \frac{\psi(z_2)}{z_2} - \frac{\psi(z_1)}{z_1} \right| = \left| \int_{z_1}^{z_2} \left( \frac{\psi(z)}{z} \right)' \, dz \right| \leq \frac{n-1}{n!} K |z_2 - z_1|
\]

(2.13)
for \( z_1, z_2 \in \mathbb{U} \) and \( z_1 \neq z_2 \). If \( z_1 \neq z_2 \), then \( g(z_1) \neq g(z_2) \), and it follows, from (2.7) and (2.13), that

\[
|f(z_1) - f(z_2)| = \left| g(z_1) - g(z_2) + g(z_1)g(z_2) \sum_{k=2}^{n-1} (a_k/k!) \left( z_2^{k-1} - z_1^{k-1} \right) + \frac{g(z_1)}{1 + g(z_2)} \sum_{k=2}^{n-1} (a_k/k!) z_2^{k-1} + g(z_2) (\psi(z_2)/z_2 - \psi(z_1)/z_1) \right|

\]

\[
\leq \left| g(z_1) - g(z_2) \right| \sum_{k=2}^{n-1} (a_k/k!) z_2^{k-1} + g(z_2) (\psi(z_2)/z_2 - \psi(z_1)/z_1) \right|

\]

\[
\leq \left| g(z_1) - g(z_2) \right| - M^2 \left| z_1 - z_2 \right| \sum_{k=2}^{n-1} (a_k/k!) \left| z_2^{k-1} - z_1^{k-1} \right| - \frac{(n-1)}{(n!)K^2} \left| z_1 - z_2 \right|

\]

\[
\leq \left| g(z_1) - g(z_2) \right| - M^2 \left| z_1 - z_2 \right| \sum_{k=2}^{n-1} (a_k/(k-1)/k!) \left| z_2^{k-1} - z_1^{k-1} \right| - \frac{(n-1)}{(n!)K^2} \left| z_1 - z_2 \right|

\]

\[
\geq 0.

\] (2.14)

Hence, \( f(z) \) is univalent in \( \mathbb{U} \).

\[\square\]

**Corollary 2.2.** Let \( f(z) \in \mathcal{A} \) with \( f(z) \neq 0 \) when \( 0 < |z| < 1 \). For any \( n \in \{3, 4, \ldots\} \), if

\[
\sum_{k=2}^{n-1} \frac{k-1}{k!} |\beta_k| + \frac{n-1}{n!} \left| \frac{d^n}{dz^n} \left( \frac{z}{f(z)} \right) \right| \leq 1 \quad (z \in \mathbb{U}),

\] (2.15)

where \( \beta_k = (d^k/dz^k)(z/f(z))\big|_{z=0} \), then \( f(z) \) is univalent in \( \mathbb{U} \). The result is sharp, where equality occurs for the Koebe function \( k(z) = z/(1 - z)^2 \) and also for functions of the following form:

\[
f(z) = \frac{z}{1 + az + z^2}, \quad (|a| \leq 2), \quad f_n(z) = \frac{z}{(1 + (1/(n-2)))z^{n-1}}.

\] (2.16)

**Proof.** Setting \( g(z) = z \) in Theorem 2.1 immediately yields (2.15). To show that the result is sharp for \( n \geq 3 \), we consider

\[
f(z) = \frac{z}{(1 + (1/(n-2)))z^{n-1}} \quad (e > 0).

\] (2.17)

A computation shows, for \( 1 \leq k \leq n-1 \), that

\[
\frac{d^k}{dz^k} \left( \frac{z}{f(z)} \right) = (n-2)^{-k}(e + n - 1)(e + n - 2) \cdots (e + n - k) \left( 1 + \frac{1}{n-2} z \right)^{e+n-k-1}.

\] (2.18)
Letting $\epsilon = 0$ in (2.17) and (2.18) implies, respectively, that $(d^n/dz^n)(z/f(z)) = 0$ and

$$|\beta_k| = \frac{(n-1)!}{(n-k-1)!(n-2)^k}. \quad (2.19)$$

This satisfies the equality in (2.15), because for $x \in \mathbb{R}$ and $n \geq 3$, an application of the binomial theorem gives

$$(1 + x)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k, \quad (2.20)$$

and so

$$\sum_{k=2}^{n-1} (k-1) \binom{n-1}{k} x^k = 1 + (n-1)(1 + x)^{n-2} x - (1 + x)^{n-1}$$

$$= 1 + (1 + x)^{n-2} [x(n-2) - 1]. \quad (2.21)$$

Choosing $x = 1/(n - 2)$ in assertion (2.21) gives the equality. However, for every $\epsilon > 0$, we have

$$f'(\frac{n-2}{n-2+\epsilon}) = 0. \quad (2.22)$$

Hence $f$ is not univalent in $U$ and the result is sharp. Moreover it can be easily checked that the equality in (2.15) holds for the given functions and the proof is complete. \qed

### 3. Special Cases and Examples

Letting $n = 2$ in inequality (2.15) gives the univalence criterion defined by (1.4), which is due to Yang and Liu [7]. Next, we reduce the result for some values of $n$ by computing the corresponding values of $\beta_k$ in terms of the coefficients. More precisely, for $n = 3$ and $n = 4$, Corollary 2.2 reduces at once to the following two remarks.

**Remark 3.1.** Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z) \neq 0$ when $0 < |z| < 1$ satisfy

$$\left| \left( \frac{z}{f(z)} \right)^n \right| \leq 3 - 3 |a_2 - a_3| \quad (z \in \mathbb{U}). \quad (3.1)$$

Then $f(z)$ is univalent in $U$. The bound in (3.1) is best possible, where equality occurs for the Koebe function and for functions of the following form:

$$f(z) = \frac{z}{1 + az + z^2} \quad (|a| \leq 2). \quad (3.2)$$
Remark 3.2. Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \) with \( f(z) \neq 0 \) for \( 0 < |z| < 1 \) satisfy
\[
\frac{d^4}{dz^4} \left( \frac{z}{f(z)} \right) \leq 8 - 8|a_2^2 - a_3| - 16|a_4 - 2a_2a_3 + a_2^2| \quad (z \in \mathbb{U}).
\] (3.3)

Then \( f(z) \) is univalent in \( \mathbb{U} \). The bound in (3.3) is best possible, where equality occurs for the Koebe function and also for functions of the following form:
\[
f(z) = \frac{z}{1 + az + z^2} \quad (|a| \leq 2), \quad f(z) = \frac{z}{(1 \pm (1/2)z)^3}.
\] (3.4)

Proof. The result follows from taking \( n = 3 \) in Corollary 2.2 and that \( |\beta_2| = 2|a_2^2 - a_3| \).

To understand the behavior of the extremal functions for our criterion (2.15), let us consider, for example, \( f(z) = z/(1 - (1/2)z)^3 \), which is an extremal function for the case \( n = 4 \). Figures 1(a) and 1(b) show the images of the unit circle under the functions \( f(z) \) and \( g(z) = z/(1 - (1/2)z)^{3.05} \), respectively. If we restrict the images around the cusps as shown in Figures 1(c) and 1(d), we see that the image of \( g \) is a curve that intersects itself in some purely real point \( u \). This means that there are two different points \( z_1 \) and \( z_2 \) that lie on the unit circle such that \( g(z_1) = g(z_2) = u \). In fact, each purely real point lies inside the closed curve of Figures 1(c) and 1(d) which is an image for two different points in \( \mathbb{U} \) having the same modulus but different arguments. However, we cannot find such points for the function \( f \), and this interprets why \( f \) is an extremal function for univalence, since the closed curve of Figure 1(d) vanishes whenever the power in the function \( g \) approaches to 3 as shown in Figure 1(c).

From Corollary 2.2, we have the following.

**Corollary 3.3.** Let
\[
f(z) = \frac{z}{1 + \sum_{k=1}^{\infty} b_k z^k} \in \mathcal{A}
\] (3.5)

with \( f(z) \neq 0 \) for \( 0 < |z| < 1 \) and
\[
\sum_{k=2}^{n} (k - 1)|b_k| + (n - 1) \sum_{k=n+1}^{\infty} \binom{k}{n} |b_k| \leq 1,
\] (3.6)

for some \( n \in \{2, 3, \ldots\} \). Then \( f(z) \) is univalent in \( \mathbb{U} \).

Proof. In view of (3.5) and by simple computation we have
\[
\frac{d^n}{dz^n} \left( \frac{z}{f(z)} \right) = n!b_n + \sum_{k=n+1}^{\infty} \frac{k!}{(k-n)!} b_k z^{k-n},
\] (3.7)
and so $\beta_m = m!b_m$, for $1 \leq m \leq n - 1$. It follows that
\[
\left| \frac{d^n}{dz^n} \left( \frac{z}{f(z)} \right) \right| \leq \sum_{k=n}^{\infty} \frac{k!|b_k|}{(k-n)!}. \tag{3.8}
\]
Hence, by applying Corollary 2.2, we get the desired result. \hfill \square

Remark 3.4. Taking $n = 2$ in Corollary 3.3 gives a result of Yang and Liu [7].

Example 3.5. From Corollary 3.3, it can be easily seen that the functions
\[
f(z) = \frac{z}{1 + \sum_{k=1}^{n} b_k z^k}, \tag{3.9}
\]
with $f(z) \neq 0$ for $0 < |z| < 1$ and $\sum_{k=2}^{n} (k-1)|b_k| \leq 1$, are univalent in $U$. 

Figure 1: Geometric description for the sharpness of the case $n = 4$. 
Acknowledgments

The authors gratefully thank the referees for their remarkable comments and gratefully acknowledge the financial support received in the form of the Research Grant UKM-ST-06-FRGS0244-2010 from the Universiti Kebangsaan Malaysia.

References

Submit your manuscripts at http://www.hindawi.com