Research Article

Algorithmic Approach to a Minimization Problem

**Yonghong Yao,¹ Shin Min Kang,²
 Yeong-Cheng Liou,³ and Zhitao Wu¹**

¹ Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China

² Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea

³ Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan

Correspondence should be addressed to Shin Min Kang, smkang@gnu.ac.kr

Received 27 April 2012; Accepted 7 May 2012

Academic Editor: Rudong Chen

Copyright © 2012 Yonghong Yao et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We first construct an implicit algorithm for solving the minimization problem $\min_{x \in \Omega} \|x\|$, where Ω is the intersection set of the solution set of some equilibrium problem, the fixed points set of a non-expansive mapping, and the solution set of some variational inequality. Further, we suggest an explicit algorithm by discretizing this implicit algorithm. We prove that the proposed implicit and explicit algorithms converge strongly to a solution of the above minimization problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H . Recall that a mapping $A : C \rightarrow H$ is called α -inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \quad (1.1)$$

A mapping $S : C \rightarrow C$ is said to be *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Denote the set of fixed points of S by $F(S)$.

Let $B : C \rightarrow H$ be a nonlinear mapping and $F : C \times C \rightarrow R$ be a bifunction. Now we concern the following equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Bz, y - z \rangle \geq 0, \quad \forall y \in C. \quad (1.2)$$

The solution set of (1.2) is denoted by $EP(F, B)$. If $B = 0$, then (1.2) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C. \quad (1.3)$$

The solution set of (1.3) is denoted by $EP(F)$. If $F = 0$, then (1.2) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Bz, y - z \rangle \geq 0, \quad \forall y \in C. \quad (1.4)$$

The solution set of variational inequality (1.4) is denoted by $VI(C, B)$.

Equilibrium problems which were introduced by Blum and Oettli [1] in 1994 have had a great impact and influence in pure and applied sciences. It has been shown that the equilibrium problems theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. Equilibrium problems include variational inequalities, fixed point, Nash equilibrium, and game theory as special cases. The equilibrium problems and the variational inequality problems have been investigated by many authors. Please see [2–35] and the references therein. The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others.

On the other hand, we also notice that it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. For instance, given a closed convex subset C of a Hilbert space H_1 and a bounded linear operator $R : H_1 \rightarrow H_2$, where H_2 is another Hilbert space. The C -constrained pseudoinverse of R , $R_C^\dagger$, is then defined as the minimum-norm solution of the constrained minimization problem:

$$R_C^\dagger(b) := \arg \min_{x \in C} \|Rx - b\|, \quad (1.5)$$

which is equivalent to the fixed point problem:

$$x = P_C(x - \lambda R^*(Rx - b)), \quad (1.6)$$

where P_C is the metric projection from H_1 onto C , R^* is the adjoint of R , $\lambda > 0$ is a constant, and $b \in H_2$ is such that $P_{\overline{R(C)}}(b) \in R(C)$.

It is therefore an interesting problem to invent some algorithms that can generate schemes which converge strongly to the minimum-norm solution of a given problem.

In this paper, we focus on the following minimization problem: find $x^* \in \Omega$ such that

$$x^* = \arg \min_{x \in \Omega} \|x\|, \quad (1.7)$$

where Ω is the intersection set of the solution set of some equilibrium problem, the fixed points set of a nonexpansive mapping, and the solution set of some variational inequality. We will suggest and analyze two very simple algorithms for solving the above minimization problem.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H . Throughout this paper, we assume that a bifunction $F : C \times C \rightarrow R$ satisfies the following conditions:

- (H1) $F(x, x) = 0$ for all $x \in C$;
- (H2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (H3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
- (H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \rightarrow C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property:

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C). \quad (2.1)$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (2.2)$$

We need the following well-known lemmas for proving our main results.

Lemma 2.1 (see [13]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $F : C \times C \rightarrow R$ be a bifunction which satisfies conditions (H1)–(H4). Let $\mu > 0$ and $x \in C$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{\mu} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \quad (2.3)$$

Further, if $T_\mu(x) = \{z \in C : F(z, y) + (1/\mu) \langle y - z, z - x \rangle \geq 0, \text{ for all } y \in C\}$, then the following hold:

- (a) T_μ is single-valued and T_μ is firmly nonexpansive, that is, for any $x, y \in C$, $\|T_\mu x - T_\mu y\|^2 \leq \langle T_\mu x - T_\mu y, x - y \rangle$;
- (b) $\text{EP}(F)$ is closed and convex and $\text{EP}(F) = F(T_\mu)$.

Lemma 2.2 (see [27]). *Let $\{x_n\}$ and $\{v_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)v_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0$.*

Lemma 2.3 (see [29]). *Let C be a closed convex subset of a real Hilbert space H and let $S : C \rightarrow C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightarrow x^*$ weakly and $(I - S)x_n \rightarrow y$ strongly, then $(I - S)x^* = y$.*

Lemma 2.4 (see [29]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n \gamma_n, \quad (2.4)$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
- (b) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main Results

In this section we will introduce two algorithms (one implicit and one explicit) for finding the minimum norm element x^* of $\Omega := \text{EP}(F, B) \cap \text{VI}(C, A) \cap F(S)$. Namely, we want to find a point x^* which solves the following minimization problem:

$$x^* = \arg \min_{x \in \Omega} \|x\|. \quad (3.1)$$

Let $S : C \rightarrow C$ be a nonexpansive mapping and $A, B : C \rightarrow H$ be α -inverse-strongly monotone and β -inverse-strongly monotone mappings, respectively. Let $F : C \times C \rightarrow R$ be a bifunction which satisfies conditions (H1)–(H4). In order to solve the minimization problem (3.1), we first construct the following implicit algorithm by using the projection method:

$$x_t = SP_C[(1-t)(I - \lambda A)T_\mu(I - \mu B)x_t], \quad \forall t \in (0, 1), \quad (3.2)$$

where T_μ is defined as Lemma 2.1 and λ, μ are two constants such that $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$. We will show that the net $\{x_t\}$ defined by (3.2) converges to a solution of the minimization problem (3.1). First, we show that the net $\{x_t\}$ is well defined. As matter of fact, for each $t \in (0, 1)$, we consider the following mapping W_t given by

$$W_t x = SP_C[(1-t)(I - \lambda A)T_\mu(I - \mu B)x], \quad \forall x \in C. \quad (3.3)$$

Since the mappings $S, P_C, I - \lambda A, T_\mu$ and $I - \mu B$ are nonexpansive, then we can check easily that $\|W_t x - W_t y\| \leq (1-t)\|x - y\|$ which implies that W_t is a contraction. Using the Banach contraction principle, there exists a unique fixed point x_t of W_t in C , that is, $x_t = W_t x_t$ which is exactly (3.2).

Next we show the first main result of the present paper.

Theorem 3.1. *Suppose that $\Omega \neq \emptyset$. Then the net $\{x_t\}$ generated by the implicit method (3.2) converges in norm, as $t \rightarrow 0$, to a solution of the minimization problem (3.1).*

Proof. Take $z \in \Omega$. First we need use the following facts:

- (1) $z = Sz = T_\mu(z - \mu Bz) = P_C(I - \lambda A)z = SP_C[(I - \lambda A)T_\mu(I - \mu B)z]$ for all $\lambda > 0, \mu > 0$.
In particular, for all $t \in (0, 1)$,

$$\begin{aligned} z &= P_C[z - (1-t)\lambda Az] \\ &= P_C[tz + (1-t)(I - \lambda A)z] \\ &= SP_C[tz + (1-t)(I - \lambda A)T_\mu(I - \mu B)z]; \end{aligned} \quad (3.4)$$

(2) $I - \lambda A$ and $I - \mu B$ are nonexpansive and for all $x, y \in C$

$$\begin{aligned}\|(I - \lambda A)x - (I - \lambda A)y\|^2 &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2; \\ \|(I - \mu B)x - (I - \mu B)y\|^2 &\leq \|x - y\|^2 + \mu(\mu - 2\beta)\|Bx - By\|^2.\end{aligned}\quad (3.5)$$

Set $u_t = T_\mu(I - \mu B)x_t$ and $y_t = (I - \lambda A)u_t$ for all $t \in (0, 1)$. It follows that

$$\begin{aligned}\|u_t - z\|^2 &= \|T_\mu(x_t - \mu Bx_t) - T_\mu(z - \mu Bz)\|^2 \\ &\leq \|(I - \mu B)x_t - (I - \mu B)z\|^2 \\ &\leq \|x_t - z\|^2 + \mu(\mu - 2\beta)\|Bx_t - Bz\|^2.\end{aligned}\quad (3.6)$$

From (3.2), we have

$$\begin{aligned}\|x_t - z\| &= \|SP_C[(1 - t)(I - \lambda A)T_\mu(I - \mu B)x_t] - SP_C[tz + (1 - t)(I - \lambda A)T_\mu(I - \mu B)z]\| \\ &\leq \|[(1 - t)(I - \lambda A)T_\mu(I - \mu B)x_t] - [tz + (1 - t)(I - \lambda A)T_\mu(I - \mu B)z]\| \\ &\leq (1 - t)\|(I - \lambda A)T_\mu(I - \mu B)x_t - (I - \lambda A)T_\mu(I - \mu B)z\| + t\|z\| \\ &\leq (1 - t)\|x_t - z\| + t\|z\|,\end{aligned}\quad (3.7)$$

that is,

$$\|x_t - z\| \leq \|z\|. \quad (3.8)$$

So, $\{x_t\}$ is bounded. Hence $\{u_t\}$, $\{Ax_t\}$, and $\{Bx_t\}$ are also bounded. Next we will use $M > 0$ to denote some possible constant appearing in the following.

From (3.7), we have

$$\begin{aligned}\|x_t - z\|^2 &\leq \|[(1 - t)(I - \lambda A)T_\mu(I - \mu B)x_t] - [tz + (1 - t)(I - \lambda A)T_\mu(I - \mu B)z]\|^2 \\ &= \|(1 - t)[(I - \lambda A)T_\mu(I - \mu B)x_t - (I - \lambda A)T_\mu(I - \mu B)z] - tz\|^2 \\ &= (1 - t)^2\|(I - \lambda A)u_t - (I - \lambda A)z\|^2 \\ &\quad - 2t(1 - t)\langle z, (I - \lambda A)u_t - (I - \lambda A)z \rangle + t^2\|z\|^2 \\ &\leq \|u_t - z\|^2 + \lambda(\lambda - 2\alpha)\|Au_t - Az\|^2 + tM \\ &\leq \|x_t - z\|^2 + \mu(\mu - 2\beta)\|Bx_t - Bz\|^2 + \lambda(\lambda - 2\alpha)\|Au_t - Az\|^2 + tM,\end{aligned}\quad (3.9)$$

that is,

$$\lambda(2\alpha - \lambda)\|Au_t - Az\|^2 + \mu(2\beta - \mu)\|Bx_t - Bz\|^2 \leq tM \longrightarrow 0. \quad (3.10)$$

Since $\lambda(2\alpha - \lambda) > 0$ and $\mu(2\beta - \mu) > 0$, we derive

$$\lim_{t \rightarrow 0} \|Au_t - Az\| = \lim_{t \rightarrow 0} \|Bx_t - Bz\| = 0. \quad (3.11)$$

From Lemma 2.1 and (2.2), we obtain

$$\begin{aligned} \|u_t - z\|^2 &= \|T_\mu(x_t - \mu Bx_t) - T_\mu(z - \mu Bz)\|^2 \\ &\leq \langle (x_t - \mu Bx_t) - (z - \mu Bz), u_t - z \rangle \\ &= \frac{1}{2} \left(\|(x_t - \mu Bx_t) - (z - \mu Bz)\|^2 + \|u_t - z\|^2 \right. \\ &\quad \left. - \|(x_t - z) - \mu(Bx_t - Bz) - (u_t - z)\|^2 \right) \\ &\leq \frac{1}{2} \left(\|x_t - z\|^2 + \|u_t - z\|^2 - \|(x_t - u_t) - \mu(Bx_t - Bz)\|^2 \right) \\ &= \frac{1}{2} \left(\|x_t - z\|^2 + \|u_t - z\|^2 - \|x_t - u_t\|^2 \right. \\ &\quad \left. + 2\mu \langle x_t - u_t, Bx_t - Bz \rangle - \mu^2 \|Bx_t - Bz\|^2 \right). \end{aligned} \quad (3.12)$$

It follows that

$$\begin{aligned} \|u_t - z\|^2 &\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2\mu \langle x_t - u_t, Bx_t - Bz \rangle - \mu^2 \|Bx_t - Bz\|^2 \\ &\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2\mu \|x_t - u_t\| \|Bx_t - Bz\| \\ &\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + M \|Bx_t - Bz\|. \end{aligned} \quad (3.13)$$

Set $z_t = P_C[(1-t)y_t]$ for all $t \in (0, 1)$. By Lemma 2.1 and (2.2), we have

$$\begin{aligned} \|z_t - z\|^2 &= \|P_C[(1-t)(I - \lambda A)u_t] - P_C[(I - \lambda A)z]\|^2 \\ &\leq \langle (1-t)(I - \lambda A)u_t - (I - \lambda A)z, z_t - z \rangle \\ &= \frac{1}{2} \left\{ \|(I - \lambda A)u_t - (I - \lambda A)z - ty_t\|^2 + \|z_t - z\|^2 \right. \\ &\quad \left. - \|(I - \lambda A)u_t - (I - \lambda A)z - (z_t - z) - ty_t\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|(I - \lambda A)u_t - (I - \lambda A)z\|^2 + tM + \|z_t - z\|^2 \right. \\ &\quad \left. - \|u_t - z_t - \lambda(Au_t - Az) - ty_t\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|u_t - z\|^2 + tM + \|z_t - z\|^2 - \|u_t - z_t\|^2 + 2\lambda \langle u_t - z_t, Au_t - Az \rangle \right. \\ &\quad \left. + 2t \langle y_t, u_t - z_t \rangle - \|\lambda A(u_t - Az) + ty_t\|^2 \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \left\{ \|u_t - z\|^2 + tM + \|z_t - z\|^2 - \|u_t - z_t\|^2 + 2\lambda \|u_t - z_t\| \|Au_t - Az\| \right. \\
&\quad \left. + 2t \|y_t\| \|u_t - z_t\| \right\},
\end{aligned} \tag{3.14}$$

that is,

$$\begin{aligned}
\|z_t - z\|^2 &\leq \|u_t - z\|^2 + tM - \|u_t - z_t\|^2 + 2\lambda \|u_t - z_t\| \|Au_t - Az\| + 2t \|y_t\| \|u_t - z_t\| \\
&\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + M \|Bx_t - Bz\| + tM - \|u_t - z_t\|^2 \\
&\quad + 2\lambda \|u_t - z_t\| \|Au_t - Az\| + 2t \|y_t\| \|u_t - z_t\|.
\end{aligned} \tag{3.15}$$

Therefore, we have

$$\begin{aligned}
\|x_t - z\|^2 &= \|Sz_t - z\|^2 \\
&\leq \|z_t - z\|^2 \\
&\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + M \|Bx_t - Bz\| + tM - \|u_t - z_t\|^2 \\
&\quad + 2\lambda \|u_t - z_t\| \|Au_t - Az\| + 2t \|y_t\| \|u_t - z_t\|.
\end{aligned} \tag{3.16}$$

Hence, we deduce

$$\begin{aligned}
&\|x_t - u_t\|^2 + \|u_t - z_t\|^2 \\
&\leq M(\|Bx_t - Bz\| + t) + 2\lambda \|u_t - z_t\| \|Au_t - Az\| + 2t \|y_t\| \|u_t - z_t\| \\
&\longrightarrow 0.
\end{aligned} \tag{3.17}$$

This denotes that

$$\lim_{t \rightarrow 0} \|x_t - u_t\| = \lim_{t \rightarrow 0} \|u_t - z_t\| = 0. \tag{3.18}$$

Note that

$$\|z_t - y_t\| = \|P_C[(1-t)y_t] - P_C y_t\| \leq t \|y_t\| \longrightarrow 0, \tag{3.19}$$

thus,

$$\begin{aligned}
\|y_t - Sy_t\| &\leq \|y_t - z_t\| + \|z_t - u_t\| + \|u_t - x_t\| + \|x_t - Sy_t\| \\
&\leq 2\|y_t - z_t\| + \|z_t - u_t\| + \|u_t - x_t\| \\
&\longrightarrow 0.
\end{aligned} \tag{3.20}$$

Next we show that $\{x_t\}$ is relatively norm compact as $t \rightarrow 0$. Let $\{t_n\} \subset (0, 1)$ be a sequence such that $t_n \rightarrow 0$ as $n \rightarrow \infty$. Put $x_n := x_{t_n}$, $u_n := u_{t_n}$ and $y_n := y_{t_n}$. From (3.20), we get

$$\|y_n - Sy_n\| \rightarrow 0. \quad (3.21)$$

By (3.2), we have

$$\begin{aligned} \|x_t - z\|^2 &= \|SP_C[(1-t)y_t] - SP_C z\|^2 \\ &\leq \|y_t - z - ty_t\|^2 \\ &= \|y_t - z\|^2 - 2t\langle y_t, y_t - z \rangle + t^2\|y_t\|^2 \\ &= \|y_t - z\|^2 - 2t\langle y_t - z, y_t - z \rangle - 2t\langle z, y_t - z \rangle + t^2\|y_t\|^2 \\ &\leq (1-2t)\|y_t - z\|^2 + 2t\langle z, z - y_t \rangle + t^2\|y_t\|^2 \\ &\leq (1-2t)\|x_t - z\|^2 + 2t\langle z, z - y_t \rangle + t^2\|y_t\|^2. \end{aligned} \quad (3.22)$$

It follows that

$$\|x_t - z\|^2 \leq \langle z, z - y_t \rangle + \frac{tM}{2}. \quad (3.23)$$

In particular,

$$\|x_n - z\|^2 \leq \langle z, z - y_n \rangle + \frac{t_n M}{2}, \quad z \in \Omega. \quad (3.24)$$

Since $\{y_n\}$ is bounded, without loss of generality, we may assume that $\{y_n\}$ converges weakly to a point $x^* \in C$. Hence, u_n and x_n also converge weakly to x^* . Noticing (3.21) we can use Lemma 2.3 to get $x^* \in F(S)$.

Now we show $x^* \in \text{EP}(F, B)$. Since $u_n = T_\mu(x_n - \mu Bx_n)$, for any $y \in C$ we have

$$F(u_n, y) + \frac{1}{\mu} \langle y - u_n, u_n - (x_n - \mu Bx_n) \rangle \geq 0. \quad (3.25)$$

From the monotonicity of F , we have

$$\frac{1}{\mu} \langle y - u_n, u_n - (x_n - \mu Bx_n) \rangle \geq F(y, u_n), \quad \forall y \in C. \quad (3.26)$$

Hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\mu} + Bx_{n_i} \right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \quad (3.27)$$

Put $v_t = ty + (1-t)x^*$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $v_t \in C$. So, from (3.27) we have

$$\begin{aligned} \langle v_t - u_{n_i}, Bv_t \rangle &\geq \langle v_t - u_{n_i}, Bv_t \rangle - \left\langle v_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\mu} + Bx_{n_i} \right\rangle + F(v_t, u_{n_i}) \\ &= \langle v_t - u_{n_i}, Bv_t - Bu_{n_i} \rangle + \langle v_t - u_{n_i}, Bu_{n_i} - Bx_{n_i} \rangle \\ &\quad - \left\langle v_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\mu} \right\rangle + F(v_t, u_{n_i}). \end{aligned} \quad (3.28)$$

Note that $\|Bu_{n_i} - Bx_{n_i}\| \leq (1/\beta)\|u_{n_i} - x_{n_i}\| \rightarrow 0$. Further, from monotonicity of B , we have $\langle v_t - u_{n_i}, Bv_t - Bu_{n_i} \rangle \geq 0$. Letting $i \rightarrow \infty$ in (3.28), we have

$$\langle v_t - x^*, Bv_t \rangle \geq F(v_t, x^*). \quad (3.29)$$

From (H1), (H4), and (3.29), we also have

$$\begin{aligned} 0 &= F(v_t, v_t) \leq tF(v_t, y) + (1-t)F(v_t, x^*) \\ &\leq tF(v_t, y) + (1-t)\langle v_t - x^*, Bv_t \rangle \\ &= tF(v_t, y) + (1-t)t\langle y - x^*, Bv_t \rangle, \end{aligned} \quad (3.30)$$

and hence

$$0 \leq F(v_t, y) + (1-t)\langle Bv_t, y - x^* \rangle. \quad (3.31)$$

Letting $t \rightarrow 0$ in (3.31), we have, for each $y \in C$,

$$0 \leq F(x^*, y) + \langle y - x^*, Bx^* \rangle. \quad (3.32)$$

This implies that $x^* \in \text{EP}(F, B)$. By the same argument as that of [13], we have $x^* \in \text{VI}(C, A)$. Therefore, $x^* \in \Omega$.

We substitute x^* for z in (3.24) to get

$$\|x_n - x^*\|^2 \leq \langle x^*, x^* - y_n \rangle + \frac{t_n}{2}M. \quad (3.33)$$

Hence, the weak convergence of $\{y_n\}$ to x^* implies that $x_n \rightarrow x^*$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \rightarrow 0$.

Now we return to (3.24) and take the limit as $n \rightarrow \infty$ to get

$$\|x^* - z\|^2 \leq \langle z, z - x^* \rangle, \quad z \in \Omega. \quad (3.34)$$

To show that the entire net $\{x_t\}$ converges to x^* , assume $x_{s_n} \rightarrow \tilde{x} \in \Omega$, where $s_n \rightarrow 0$. In (3.34), we take $z = \tilde{x}$ to get

$$\|x^* - \tilde{x}\|^2 \leq \langle \tilde{x}, \tilde{x} - x^* \rangle. \quad (3.35)$$

Interchange x^* and $\tilde{x}$ to obtain

$$\|\tilde{x} - x^*\|^2 \leq \langle x^*, x^* - \tilde{x} \rangle. \quad (3.36)$$

Adding up (3.35) and (3.36) yields

$$2\|x^* - \tilde{x}\|^2 \leq \|x^* - \tilde{x}\|^2, \quad (3.37)$$

which implies that $\tilde{x} = x^*$.

We note that (3.34) is equivalent to

$$\|x^*\|^2 \leq \langle x^*, z \rangle, \quad z \in \Omega. \quad (3.38)$$

This clearly implies that

$$\|x^*\| \leq \|z\|, \quad z \in \Omega. \quad (3.39)$$

Therefore, x^* solves the minimization problem (3.1). This completes the proof. $\square$

Next we introduce an explicit algorithm for finding a solution of the minimization problem (3.1). This scheme is obtained by discretizing the implicit scheme (3.2). We will show the strong convergence of this algorithm.

Theorem 3.2. *Suppose that $\Omega \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) SP_C [(1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)x_n], \quad n \geq 0, \quad (3.40)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ satisfying the following conditions:

- (a) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (b) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to a solution of the minimization problem (3.1).

Proof. Take $z \in \Omega$. First we need use the following fact:

$z = Sz = T_\mu(I - \mu B)z = P_C(I - \lambda A)z = SP_C[(I - \lambda A)T_\mu(I - \mu B)z]$ for all $\lambda > 0, \mu > 0$. In particular,

$$\begin{aligned} z &= P_C[z - (1 - \alpha_n)\lambda Az] \\ &= P_C[\alpha_n z + (1 - \alpha_n)(I - \lambda A)z] \\ &= SP_C[\alpha_n z + (1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)z], \quad \forall n \geq 0. \end{aligned} \quad (3.41)$$

Set $u_n = T_\mu(x_n - \mu Bx_n)$, $y_n = (I - \lambda A)u_n$ and $z_n = P_C[(1 - \alpha_n)y_n]$ for all $n \geq 0$. From (3.40), we get

$$\begin{aligned} \|x_{n+1} - z\| &= \|\beta_n(x_n - z) + (1 - \beta_n)(SP_C[(1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)x_n] \\ &\quad - SP_C[\alpha_n z + (1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)z])\| \\ &\leq \beta_n\|x_n - z\| + (1 - \beta_n)\|(1 - \alpha_n)[(I - \lambda A)T_\mu(I - \mu B)x_n \\ &\quad - (I - \lambda A)T_\mu(I - \mu B)z] - \alpha_n z\| \\ &\leq \beta_n\|x_n - z\| + (1 - \beta_n)[(1 - \alpha_n)\|x_n - z\| + \alpha_n\|z\|] \\ &= [1 - (1 - \beta_n)\alpha_n]\|x_n - z\| + \alpha_n(1 - \beta_n)\|z\| \\ &\leq \max\{\|x_n - z\|, \|z\|\}. \end{aligned} \quad (3.42)$$

By induction, we obtain, for all $n \geq 0$,

$$\|x_n - z\| \leq \max\{\|x_0 - z\|, \|z\|\}. \quad (3.43)$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ are all bounded. We will use $M > 0$ to denote some possible constant appearing in the following.

Define $x_{n+1} = \beta_n x_n + (1 - \beta_n)v_n$ for all $n \geq 0$. It follows that

$$\begin{aligned} \|v_{n+1} - v_n\| &= \|SP_C[(1 - \alpha_{n+1})y_{n+1}] - SP_C[(1 - \alpha_n)y_n]\| \\ &\leq \|(1 - \alpha_{n+1})y_{n+1} - (1 - \alpha_n)y_n\| \\ &\leq \|y_{n+1} - y_n\| + \alpha_{n+1}\|y_{n+1}\| + \alpha_n\|y_n\| \\ &\leq \|(I - \lambda A)u_{n+1} - (I - \lambda A)u_n\| + M(\alpha_{n+1} + \alpha_n) \\ &\leq \|u_{n+1} - u_n\| + M(\alpha_{n+1} + \alpha_n) \\ &= \|T_\mu(I - \mu B)x_{n+1} - T_\mu(I - \mu B)x_n\| + M(\alpha_{n+1} + \alpha_n) \\ &\leq \|x_{n+1} - x_n\| + M(\alpha_{n+1} + \alpha_n). \end{aligned} \quad (3.44)$$

This together with (a) implies that

$$\limsup_{n \rightarrow \infty} (\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (3.45)$$

Hence by Lemma 2.2, we get

$$\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0. \quad (3.46)$$

Therefore,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \|v_n - x_n\| = 0. \quad (3.47)$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\beta_n(x_n - z) + (1 - \beta_n)(v_n - z)\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|P_C[(1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)x_n] \\ &\quad - P_C[\alpha_n z + (1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)z]\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|(1 - \alpha_n)[(I - \lambda A)T_\mu(I - \mu B)x_n \\ &\quad - (I - \lambda A)T_\mu(I - \mu B)z] - \alpha_n z\|^2 \\ &= \beta_n \|x_n - z\|^2 + (1 - \beta_n) \left[(1 - \alpha_n)^2 \|(I - \lambda A)u_n - (I - \lambda A)z\|^2 \right. \\ &\quad \left. - 2\alpha_n(1 - \alpha_n) \langle (I - \lambda A)u_n - (I - \lambda A)z, z \rangle + \alpha_n^2 \|z\|^2 \right] \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \left[\|u_n - z\|^2 + \lambda(\lambda - 2\alpha) \|Au_n - Az\|^2 + \alpha_n M \right] \\ &= \beta_n \|x_n - z\|^2 + (1 - \beta_n) \left[\|T_\mu(I - \mu B)x_n - T_\mu(I - \mu B)z\|^2 \right. \\ &\quad \left. + \lambda(\lambda - 2\alpha) \|Au_n - Az\|^2 + \alpha_n M \right] \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \left[\|x_n - z\|^2 + \mu(\mu - 2\beta) \|Bx_n - Bz\|^2 \right. \\ &\quad \left. + \lambda(\lambda - 2\alpha) \|Au_n - Az\|^2 + \alpha_n M \right]. \end{aligned} \quad (3.48)$$

It follows that

$$\begin{aligned} &(1 - \beta_n) \lambda(2\alpha - \lambda) \|Au_n - Az\|^2 + (1 - \beta_n) \mu(2\beta - \mu) \|Bx_n - Bz\|^2 \\ &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M \\ &\leq (\|x_n - z\| + \|x_{n+1} - z\|) \|x_n - x_{n+1}\| + \alpha_n M. \end{aligned} \quad (3.49)$$

Since $\liminf_{n \rightarrow \infty} (1 - \beta_n) \lambda (2\alpha - \lambda) > 0$, $\liminf_{n \rightarrow \infty} (1 - \beta_n) \mu (2\beta - \mu) > 0$, $\|x_n - x_{n+1}\| \rightarrow 0$ and $\alpha_n \rightarrow 0$, we derive

$$\lim_{n \rightarrow \infty} \|Au_n - Az\| = \lim_{n \rightarrow \infty} \|Bx_n - Bz\| = 0. \quad (3.50)$$

From Lemma 2.1 and (2.2), we obtain

$$\begin{aligned} & \|u_n - z\|^2 \\ &= \|T_\mu(x_n - \mu Bx_n) - T_\mu(z - \mu Bz)\|^2 \\ &\leq \langle (x_n - \mu Bx_n) - (z - \mu Bz), u_n - z \rangle \\ &= \frac{1}{2} \left(\|(x_n - \mu Bx_n) - (z - \mu Bz)\|^2 + \|u_n - z\|^2 - \|(x_n - u_n) - \mu(Bx_n - Bz)\|^2 \right) \\ &\leq \frac{1}{2} \left(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n) - \mu(Bx_n - Bz)\|^2 \right) \\ &= \frac{1}{2} \left(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 + 2\mu \langle x_n - u_n, Bx_n - Bz \rangle - \mu^2 \|Bx_n - Bz\|^2 \right). \end{aligned} \quad (3.51)$$

It follows that

$$\begin{aligned} \|u_n - z\|^2 &\leq \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2\mu \langle x_n - u_n, Bx_n - Bz \rangle \\ &\quad - \mu^2 \|Bx_n - Bz\|^2 \\ &\leq \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2\mu \|x_n - u_n\| \|Bx_n - Bz\| \\ &\leq \|x_n - z\|^2 - \|x_n - u_n\|^2 + M \|Bx_n - Bz\|. \end{aligned} \quad (3.52)$$

Again, by Lemma 2.1 and (2.2), we have

$$\begin{aligned} \|z_n - z\|^2 &= \|P_C[(1 - \alpha_n)(I - \lambda A)u_n] - P_C[(I - \lambda A)z]\|^2 \\ &\leq \langle (1 - \alpha_n)(I - \lambda A)u_n - (I - \lambda A)z, z_n - z \rangle \\ &= \frac{1}{2} \left\{ \|(I - \lambda A)u_n - (I - \lambda A)z - \alpha_n y_n\|^2 + \|z_n - z\|^2 \right. \\ &\quad \left. - \|(I - \lambda A)u_n - (I - \lambda A)z - (z_n - z) - \alpha_n y_n\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|(I - \lambda A)u_n - (I - \lambda A)z\|^2 + \alpha_n M + \|z_n - z\|^2 \right. \\ &\quad \left. - \|u_n - z_n - \lambda(Au_n - Az) - \alpha_n y_n\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|u_n - z\|^2 + \alpha_n M + \|z_n - z\|^2 - \|u_n - z_n\|^2 + 2\lambda \langle u_n - z_n, Au_n - Az \rangle \right. \\ &\quad \left. + 2\alpha_n \langle y_n, u_n - z_n \rangle - \|\lambda(Au_n - Az) + \alpha_n y_n\|^2 \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \left\{ \|u_n - z\|^2 + \alpha_n M + \|z_n - z\|^2 - \|u_n - z_n\|^2 + 2\lambda \|u_n - z_n\| \|Au_n - Az\| \right. \\
&\quad \left. + 2\alpha_n \|y_n\| \|u_n - z_n\| \right\},
\end{aligned} \tag{3.53}$$

that is,

$$\begin{aligned}
\|z_n - z\|^2 &\leq \|u_n - z\|^2 + \alpha_n M - \|u_n - z_n\|^2 + 2\lambda \|u_n - z_n\| \|Au_n - Az\| \\
&\quad + 2\alpha_n \|y_n\| \|u_n - z_n\| \\
&\leq \|x_n - z\|^2 - \|x_n - u_n\|^2 + M \|Bx_n - Bz\| + \alpha_n M - \|u_n - z_n\|^2 \\
&\quad + 2\lambda \|u_n - z_n\| \|Au_n - Az\| + 2\alpha_n \|y_n\| \|u_n - z_n\|.
\end{aligned} \tag{3.54}$$

Hence,

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|z_n - z\|^2 \\
&\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \left[\|x_n - z\|^2 - \|x_n - u_n\|^2 + M \|Bx_n - Bz\| \right. \\
&\quad \left. + \alpha_n M - \|u_n - z_n\|^2 + 2\lambda \|u_n - z_n\| \|Au_n - Az\| \right. \\
&\quad \left. + 2\alpha_n \|y_n\| \|u_n - z_n\| \right] \\
&\leq \|x_n - z\|^2 - (1 - \beta_n) \|x_n - u_n\|^2 + (\|Bx_n - Bz\| + \alpha_n) M \\
&\quad - (1 - \beta_n) \|u_n - z_n\|^2 + 2\lambda \|u_n - z_n\| \|Au_n - Az\| + 2\alpha_n \|y_n\| \|u_n - z_n\|.
\end{aligned} \tag{3.55}$$

It follows that

$$\begin{aligned}
&(1 - \beta_n) \left(\|x_n - u_n\|^2 + \|u_n - z_n\|^2 \right) \\
&\leq \|x_n - x_{n+1}\| (\|x_n - z\| + \|x_{n+1} - z\|) + (\|Bx_n - Bz\| + \alpha_n) M \\
&\quad + 2\lambda \|u_n - z_n\| \|Au_n - Az\| + 2\alpha_n \|y_n\| \|u_n - z_n\|.
\end{aligned} \tag{3.56}$$

Since $\liminf_{n \rightarrow \infty} (1 - \beta_n) > 0$, $\alpha_n \rightarrow 0$, $\|x_{n+1} - x_n\| \rightarrow 0$, $\|Au_n - Az\| \rightarrow 0$ and $\|Bx_n - Bz\| \rightarrow 0$, we derive that

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = \lim_{n \rightarrow \infty} \|u_n - z_n\| = 0. \tag{3.57}$$

Note that

$$\|z_n - y_n\| = \|P_C[(1 - \alpha_n)y_n] - P_C y_n\| \leq \alpha_n \|y_n\| \rightarrow 0, \tag{3.58}$$

therefore,

$$\begin{aligned}
\|Sy_n - y_n\| &\leq \|Sy_n - Sz_n\| + \|Sz_n - x_n\| + \|x_n - u_n\| + \|u_n - z_n\| + \|z_n - y_n\| \\
&\leq 2\|y_n - z_n\| + \|Sz_n - x_n\| + \|x_n - u_n\| + \|u_n - z_n\| \\
&\longrightarrow 0.
\end{aligned} \tag{3.59}$$

Next we prove

$$\limsup_{n \rightarrow \infty} \langle x^*, x^* - y_n \rangle \leq 0, \tag{3.60}$$

where $x^* = P_\Omega(0)$ is a solution of the minimization problem (3.1).

Indeed, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle x^*, x^* - y_n \rangle = \lim_{i \rightarrow \infty} \langle x^*, x^* - y_{n_i} \rangle. \tag{3.61}$$

Without loss of generality, we may further assume that $y_{n_i} \rightarrow \tilde{x}$ weakly. By the same argument as that of Theorem 3.1, we can deduce that $\tilde{x} \in \Omega$. Therefore,

$$\limsup_{n \rightarrow \infty} \langle x^*, x^* - y_n \rangle = \langle x^*, x^* - \tilde{x} \rangle \leq 0. \tag{3.62}$$

Finally, we prove $x_n \rightarrow x^*$. As a matter of fact, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\beta_n(x_n - x^*) + (1 - \beta_n)(SP_C[(1 - \alpha_n)y_n] - SP_C[x^*])\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\|(1 - \alpha_n)(y_n - x^*) - \alpha_n x^*\|^2 \\
&= \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\left[(1 - \alpha_n)^2\|y_n - x^*\|^2 - 2\alpha_n(1 - \alpha_n)\langle x^*, y_n - x^* \rangle + \alpha_n^2\|x^*\|^2\right] \\
&\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\left[(1 - \alpha_n)\|x_n - x^*\|^2 - 2\alpha_n(1 - \alpha_n)\langle x^*, y_n - x^* \rangle + \alpha_n^2\|x^*\|^2\right] \\
&\leq [1 - 2(1 - \beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n(1 - \alpha_n)(1 - \beta_n)\langle x^*, x^* - y_n \rangle + (1 - \beta_n)\alpha_n^2 M \\
&= (1 - \gamma_n)\|x_n - x^*\|^2 + \delta_n \gamma_n,
\end{aligned} \tag{3.63}$$

where $\gamma_n = 2(1 - \beta_n)\alpha_n$ and $\delta_n = (1 - \alpha_n)\langle x^*, x^* - y_n \rangle + (\alpha_n M/2)$. It is clear that $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Hence, all conditions of Lemma 2.4 are satisfied. Therefore, we immediately deduce that $x_n \rightarrow x^*$ strongly. This completes the proof. $\square$

Acknowledgments

Y. Yao was supported in part by NSFC 11071279 and NSFC 71161001-G0105. Y. Liou was supported in part by NSC 100-2221-E-230-012.

References

- [1] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
- [2] H. Attouch and R. Cominetti, "A dynamical approach to convex minimization coupling approximation with the steepest descent method," *Journal of Differential Equations*, vol. 128, no. 2, pp. 519–540, 1996.
- [3] H. H. Bauschke and J. M. Borwein, "On projection algorithms for solving convex feasibility problems," *SIAM Review*, vol. 38, no. 3, pp. 367–426, 1996.
- [4] H. H. Bauschke, P. L. Combettes, and D. R. Luke, "Finding best approximation pairs relative to two closed convex sets in Hilbert spaces," *Journal of Approximation Theory*, vol. 127, no. 2, pp. 178–192, 2004.
- [5] A. Bnouhachem, M. A. Noor, and M. Khalfaoui, "Modified descent-projection method for solving variational inequalities," *Applied Mathematics and Computation*, vol. 190, no. 2, pp. 1691–1700, 2007.
- [6] D. Butnariu, Y. Censor, and S. Reich, *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, Elsevier, New York, NY, USA, 2001.
- [7] L. C. Ceng, S. Schaible, and J. C. Yao, "Implicit iteration scheme with perturbed mapping for equilibrium problems and fixed point problems of finitely many nonexpansive mappings," *Journal of Optimization Theory and Applications*, vol. 139, no. 2, pp. 403–418, 2008.
- [8] Y. Censor and A. Lent, "Short Communication: cyclic subgradient projections," *Mathematical Programming*, vol. 24, no. 1, pp. 233–235, 1982.
- [9] S.-S. Chang, H. W. Joseph Lee, and C. K. Chan, "A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization," *Nonlinear Analysis A*, vol. 70, no. 9, pp. 3307–3319, 2009.
- [10] V. Colao, G. L. Acedo, and G. Marino, "An implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of non-expansive mappings," *Nonlinear Analysis A*, vol. 71, no. 7–8, pp. 2708–2715, 2009.
- [11] V. Colao, G. Marino, and H.-K. Xu, "An iterative method for finding common solutions of equilibrium and fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 340–352, 2008.
- [12] P. L. Combettes, "Inconsistent signal feasibility problems: least-squares solutions in a product space," *IEEE Transactions on Signal Processing*, vol. 42, pp. 2955–2966, 1994.
- [13] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.
- [14] Y.-P. Fang, R. Hu, and N.-J. Huang, "Well-posedness for equilibrium problems and for optimization problems with equilibrium constraints," *Computers & Mathematics with Applications*, vol. 55, no. 1, pp. 89–100, 2008.
- [15] C. S. Hu and G. Cai, "Viscosity approximation schemes for fixed point problems and equilibrium problems and variational inequality problems," *Nonlinear Analysis A*, vol. 72, no. 3–4, pp. 1792–1808, 2010.
- [16] A. N. Iusem and W. Sosa, "Iterative algorithms for equilibrium problems," *Optimization*, vol. 52, no. 3, pp. 301–316, 2003.
- [17] C. Jaiboon and P. Kumam, "Strong convergence theorems for solving equilibrium problems and fixed point problems of ξ -strict pseudo-contraction mappings by two hybrid projection methods," *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 722–732, 2010.
- [18] J. S. Jung, "Strong convergence of composite iterative methods for equilibrium problems and fixed point problems," *Applied Mathematics and Computation*, vol. 213, no. 2, pp. 498–505, 2009.
- [19] L.-J. Lin and Y.-J. Huang, "Generalized vector quasi-equilibrium problems with applications to common fixed point theorems and optimization problems," *Nonlinear Analysis A*, vol. 66, no. 6, pp. 1275–1289, 2007.

- [20] A. Moudafi, "On the convergence of splitting proximal methods for equilibrium problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 359, no. 2, pp. 508–513, 2009.
- [21] A. Moudafi and M. Théra, "Proximal and dynamical approaches to equilibrium problems," in *Ill-Posed Variational Problems and Regularization Techniques (Trier, 1998)*, vol. 477 of *Lecture Notes in Economics and Mathematical Systems*, pp. 187–201, Springer, Berlin, Germany, 1999.
- [22] M. A. Noor, "Projection-proximal methods for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 53–62, 2006.
- [23] J.-W. Peng and J.-C. Yao, "Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems," *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1816–1828, 2009.
- [24] S. Plubtieng and R. Punpaeng, "A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 455–469, 2007.
- [25] X. Qin, Y. J. Cho, and S. M. Kang, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications," *Nonlinear Analysis A*, vol. 72, no. 1, pp. 99–112, 2010.
- [26] Y. Shehu, "Fixed point solutions of generalized equilibrium problems for nonexpansive mappings," *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 892–898, 2010.
- [27] T. Suzuki, "Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces," *Fixed Point Theory and Applications*, vol. 2005, no. 1, pp. 103–123, 2005.
- [28] F. Q. Xia and X. P. Ding, "Predictor-corrector algorithms for solving generalized mixed implicit quasi-equilibrium problems," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 173–179, 2007.
- [29] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.
- [30] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis A*, vol. 69, no. 3, pp. 1025–1033, 2008.
- [31] P. Tseng, "Applications of a splitting algorithm to decomposition in convex programming and variational inequalities," *SIAM Journal on Control and Optimization*, vol. 29, no. 1, pp. 119–138, 1991.
- [32] I. Yamada and N. Ogura, "Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 25, no. 7-8, pp. 619–655, 2004.
- [33] Y. Yao, Y.-C. Liou, and J.-C. Yao, "An iterative algorithm for approximating convex minimization problem," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 648–656, 2007.
- [34] Y. Yao and J.-C. Yao, "On modified iterative method for nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1551–1558, 2007.
- [35] H. Zegeye, E. U. Ofoedu, and N. Shahzad, "Convergence theorems for equilibrium problem, variational inequality problem and countably infinite relatively quasi-nonexpansive mappings," *Applied Mathematics and Computation*, vol. 216, no. 12, pp. 3439–3449, 2010.