Research Article

Minimax Theorems for Set-Valued Mappings under Cone-Convexities

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The aim of this paper is to study the minimax theorems for set-valued mappings with or without linear structure. We define several kinds of cone-convexities for set-valued mappings, give some examples of such set-valued mappings, and study the relationships among these cone-convexities. By using our minimax theorems, we derive some existence results for saddle points of set-valued mappings. Some examples to illustrate our results are also given.

1. Introduction

The minimax theorems for real-valued functions were introduced by Fan [1, 2] in the early fifties. Since then, these were extended and generalized in many different directions because of their applications in variational analysis, game theory, mathematical economics, fixed-point theory, and so forth (see, for example, [3–11] and the references therein). The minimax theorems for vector-valued functions have been studied in [4, 9, 10] with applications to vector saddle point problems. However, the minimax theorems for set-valued bifunctions have been studied only in few papers, namely, [4–8] and the references therein.

In this paper, we establish some new minimax theorems for set-valued mappings. Section 2 deals with preliminaries which will be used in rest of the paper. Section 3 denotes the cone-convexities of set-valued mappings. In Section 4, we establish some minimax theorems by using separation theorems, Fan-Browder fixed-point theorem. In the last

section, we discuss some existence results for different kinds of saddle points for set-valued mappings.

2. Preliminaries

Throughout the paper, unless otherwise specified, we assume that X, Y are two nonempty subsets, and $\mathcal{Z}$ is a real Hausdorff topological vector space, C is a closed convex pointed cone in $\mathcal{Z}$ with $\text{int } C \neq \emptyset$. Let $\mathcal{Z}^*$ be the topological dual space of $\mathcal{Z}$, and let

$$C^* = \{g \in \mathcal{Z}^* : g(c) \geq 0 \forall c \in C\}. \quad (2.1)$$

We present some fundamental concepts which will be used in the sequel.

Definition 2.1 (see [3, 4, 8]). Let A be a nonempty subset of $\mathcal{Z}$. A point $z \in A$ is called a

- (a) *minimal point* of A if $A \cap (z - C) = \{z\}$; $\text{Min } A$ denotes the set of all minimal points of A ;
- (b) *maximal point* of A if $A \cap (z + C) = \{z\}$; $\text{Max } A$ denotes the set of all maximal points of A ;
- (c) *weakly minimal point* of A if $A \cap (z - \text{int } C) = \emptyset$; $\text{Min}_w A$ denotes the set of all weakly minimal points of A ;
- (d) *weakly maximal point* of A if $A \cap (z + \text{int } C) = \emptyset$; $\text{Max}_w A$ denotes the set of all weakly maximal points of A .

It can be easily seen that $\text{Min } A \subset \text{Min}_w A$ and $\text{Max } A \subset \text{Max}_w A$.

Lemma 2.2 (see [3, 4]). Let A be a nonempty compact subset of $\mathcal{Z}$. Then,

- (a) $\text{Min } A \neq \emptyset$;
- (b) $A \subset \text{Min } A + C$;
- (c) $\text{Max } A \neq \emptyset$;
- (d) $A \subset \text{Max } A - C$.

Following [6], we denote both Max and Max_w by $\max$ (both Min and Min_w by $\min$) in $\mathbb{R}$ since both Max and Max_w (both Min and Min_w) are the same in $\mathbb{R}$.

Definition 2.3. Let $\mathcal{X}, \mathcal{Y}$ be Hausdorff topological spaces. A set-valued map $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ with nonempty values is said to be

- (a) *upper semicontinuous* at $x_0 \in \mathcal{X}$ if for every $x_0 \in \mathcal{X}$ and for every open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$;
- (b) *lower semi-continuous* at $x_0 \in \mathcal{X}$ if for any sequence $\{x_n\} \subset \mathcal{X}$ such that $x_n \rightarrow x_0$ and any $y_0 \in F(x_0)$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \rightarrow y_0$;
- (c) *continuous* at $x_0 \in \mathcal{X}$ if F is upper semi-continuous as well as lower semi-continuous at x_0 .

We present the following fundamental lemmas which will be used in the sequel.

Lemma 2.4 (see [9, Lemma 3.1]). Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be three topological spaces. Let $\mathcal{Y}$ be compact, $F : \mathcal{X} \times \mathcal{Y} \rightrightarrows \mathcal{Z}$ a set-valued mapping, and the set-valued mapping $T : \mathcal{X} \rightrightarrows \mathcal{Z}$ defined by

$$T(x) = \bigcup_{y \in \mathcal{Y}} F(x, y), \quad \forall x \in \mathcal{X}. \quad (2.2)$$

- (a) If F is upper semi-continuous on $\mathcal{X} \times \mathcal{Y}$, then T is upper semi-continuous on $\mathcal{X}$.
- (b) If F is lower semi-continuous on $\mathcal{X}$, so is T .

Lemma 2.5 (see [9, Lemma 3.2]). Let $\mathcal{Z}$ be a Hausdorff topological vector space, $F : \mathcal{Z} \rightrightarrows \mathbb{R}$ a set-valued mapping with nonempty compact values, and the functions $p, q : \mathcal{Z} \rightarrow \mathbb{R}$ defined by $p(z) = \max F(z)$ and $q(z) = \min F(z)$.

- (a) If F is upper semi-continuous, so is p .
- (b) If F is lower semi-continuous, so is p .
- (c) If F is continuous, so are p and q .

We shall use the following nonlinear scalarization function to establish our results.

Definition 2.6 (see [6, 10]). Let $k \in \text{int } C$ and $v \in \mathcal{Z}$. The Gerstewitz function $\xi_{kv} : \mathcal{Z} \rightarrow \mathbb{R}$ is defined by

$$\xi_{kv}(u) = \min\{t \in \mathbb{R} : u \in v + tk - C\}. \quad (2.3)$$

We present some fundamental properties of the scalarization function.

Proposition 2.7 (see [6, 10]). Let $k \in \text{int } C$ and $v \in \mathcal{Z}$. The Gerstewitz function $\xi_{kv} : \mathcal{Z} \rightarrow \mathbb{R}$ has the following properties:

- (a) $\xi_{kv}(u) < r \Leftrightarrow u \in v + rk - \text{int } C$;
- (b) $\xi_{kv}(u) \leq r \Leftrightarrow u \in v + rk - C$;
- (c) $\xi_{kv}(u) = 0 \Leftrightarrow u \in v - \partial C$, where ∂C is the topological boundary of C ;
- (d) $\xi_{kv}(u) > r \Leftrightarrow u \notin v + rk - C$;
- (e) $\xi_{kv}(u) \geq r \Leftrightarrow u \notin v + rk - \text{int } C$;
- (f) $\xi_{kv}(\cdot)$ is a convex function;
- (g) $\xi_{kv}(\cdot)$ is an increasing function, that is, $u_2 - u_1 \in \text{int } S \Rightarrow \xi_{kv}(u_1) < \xi_{kv}(u_2)$;
- (h) $\xi_{kv}(\cdot)$ is a continuous function.

Theorem 2.8 (Fan-Browder fixed-point theorem (see [12])). Let X be a nonempty compact convex subset of a Hausdorff topological vector space and let $T : X \rightrightarrows X$ be a set-valued mapping with nonempty convex values and open fibers, that is, $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open for all $y \in X$. Then, T has a fixed point.

3. Cone-Convexities

In this section, we present different kinds of cone-convexities for set-valued mappings and give some relations among them. Some examples of such set-valued mappings are also given.

Definition 3.1. Let X be a nonempty convex subset of a topological vector space $\mathcal{W}$. A set-valued mapping $F : X \rightrightarrows \mathcal{Z}$ is said to be

- (a) *above -C-convex* [4] (resp., *above-C-concave* [5]) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$,

$$\begin{aligned} F(\lambda x_1 + (1 - \lambda)x_2) &\subset \lambda F(x_1) + (1 - \lambda)F(x_2) - C, \\ (\text{resp., } \lambda F(x_1) + (1 - \lambda)F(x_2) &\subset F(\lambda x_1 + (1 - \lambda)x_2) - C); \end{aligned} \quad (3.1)$$

- (b) *below-C-convex* [13] (resp., *below-C-concave* [9, 13]) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda F(x_1) + (1 - \lambda)F(x_2) &\subset F(\lambda x_1 + (1 - \lambda)x_2) + C \\ (\text{resp., } F(\lambda x_1 + (1 - \lambda)x_2) &\subset \lambda F(x_1) + (1 - \lambda)F(x_2) + C); \end{aligned} \quad (3.2)$$

- (c) *above-C-quasi-convex* (resp., *below-C-quasiconcave*) [7, Definition 2.3] on X if the set

$$\begin{aligned} \text{Lev}_{F \leq}(z) &:= \{x \in X : F(x) \subset z - C\} \\ (\text{resp., } \text{Lev}_{F \geq}(z) &:= \{x \in X : F(x) \subset z + C\}), \end{aligned} \quad (3.3)$$

is convex for all $z \in \mathcal{Z}$;

- (d) *above-properly C-quasiconvex* (resp., *above-properly C-quasiconcave* [6]) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$, either

$$\begin{aligned} F(\lambda x_1 + (1 - \lambda)x_2) &\subset F(x_1) - C \\ (\text{resp., } F(x_1) &\subset F(\lambda x_1 + (1 - \lambda)x_2) - C) \end{aligned} \quad (3.4)$$

or

$$\begin{aligned} F(\lambda x_1 + (1 - \lambda)x_2) &\subset F(x_2) - C \\ (\text{resp., } F(x_2) &\subset F(\lambda x_1 + (1 - \lambda)x_2) - C); \end{aligned} \quad (3.5)$$

- (e) *below-properly C-quasiconvex* [7] (resp., *below-properly C-quasiconcave*) on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$, either

$$\begin{aligned} F(x_1) &\subset F(\lambda x_1 + (1 - \lambda)x_2) + C \\ (\text{resp., } F(\lambda x_1 + (1 - \lambda)x_2) &\subset F(x_1) + C) \end{aligned} \quad (3.6)$$

or

$$\begin{aligned} F(x_2) &\subset F(\lambda x_1 + (1 - \lambda)x_2) + C \\ (\text{resp., } F(\lambda x_1 + (1 - \lambda)x_2) &\subset F(x_2) + C); \end{aligned} \quad (3.7)$$

(f) *above-naturally C-quasiconvex* [6] on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \subset \text{co}\{F(x_1) \cup F(x_2)\} - C, \quad (3.8)$$

where $\text{co } A$ denotes the convex hull of a set A ;

(g) *above -C-convex-like* (resp., *above-C-concave-like*) on X (X is not necessarily convex) if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$, there is an $x' \in X$ such that

$$\begin{aligned} F(x') &\subset \lambda F(x_1) + (1 - \lambda)F(x_2) - C \\ (\text{resp., } \lambda F(x_1) + (1 - \lambda)F(x_2) &\subset F(x') - C); \end{aligned} \quad (3.9)$$

(h) *below -C-convex-like* [13] (resp., *below -C-concave-like*) on X (X is not necessarily convex) if for all $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$, there is an $x' \in X$ such that

$$\begin{aligned} \lambda F(x_1) + (1 - \lambda)F(x_2) &\subset F(x') + C \\ (\text{resp., } F(x') &\subset \lambda F(x_1) + (1 - \lambda)F(x_2) + C). \end{aligned} \quad (3.10)$$

It is obvious that every above-C-convex set-valued mapping or above-properly C-quasi-convex set-valued mapping is an above-naturally C-quasi-convex set-valued mapping, and every above-C-convex (above-C-concave) set-valued mapping is an above-C-convex-like (above-C-concave-like) set-valued mapping. Similar relations hold for cases below.

Remark 3.2. The definition of above-properly C-quasi-convex (above-properly C-quasi-concave) set-valued mapping is different from the one mentioned in [7, Definition 2.3] or [5, 6]. The following Examples 3.3 and 3.4 illustrate the reason why they are different from the one mentioned in [5–7]. However, if F is a vector-valued mapping or a single-valued mapping, both mappings reduce to the ordinary definition of a properly C-quasi-convex mapping for vector-valued functions [7]. The above-C-convexity in Definition 3.1 is also different from the below-C-convexity used in [5, 9].

Example 3.3. Consider $C = \{(s, t) \in \mathbb{R}^2 : s \geq 0, t \geq 0\}$. Let $F : [x_1, x_2] \subset \mathbb{R} \rightrightarrows \mathbb{R}^2$ be a set-valued mapping defined by

$$\begin{aligned} F(x_1) &:= \left\{ (s, t) \in \mathbb{R}^2 : (s - 2)^2 + (t - 4)^2 = 1, 2 \leq s \leq 3, 4 \leq t \leq 5 \right\} \cup \{(s, 5) : -1 \leq s \leq 2\}, \\ F(x_2) &:= \left\{ (s, t) \in \mathbb{R}^2 : (s - 6)^2 + (t + 2)^2 = 1, 6 \leq s \leq 7, -2 \leq t \leq -1 \right\}, \end{aligned} \quad (3.11)$$

and for all $\lambda \in (0, 1)$,

$$F(\lambda x_1 + (1 - \lambda)x_2) := \left\{ (s, t) \in \mathbb{R}^2 : (s - 2)^2 + (t - 2)^2 = 4, 0 \leq s \leq 2, 0 \leq t \leq 2 \right\}. \quad (3.12)$$

Then F is an above-properly C -quasi-convex set-valued mapping, but it is not below-properly C -quasi-convex.

On the other hand, let $G : [x_1, x_2] \subset \mathbb{R} \rightrightarrows \mathbb{R}^2$ be a set-valued mapping defined by

$$\begin{aligned} G(x_1) &:= \left\{ (s, t) \in \mathbb{R}^2 : (s - 1)^2 + (t - 4)^2 = 1, 1 \leq s \leq 2, 4 \leq t \leq 5 \right\}, \\ G(x_2) &:= \left\{ (s, t) \in \mathbb{R}^2 : (s - 6)^2 + (t + 2)^2 = 1, 6 \leq s \leq 7, -2 \leq t \leq -1 \right\}, \end{aligned} \quad (3.13)$$

and for all $\lambda \in (0, 1)$,

$$\begin{aligned} G(\lambda x_1 + (1 - \lambda)x_2) &:= \left\{ (s, t) \in \mathbb{R}^2 : (s - 2)^2 + (t - 2)^2 = 4, 0 \leq s \leq 2, 0 \leq t \leq 2 \right\} \\ &\quad \bigcup \{(s, 0) : 2 \leq s \leq 3\}. \end{aligned} \quad (3.14)$$

Then, G is a below-properly C -quasi-convex set-valued mapping, but it is not above-properly C -quasi-convex.

Example 3.4. Let $C = \{(s, t) : s \geq 0, t \geq 0\}$. Define $F : [-1, 1] \rightrightarrows \mathbb{R}^2$ by

$$F(x) = \left\{ (x, t) : 1 - x^2 \leq t \leq 1 \right\}, \quad \forall x \in [-1, 1]. \quad (3.15)$$

Then F is continuous, above- C -quasi-convex, below- C -quasi-concave, above-properly C -quasi-convex, and above-properly C -quasi-concave, but it is not below-properly C -quasi-concave.

Proposition 3.5. *Let X be a nonempty set (not necessarily convex) and for a given set-valued mapping $F : X \rightrightarrows \mathcal{Z}$ with nonempty compact values, define a set-valued mapping $M : X \rightrightarrows \mathcal{Z}$ as*

$$M(x) = \text{Max}_w F(x), \quad \forall x \in X. \quad (3.16)$$

- (a) *If F is above- C -convex-like, then M is so.*
- (b) *If X is a topological space and F is a continuous mapping, then M is upper semicontinuous with nonempty compact values on X .*

Proof. (a) Let F be above- C -convex-like, and let $x_1, x_2 \in X$ be arbitrary. Since F is above- C -convex-like, for any $\alpha \in [0, 1]$, there exists $x' \in X$ such that

$$F(x') \subset \alpha F(x_1) + (1 - \alpha)F(x_2) - C. \quad (3.17)$$

By Lemma 2.2,

$$\text{Max}_w F(x') \subset \alpha \text{Max}_w F(x_1) + (1 - \alpha) \text{Max}_w F(x_2) - C. \quad (3.18)$$

Therefore, $x \mapsto \text{Max}_w F(x)$ is above-C-convex-like.

(b) The upper semicontinuity of M was deduced in [4, Lemma 2]. $\square$

Proposition 3.6. *Let X be a nonempty convex set, and let $F : X \rightrightarrows \mathcal{Z}$ be a set-valued mapping with nonempty compact values. Then, the set-valued mapping $M : X \rightrightarrows \mathcal{Z}$ defined by*

$$M(x) = \text{Max}_w F(x), \quad \forall x \in X, \quad (3.19)$$

is above-C-quasiconvex if F is so.

The following result can be easily derived, and therefore, we omit the proof.

Proposition 3.7. *Let X be a nonempty convex set and $F : X \rightrightarrows \mathbb{R}$ be above- $\mathbb{R}_+$ -concave. Then the set-valued mapping $x \mapsto \max F(x)$ is above- $\mathbb{R}_+$ -concave and below- $\mathbb{R}_+$ -quasiconcave. Furthermore, if $F : X \rightrightarrows \mathbb{R}$ is above-properly $\mathbb{R}_+$ -quasiconcave, then the set-valued mapping $x \mapsto \max F(x)$ is also above-properly $\mathbb{R}_+$ -quasiconcave and below- $\mathbb{R}_+$ -quasiconcave.*

Let $\xi \in C^*$ and $F : X \rightrightarrows \mathcal{Z}$ be a set-valued mapping. Then, the composition mapping $\xi \circ F : X \rightrightarrows \mathbb{R}$ is defined by

$$(\xi \circ F)(x) = \xi(F(x)) = \bigcup_{y \in F(x)} \xi(y), \quad \forall x \in X. \quad (3.20)$$

Clearly, the composition mapping $\xi \circ F : X \rightrightarrows \mathbb{R}$ is also a set-valued mapping.

Proposition 3.8. *Let X be a nonempty set, $F : X \rightrightarrows \mathcal{Z}$ a set-valued mapping, and $\xi \in C^*$.*

- (a) *If F is above-C-convex-like, then $\xi \circ F$ is above- $\mathbb{R}_+$ -convex-like.*
- (b) *If F is below-C-concave-like, then $\xi \circ F$ is below- $\mathbb{R}_+$ -concave-like.*
- (c) *If X is a topological space and F is upper semi-continuous, then so is $\xi \circ F$.*

Proof. (a) By the definition of above-C-convex-like set-valued mapping $F : X \rightrightarrows \mathcal{Z}$, for any $x_1, x_2 \in X$ and all $\lambda \in [0, 1]$, there exists $x' \in X$ such that $F(x') \subset \lambda F(x_1) + (1 - \lambda)F(x_2) - C$. For any $y \in F(x')$, there exist $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ such that

$$\lambda y_1 + (1 - \lambda)y_2 \in y - C. \quad (3.21)$$

For any $\xi \in C^*$, we have $\xi(y) \leq \lambda \xi(y_1) + (1 - \lambda)\xi(y_2)$. Hence, $\xi(F(x')) \subset \lambda \xi(F(x_1)) + (1 - \lambda)\xi(F(x_2)) - \mathbb{R}_+$. Thus, $\xi \circ F$ is above- $\mathbb{R}_+$ -convex-like.

The proof of (b) and (c) is easy, and therefore, we omit it. $\square$

Proposition 3.9. *Let X be a nonempty convex set and $\xi \in C^*$.*

- (a) *If $F : X \rightrightarrows \mathcal{Z}$ is above- C -concave (above-properly C -quasi-concave), then $\xi \circ F : X \rightrightarrows \mathbb{R}$ is above- $\mathbb{R}_+$ -concave (above-properly $\mathbb{R}_+$ -quasi-concave).*
- (b) *If $F : X \rightrightarrows \mathcal{Z}$ is above-properly C -quasi-convex, then $\xi \circ F : X \rightrightarrows \mathbb{R}$ is above- $\mathbb{R}_+$ -quasi-convex and above-properly $\mathbb{R}_+$ -quasi-convex.*
- (c) *If $F : X \rightrightarrows \mathcal{Z}$ is above- C -convex, then $\xi \circ F : X \rightrightarrows \mathbb{R}$ is above- $\mathbb{R}_+$ -convex and above- $\mathbb{R}_+$ -quasi-convex.*

Lemma 3.10. *Let $\mathcal{Z}$ be a real Hausdorff topological vector space and C a closed convex pointed cone in $\mathcal{Z}$ with $\text{int } C \neq \emptyset$. Let X be a nonempty compact subset of a topological space $\mathcal{X}$, and let $F : X \rightrightarrows \mathcal{Z}$ be an upper semi-continuous set-valued mapping with nonempty compact values. Then, for any $\xi \in C^*$, there exists $y \in \text{Max}_w F(X)$ such that $\xi(y) = \max \bigcup_{x \in X} \xi(F(x))$.*

Proof. For any given $\xi \in C^*$, the mapping $x \mapsto \xi(F(x))$ is upper semi-continuous by Proposition 3.8 (c). By the compactness of X , there exist $x_0 \in X$ and $y_0 \in F(x_0)$ such that $\xi(y_0) = \max \bigcup_{x \in X} \xi(F(x))$. By Lemma 2.2, there exists $y \in \text{Max}_w \bigcup_{x \in X} F(x)$ such that $y_0 - y \in -C$, and hence $\xi(y) \geq \xi(y_0)$. On the other hand, $y \in \text{Max}_w \bigcup_{x \in X} F(x) \subset F(X)$, we know that $\xi(y) \in \xi(F(X))$, and then $\xi(y) \leq \max \bigcup_{x \in X} \xi(F(x)) = \xi(y_0)$. Therefore, the conclusion holds. $\square$

Proposition 3.11. *Let X be a nonempty convex set. If $F : X \rightrightarrows \mathcal{Z}$ is above-properly C -quasi-convex, then it is above- C -quasi-convex.*

Proof. For any $z \in \mathcal{Z}$, let $x_1, x_2 \in \text{Lev}_{F \leq}(z)$. Then, $F(x_1)$ and $F(x_2)$ are subsets of $z - C$. Since F is above-properly C -quasi-convex, for any $\lambda \in [0, 1]$, $F(\lambda x_1 + (1 - \lambda)x_2)$ is contained in either $F(x_1) - C$ or $F(x_2) - C$, and hence, in $z - C$. Thus, the set $\text{Lev}_{F \leq}(z)$ is convex, and therefore, F is above- C -quasi-convex. $\square$

Proposition 3.12. *Let X be a nonempty convex set. If $F : X \rightrightarrows \mathcal{Z}$ is above-naturally C -quasi-convex, then it is above- C -quasi-convex.*

Proof. Let z, x_1 , and x_2 be the same as given as in Proposition 3.11. Then, $\text{co}\{F(x_1) \cup F(x_2)\} \subset z - C$ since $z - C$ is convex. By the above-naturally C -quasi-convexity, $F(\lambda x_1 + (1 - \lambda)x_2) \subset z - C$ for all $\lambda \in [0, 1]$. Thus, the set $\text{Lev}_{F \leq}(z)$ is convex, and therefore, F is above- C -quasi-convex. $\square$

Proposition 3.13. *Let X be a nonempty convex set. If $F : X \rightrightarrows \mathcal{Z}$ is above-naturally C -quasi-convex, then $\xi \circ F$ is above-naturally $\mathbb{R}_+$ -quasi-convex for any $\xi \in C^*$.*

Proof. Let $\xi \in C^*$ be given. From the above-naturally C -quasi-convexity of F , for any $x_1, x_2 \in X$ and any $\lambda \in [0, 1]$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \subset \text{co}\{F(x_1) \cup F(x_2)\} - C. \quad (3.22)$$

For any $y \in F(\alpha x_1 + (1 - \alpha)x_2)$, there is a $w \in \text{co}\{F(x_1) \cup F(x_2)\}$ such that $y \in w - C$. Then there exist $y_i \in F(x_1) \cup F(x_2)$ and $\lambda_i \in [0, 1]$, $1 \leq i \leq n$ such that $w = \sum_{i=1}^n \lambda_i y_i$. Hence, $\xi(w) = \sum_{i=1}^n \lambda_i \xi(y_i)$, and

$$\xi(y) \in \xi(w) - \mathbb{R}_+ = \sum_{i=1}^n \lambda_i \xi(y_i) - \mathbb{R}_+ \subset \text{co}\{\xi(F(x_1)) \cup \xi(F(x_2))\} - \mathbb{R}_+. \quad (3.23)$$

Therefore, $\xi \circ F$ is a above-naturally $\mathbb{R}_+$ -quasi-convex. $\square$

Proposition 3.14. *Let $F : X \rightrightarrows \mathcal{Z}$ be a set-valued mapping with nonempty compact values. For any $\xi \in C^*$,*

- (a) *if $\xi(d) = \min \bigcup_{x \in X} \xi(F(x))$ for some $d \in \mathcal{Z}$, then $d \in \text{Min}_w \bigcup_{x \in X} F(x)$;*
- (b) *if $\xi(e) = \max \bigcup_{x \in X} \xi(F(x))$ for some $e \in \mathcal{Z}$, then $e \in \text{Max}_w \bigcup_{x \in X} F(x)$.*

Proof. Let $\xi(d) = \min \bigcup_{x \in X} \xi(F(x))$. Suppose that $d \notin \text{Min}_w \bigcup_{x \in X} F(x)$. Then

$$\left(\bigcup_{x \in X} F(x) \right) \cap (d - \text{int } C) \neq \emptyset. \quad (3.24)$$

Then, there exists $w \in \bigcup_{x \in X} F(x)$ and $w \in d - \text{int } C$. Therefore, there exists $s \in X$ such that $w \in F(s)$ and $d - w \in \text{int } C$. Since $\xi \in C^*$, $\xi(d) > \xi(w)$ and $\xi(w) \geq \min \bigcup_{x \in X} \xi(F(x))$. This implies that $\xi(d) > \min \bigcup_{x \in X} \xi(F(x))$, which is a contradiction. This proves (a).

Analogously, we can prove (b), so we omit it. $\square$

Remark 3.15. Propositions 3.8 and 3.9, Lemma 3.10, and Propositions 3.13 and 3.14 are always true except Proposition 3.8 (b) if we replace ξ by any Gerstewitz function.

4. Minimax Theorems for Set-Valued Mappings

In this section, we establish some minimax theorems for set-valued mappings with or without linear structure.

Theorem 4.1. *Let X, Y be two nonempty compact subsets (not necessarily convex) of real Hausdorff topological spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ be lower semi-continuous on X and upper semi-continuous on Y such that for all $(x, y) \in X \times Y$, $F(x, y)$ is nonempty compact and satisfies the following conditions:*

- (i) *for each $x \in X$, $y \mapsto F(x, y)$ is below- $\mathbb{R}_+$ -concave-like on Y ;*
- (ii) *for each $y \in Y$, $x \mapsto F(x, y)$ is above- $\mathbb{R}_+$ -convex-like on X .*

Then,

$$\max_{y \in Y} \min_{x \in X} \bigcup_{x \in X} F(x, y) = \min_{x \in X} \max_{y \in Y} \bigcup_{y \in Y} F(x, y). \quad (4.1)$$

Proof. Since

$$\max_{y \in Y} \min_{x \in X} F(x, y) \leq \min_{x \in X} \max_{y \in Y} F(x, y), \quad (4.2)$$

it is sufficient to prove that

$$\max_{y \in Y} \min_{x \in X} F(x, y) \geq \min_{x \in X} \max_{y \in Y} F(x, y). \quad (4.3)$$

Choose any $\alpha \in \mathbb{R}$ such that $\alpha < \min_{x \in X} \max_{y \in Y} F(x, y)$. For any $y \in Y$, let

$$\text{Lev}_{F \leq}(y; \alpha) = \{x \in X : F(x, y) \leq \alpha - \mathbb{R}_+\}. \quad (4.4)$$

Then, by the lower semi-continuity of the set-valued mapping $x \mapsto F(x, y)$, the set $\text{Lev}_{F \leq}(y; \alpha)$ is closed, hence it is compact for all $y \in Y$. By the choice of α , we have

$$\bigcap_{y \in Y} \text{Lev}_{F \leq}(y; \alpha) = \emptyset. \quad (4.5)$$

Since X is compact and the collection $\{X \setminus \text{Lev}_{F \leq}(y; \alpha) : y \in Y\}$ covers X , there exist finite number of points $y_1, y_2, \dots, y_m$ in Y such that

$$X \subset \bigcup_{i=1}^m (X \setminus \text{Lev}_{F \leq}(y_i; \alpha)) \quad (4.6)$$

or

$$\bigcap_{i=1}^m \text{Lev}_{F \leq}(y_i; \alpha) = \emptyset. \quad (4.7)$$

This implies that

$$\max_{i=1}^m F(x, y_i) > \alpha, \quad \forall x \in X, \quad (4.8)$$

and therefore,

$$\min_{x \in X} \max_{i=1}^m F(x, y_i) > \alpha. \quad (4.9)$$

Following the idea of Borwein and Zhuang [14], let

$$\mathfrak{E} := \left\{ (z, r) \in \mathbb{R}^{m+1} : \text{there is } x \in X, F(x, y_i) \leq r + z_i - \mathbb{R}_+, i = 1, 2, \dots, m \right\}, \quad (4.10)$$

where $\mathbf{z} = (z_1, z_2, \dots, z_m)$. Then the set $\mathfrak{E}$ is convex, so is $\text{int } \mathfrak{E}$. We note that the interior $\text{int } \mathfrak{E}$ of $\mathfrak{E}$ is nonempty since

$$\left(\mathbf{0}, 1 + \max \bigcup_{i=1}^m F(x, y_i) \right) \in \text{int } \mathfrak{E}, \quad \forall x \in X. \quad (4.11)$$

Since $(\mathbf{0}, \alpha) \notin \mathfrak{E}$, by separation hyperplane theorem [15, Theorem 14.2], there is a $(\Xi, \varepsilon) \neq \mathbf{0} \times \{0\}$ such that

$$\langle (\Xi, \varepsilon), (\mathbf{z}, r) \rangle \geq \langle (\Xi, \varepsilon), (\mathbf{0}, \alpha) \rangle, \quad \forall (\mathbf{z}, r) \in \mathfrak{E}, \quad (4.12)$$

where $\Xi = (\lambda_1, \lambda_2, \dots, \lambda_m)$, that is,

$$\Xi \mathbf{z} + \varepsilon r \geq \varepsilon \alpha, \quad \forall (\mathbf{z}, r) \in \mathfrak{E}. \quad (4.13)$$

By (4.11), (4.13), and the choice of α , we have that $\varepsilon > 0$. Furthermore, from the fact

$$\prod_{i=1}^m (F(x, y_i) + r) \times \{-r\} \subset \mathfrak{E}, \quad (4.14)$$

we have

$$(\eta_{x,1} + r, \eta_{x,2} + r, \dots, \eta_{x,m} + r, -r) \in \mathfrak{E}, \quad \forall \eta_{x,i} \in F(x, y_i). \quad (4.15)$$

Hence, by (4.13), we have

$$\sum_{i=1}^m \lambda_i (\eta_{x,i} + r) + \varepsilon(-r) \geq \varepsilon \alpha \quad (4.16)$$

or

$$\sum_{i=1}^m \left(\frac{\lambda_i}{\varepsilon} \right) \eta_{x,i} + \left(\frac{\sum_{i=1}^m \lambda_i}{\varepsilon} - 1 \right) r \geq \alpha, \quad \forall x \in X, r \in \mathbb{R}. \quad (4.17)$$

Thus, we have $\sum_{i=1}^m (\lambda_i / \varepsilon) = 1$. Hence, by (4.17), we have

$$\sum_{i=1}^m \left(\frac{\lambda_i}{\varepsilon} \right) F(x, y_i) \subset \alpha + \mathbb{R}_+. \quad (4.18)$$

Since $F(x, y)$ is below- $\mathbb{R}_+$ -concave-like in y , there is $y' \in Y$ such that

$$F(x, y') \subset \sum_{i=1}^m \left(\frac{\lambda_i}{\varepsilon} \right) F(x, y_i) + \mathbb{R}_+, \quad \forall x \in X. \quad (4.19)$$

Therefore,

$$\bigcup_{x \in X} F(x, y') \subset \alpha + \mathbb{R}_+, \quad (4.20)$$

and hence,

$$\max_{y \in Y} \min_{x \in X} \bigcup_{x \in X} F(x, y) \geq \alpha. \quad (4.21)$$

This completes the proof. $\square$

Remark 4.2. Theorem 4.1 is a modification of [14, Theorem A]. If F is a real-valued function, then Theorem 4.1 reduces to the well-known minimax theorem due to Fan [2].

We next establish a minimax theorem for set-valued mappings defined on the sets with linear structure.

Theorem 4.3. *Let X, Y be two nonempty compact convex subsets of real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ be lower semi-continuous on X and upper semi-continuous on Y such that for all $(x, y) \in X \times Y$, $F(x, y)$ is nonempty compact, and satisfies the following conditions:*

- (i) *for each $y \in Y$, $x \mapsto F(x, y)$ is above- $\mathbb{R}_+$ -quasi-convex on X ;*
- (ii) *for each $x \in X$, $y \mapsto F(x, y)$ is above- $\mathbb{R}_+$ -concave, or above-properly $\mathbb{R}_+$ -quasi-concave on Y ;*
- (iii) *for each $y \in Y$, there is a $x_y \in X$ such that*

$$\max F(x_y, y) \leq \max_{y \in Y} \min_{x \in X} \bigcup_{x \in X} F(x, y). \quad (4.22)$$

Then,

$$\min_{x \in X} \max_{y \in Y} \bigcup_{y \in Y} F(x, y) = \max_{y \in Y} \min_{x \in X} \bigcup_{x \in X} F(x, y). \quad (4.23)$$

Proof. We only need to prove that

$$\max_{y \in Y} \min_{x \in X} \bigcup_{x \in X} F(x, y) < \min_{x \in X} \max_{y \in Y} \bigcup_{y \in Y} F(x, y) \quad (4.24)$$

is impossible, since it is always true that

$$\max_{y \in Y} \min_{x \in X} \bigcup_{x \in X} F(x, y) \leq \min_{x \in X} \max_{y \in Y} \bigcup_{y \in Y} F(x, y). \quad (4.25)$$

Suppose that there is an $\alpha \in \mathbb{R}$ such that

$$\max_{y \in Y} \min_{x \in X} F(x, y) < \alpha < \min_{x \in X} \max_{y \in Y} F(x, y). \quad (4.26)$$

Define $G : X \times Y \rightrightarrows X \times Y$ by

$$G(x, y) = \{s \in X : \max F(s, y) < \alpha\} \times \{t \in Y : \max F(x, t) > \alpha\}. \quad (4.27)$$

For each $x \in X$, $\max_{y \in Y} F(x, y) \geq \min_{x \in X} \max_{y \in Y} F(x, y) > \alpha$. Since Y is compact and the set-valued mapping $y \mapsto \max F(x, y)$ is upper semi-continuous, there is a $t \in Y$ such that $\max F(x, t) = \max_{y \in Y} F(x, y) > \alpha$.

On the other hand, from the condition (iii), for each $y \in Y$, there is a $x_y \in X$ such that $\max F(x_y, y) < \alpha$. Hence, for each $(x, y) \in X \times Y$, $G(x, y) \neq \emptyset$. By (i) and Proposition 3.6, the mapping $x \rightarrow \max F(x, y)$ is above- $\mathbb{R}_+$ -quasi-convex on X . By (ii) and Proposition 3.7, the mapping $y \rightarrow \max F(x, y)$ is below- $\mathbb{R}_+$ -quasi-concave on Y . Hence, for each $(x, y) \in X \times Y$, the set $G(x, y)$ is convex. From the lower semi-continuities on X and upper semi-continuity on Y of F , the set

$$G^{-1}(s, t) = \{x \in X : \max F(x, t) > \alpha\} \times \{y \in Y : \max F(s, y) < \alpha\} \quad (4.28)$$

is open in $X \times Y$. By Fan-Browder fixed-point Theorem 2.8, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$(\bar{x}, \bar{y}) \in G(\bar{x}, \bar{y}), \quad (4.29)$$

that is,

$$\max F(\bar{x}, \bar{y}) > \alpha > \max F(\bar{x}, \bar{y}), \quad (4.30)$$

which is a contradiction. This completes the proof. $\square$

Remark 4.4. [5, Propositions 2.7 and 2.1] can be deduced from Theorem 4.3. Indeed, in [5, Proposition 2.1], the above-naturally C -quasi-convexity is used. By Proposition 3.12, the condition (i) of Theorem 4.3 holds. Hence the conclusion of Proposition 2.1 in [5] holds. We also note that, in Theorem 4.3, the mapping F need not be continuous on $X \times Y$. Hence Theorem 4.3 is a slight generalization of [7, Theorem 3.1].

Theorem 4.5. *Let X and Y be nonempty compact (not necessarily convex) subsets of real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let the mapping $F : X \times Y \rightrightarrows \mathcal{Z}$ be upper semi-continuous with nonempty compact values and lower semi-continuous on X such that*

- (i) *for each $x \in X$, $y \rightarrow F(x, y)$ is below- C -concave-like on Y ;*
- (ii) *for each $y \in Y$, $x \rightarrow F(x, y)$ is above- C -convex-like on X ;*

(iii) for every $y \in Y$,

$$\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) \subset \text{Min}_w \bigcup_{x \in X} F(x, y) + C. \quad (4.31)$$

Then for any

$$z_1 \in \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y), \quad (4.32)$$

there is a

$$z_2 \in \text{Min} \left(\text{co} \left\{ \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) \right\} \right) \quad (4.33)$$

such that

$$z_1 \in z_2 + C, \quad (4.34)$$

that is,

$$\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) \subset \text{Min} \left(\text{co} \left\{ \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) \right\} \right) + C. \quad (4.35)$$

Proof. Let $\Gamma(x) := \text{Max}_w \bigcup_{y \in Y} F(x, y)$ for all $x \in X$. From Lemma 2.4 and Proposition 3.5, the set-valued mapping $x \mapsto \Gamma(x)$ is upper semi-continuous with nonempty compact values on X . Hence the set $\Gamma(X)$ is compact, and so is $\text{co}\{\Gamma(X)\}$. Then $\text{co}\{\Gamma(X)\} + C$ is a closed convex set with nonempty interior. Suppose that $v \notin \text{co}\{\Gamma(X)\} + C$. By separation hyperplane theorem [15, Theorem 14.2], there exist $k \in \mathbb{R}$, $\varepsilon > 0$ and a nonzero continuous linear functional $\xi : Z \rightarrow \mathbb{R}$ such that

$$\xi(v) \leq k - \varepsilon < k \leq \xi(u + c), \quad \text{for every } u \in \text{co}\{\Gamma(X)\}, \quad c \in C. \quad (4.36)$$

Therefore,

$$\xi(c) > \xi(v - u), \quad \text{for every } u \in \text{co}\{\Gamma(X)\}, \quad c \in C. \quad (4.37)$$

This implies that $\xi \in C^*$ and $\xi(v) < \xi(u)$ for all $u \in \text{co}\{\Gamma(X)\}$.

Let $g := \xi F : X \times Y \rightrightarrows \mathbb{R}$. From Lemma 3.10, for each fixed $x \in X$, there exist $y_x^* \in Y$ and $f(x, y_x^*) \in F(x, y_x^*)$ with $f(x, y_x^*) \in \Gamma(x)$ such that $\xi(f(x, y_x^*)) = \max_{y \in Y} \xi(F(x, y))$. Choosing $c = 0$ and $u = f(x, y_x^*)$ in (4.36), we have

$$\max_{y \in Y} \xi(F(x, y)) = \xi(f(x, y_x^*)) \geq k > k - \varepsilon \geq \xi(v), \quad \forall x \in X. \quad (4.38)$$

Therefore,

$$\min_{x \in X} \bigcup \max_{y \in Y} \xi(F(x, y)) > \xi(v). \quad (4.39)$$

By the conditions (i), (ii) and Proposition 3.8, the set-valued mapping $y \mapsto \xi(F(x, y))$ is below- $\mathbb{R}_+$ -concave-like on Y for all $x \in X$, and the set-valued mapping $x \mapsto \xi(F(x, y))$ is above- $\mathbb{R}_+$ -convex-like on X for all $y \in Y$. From Theorem 4.1, we have

$$\max_{y \in Y} \bigcup \min_{x \in X} \xi(F(x, y)) > \xi(v). \quad (4.40)$$

Since Y is compact, there is an $y' \in Y$ such that $\min_{x \in X} \xi(F(x, y')) > \xi(v)$. For any $x \in X$ and all $g(x, y') \in F(x, y')$, we have

$$\xi(g(x, y')) > \xi(v), \quad (4.41)$$

that is,

$$\xi(g(x, y') - v) > 0, \quad \forall x \in X, g(x, y') \in F(x, y'). \quad (4.42)$$

Thus, $v \notin \bigcup_{x \in X} F(x, y') + C$, and hence,

$$v \notin \text{Min}_w \bigcup_{x \in X} F(x, y') + C. \quad (4.43)$$

If $v \in \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y)$, by the condition (iii), $v \in \text{Min}_w \bigcup_{x \in X} F(x, y') + C$ which contradicts (4.43). Hence, for every $v \in \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y)$,

$$v \in \text{co} \left\{ \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) \right\} + C, \quad (4.44)$$

that is,

$$\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) \subset \text{co} \left\{ \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) \right\} + C \quad (4.45)$$

or

$$\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) \subset \text{Min} \left(\text{co} \left\{ \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) \right\} \right) + C. \quad (4.46)$$

□

The following examples illustrate Theorem 4.5.

Example 4.6. Let $X = Y = \{0\} \cup \{1/n : n \in \mathbb{N}\}$, $C = \mathbb{R}_+^2$ and

$$F(x, y) = \left\{ (s, t) \in \mathbb{R}^2 : s = x^2, t = 1 - y^2 \right\}, \quad \forall (x, y) \in X \times Y. \quad (4.47)$$

It is obviously that F is below- $\mathbb{R}_+^2$ -concave-like on Y and above- $\mathbb{R}_+^2$ -convex-like on X . We now verify the condition (iii) of Theorem 4.5. Indeed, for any $y \in Y$,

$$\begin{aligned} \bigcup_{x \in X} F(x, y) &= \left(\{0\} \cup \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \right) \times \{1 - y^2\}, \\ \text{Min}_w \bigcup_{x \in X} F(x, y) &= \left(\{0\} \cup \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \right) \times \{1 - y^2\}. \end{aligned} \quad (4.48)$$

Then,

$$\begin{aligned} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &= \left(\{0\} \cup \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \right) \times \left(\{1\} \cup \left\{ 1 - \frac{1}{n^2} : n \in \mathbb{N} \right\} \right), \\ \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &= \{(1, 1)\}. \end{aligned} \quad (4.49)$$

Thus, for every $y \in Y$,

$$\begin{aligned} \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &\subset \left(\{0\} \cup \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \right) \times \{1 - y^2\} + C \\ &= \text{Min}_w \bigcup_{x \in X} F(x, y) + C, \end{aligned} \quad (4.50)$$

and the condition (iii) of Theorem 4.5 holds.

Furthermore, for any $x \in X$,

$$\begin{aligned} \bigcup_{y \in Y} F(x, y) &= \{x^2\} \times \left(\{1\} \cup \left\{ 1 - \frac{1}{n^2} : n \in \mathbb{N} \right\} \right), \\ \text{Max}_w \bigcup_{y \in Y} F(x, y) &= \{x^2\} \times \left(\{1\} \cup \left\{ 1 - \frac{1}{n^2} : n \in \mathbb{N} \right\} \right). \end{aligned} \quad (4.51)$$

Then,

$$\begin{aligned} \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) &= \left(\{0\} \cup \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \right) \times \left(\{1\} \cup \left\{ 1 - \frac{1}{n^2} : n \in \mathbb{N} \right\} \right), \\ \text{co} \left\{ \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) \right\} &= [0, 1] \times [0, 1]. \end{aligned} \quad (4.52)$$

Thus,

$$\begin{aligned} \text{Min} \left(\text{co} \left\{ \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) \right\} \right) &= \{(0, 0)\}, \\ \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &= \{(1, 1)\} \subset \text{Min} \left(\text{co} \left\{ \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) \right\} \right) + C. \end{aligned} \quad (4.53)$$

Hence, the conclusion of Theorem 4.5 holds.

Example 4.7. Let $X = [0, 1]$, $Y = [-1, 0]$, $C = \mathbb{R}_+^2$, and $G : Y \rightrightarrows Y$ be defined by

$$G(y) = \begin{cases} [-1, 0], & y = 0, \\ \{0\}, & y \neq 0. \end{cases} \quad (4.54)$$

Let $F(x, y) = \{x^2\} \times G(y)$ for all $(x, y) \in X \times Y$. Then G is upper semi-continuous, but not lower semi-continuous on $\mathbb{R}$, and F is not continuous but is upper semi-continuous on $X \times Y$. Moreover, F has nonempty compact values and is lower semi-continuous on X . It is easy to see that F is below- C -concave-like on Y and is above- C -convex-like on X . We verify the condition (iii) of Theorem 4.5. Indeed, for all $y \in Y$, $\bigcup_{x \in X} F(x, y) = [0, 1] \times G(y)$.

$$\text{Min}_w \bigcup_{x \in X} F(x, y) = \begin{cases} [0, 1] \times \{0\}, & y \neq 0, \\ (\{0\} \times [-1, 0]) \cup ([0, 1] \times \{-1\}), & y = 0. \end{cases} \quad (4.55)$$

Then,

$$\begin{aligned} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &= (\{0\} \times [-1, 0]) \cup ([0, 1] \times \{-1\}) \cup ([0, 1] \times \{0\}), \\ \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &= \{(1, 0)\} \subset \text{Min}_w \bigcup_{x \in X} F(x, y) + C. \end{aligned} \quad (4.56)$$

Therefore, the condition (iii) of Theorem 4.5 holds.

Since

$$F(x, y) = \begin{cases} \{x^2\} \times [-1, 0], & y = 0, \\ \{x^2\} \times \{0\}, & y \neq 0, \end{cases} \quad (4.57)$$

for all $(x, y) \in X \times Y$, and $\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) = \{(1, 0)\}$, for each $y \in Y$, we can choose $x_y = 0 \in X$ such that

$$\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) \subset F(x_y, y) + C. \quad (4.58)$$

Furthermore,

$$\begin{aligned}
 \bigcup_{y \in Y} F(x, y) &= \{x^2\} \times \left(\bigcup_{y \in Y} G(y) \right) \\
 &= \{x^2\} \times ([-1, 0] \cup \{0\}) \\
 &= \{x^2\} \times [-1, 0], \\
 \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) &= [0, 1] \times [-1, 0].
 \end{aligned} \tag{4.59}$$

Therefore,

$$\begin{aligned}
 \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &= \{(1, 0)\} \subset \{(0, -1)\} + C \\
 &= \text{Min} \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) + C.
 \end{aligned} \tag{4.60}$$

Hence, the conclusion of Theorem 4.5 holds.

Remark 4.8. Theorem 3.1 in [5] Theorem 3.1 in [6], or Theorem 4.2 in [7] cannot be applied to Examples 4.6 and 4.7 because of the following reasons:

- (i) the two sets X and Y are not convex in Example 4.6;
- (ii) F is not continuous on $X \times Y$ in Examples 4.6 and 4.7.

Theorem 4.9. *Let X, Y be two nonempty compact convex subsets of real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that the set-valued mapping $F : X \times Y \rightrightarrows \mathcal{Z}$ has nonempty compact values, and it is continuous on Y and lower semi-continuous on X such that*

- (i) *for each $y \in Y$, $x \mapsto F(x, y)$ is above-naturally C -quasi-convex on X ;*
- (ii) *for each $x \in X$, $y \mapsto F(x, y)$ is above- C -concave or above-properly C -quasi-concave on Y ;*
- (iii) *for every $y \in Y$,*

$$\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) \subset \text{Min}_w \bigcup_{x \in X} F(x, y) + C; \tag{4.61}$$

that

$$\max h(F(x_y, y)) \leq \max \bigcup_{y \in Y} \min \bigcup_{x \in X} h(F(x, y)). \tag{4.62}$$

Then, for any $z_1 \in \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y)$, there is a

$$z_2 \in \text{Min} \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) \quad (4.63)$$

such that $z_1 \in z_2 + C$, that is,

$$\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) \subset \text{Min} \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) + C. \quad (4.64)$$

Proof. Let $\Gamma(x)$ be defined as the same as in the proof of Theorem 4.5. Following the same perspective as in the proof of Theorem 4.5, suppose that $v \notin \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) + C$. For any $k \in \text{int } C$ and Gerstewitz function $\xi_{kv} : \mathcal{Z} \Rightarrow \mathbb{R}$. By Proposition 2.7(d), we have

$$\xi_{kv}(u) > 0, \quad \text{for every } u \in \Gamma(X). \quad (4.65)$$

Let $g := \xi_{kv} \circ F : X \times Y \Rightarrow \mathbb{R}$. From Lemma 3.10, for the mapping ξ_{kv} and Remark 3.15, for each $x \in X$, there exist $y_x^* \in Y$ and $f(x, y_x^*) \in F(x, y_x^*)$ with $f(x, y_x^*) \in \text{Max}_w \bigcup_{y \in Y} F(x, y)$ such that $\xi_{kv} f(x, y_x^*) = \max \bigcup_{y \in Y} \xi_{kv}(F(x, y))$. Choosing $u = f(x, y_x^*)$ in (4.65), we have

$$\max \bigcup_{y \in Y} \xi_{kv}(F(x, y)) > 0, \quad \forall x \in X. \quad (4.66)$$

Therefore,

$$\min \bigcup_{x \in X} \max \bigcup_{y \in Y} \xi_{kv}(F(x, y)) > 0. \quad (4.67)$$

By conditions (i), (ii) and Remark 3.15, the set-valued mapping $y \mapsto \xi_{kv}(F(x, y))$ is upper semi-continuous, and either above- $\mathbb{R}_+$ -concave or above-properly $\mathbb{R}_+$ -quasi-concave on Y , and the set-valued mapping $x \mapsto \xi_{kv}(F(x, y))$ is lower semi-continuous and above- $\mathbb{R}_+$ -quasi-convex on X . From Theorem 4.3, we have

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} \xi_{kv}(F(x, y)) > 0. \quad (4.68)$$

Since the set-valued mapping $y \mapsto F(x, y)$ is lower semi-continuous on Y , by Lemma 2.4 (b) and Lemma 2.5 (b), the set-valued mapping $y \mapsto \min \bigcup_{x \in X} \xi_{kv}(F(x, y))$ is upper semi-continuous on Y . By the compactness of Y , there exists $y' \in Y$ such that $\min \bigcup_{x \in X} \xi_{kv}(F(x, y')) > 0$. For all $x \in X$ and all $g(x, y') \in F(x, y')$, we have $\xi_{kv}(g(x, y')) > 0$. Thus, $v \notin \bigcup_{x \in X} F(x, y') + C$, and hence,

$$v \notin \text{Min}_w \bigcup_{x \in X} F(x, y') + C. \quad (4.69)$$

If $v \in \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y)$, by the condition (iii), $v \in \text{Min}_w \bigcup_{x \in X} F(x, y') + C$ which contradicts (4.69). Hence, for every $v \in \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y)$,

$$v \in \text{Min} \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) + C, \quad (4.70)$$

that is,

$$\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) \subset \text{Min} \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) + C. \quad (4.71)$$

This completes the proof. $\square$

The following example illustrates Theorem 4.9.

Example 4.10. Let $X = Y = [0, 1]$, $C = \mathbb{R}_+^2$ and $G : X \rightrightarrows Y$ be a set-valued mapping defined as

$$G(x) = \begin{cases} [0, 1], & x \neq 0, \\ \{0\}, & x = 0. \end{cases} \quad (4.72)$$

Let $F(x, y) = G(x) \times \{-y^2\}$ for all $(x, y) \in X \times Y$. Then G is lower semi-continuous, but not upper semi-continuous on $\mathbb{R}$, and F is continuous on Y , and F has nonempty compact values and is lower semi-continuous on X . It is easy to see that F is above- C -concave or above-properly C -quasi-concave on Y and is above-naturally C -quasi-convex on X .

We verify the condition (iii) of Theorem 4.9. Indeed, for all $y \in Y$, $\bigcup_{x \in X} F(x, y) = [0, 1] \times \{-y^2\}$ and $\text{Min}_w \bigcup_{x \in X} F(x, y) = [0, 1] \times \{-y^2\}$. Hence,

$$\begin{aligned} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &= [0, 1] \times [-1, 0], \\ \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &= \{(1, 0)\} \subset \text{Min}_w \bigcup_{x \in X} F(x, y) + C. \end{aligned} \quad (4.73)$$

Therefore, the condition (iii) of Theorem 4.9 holds.

Since $\text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) = \{(1, 0)\}$ for any $y \in Y$, we can choose $x_y = 0 \in X$ such that

$$F(x_y, y) = \{(0, -y^2)\} \subset \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) - C. \quad (4.74)$$

For any continuous increasing function h , the condition (iv) of Theorem 4.9 holds.

Furthermore, since for each $x \in X$,

$$\begin{aligned} \bigcup_{y \in Y} F(x, y) &= G(x) \times [-1, 0], \\ \text{Max}_w \bigcup_{y \in Y} F(x, y) &= \begin{cases} \{0\} \times [-1, 0], & x = 0, \\ (\{1\} \times [-1, 0]) \cup ([0, 1] \times \{0\}), & x \neq 0, \end{cases} \end{aligned} \quad (4.75)$$

we have

$$\begin{aligned} \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) &= (\{0\} \times [-1, 0]) \cup ([0, 1] \times \{0\}) \cup (\{1\} \times [-1, 0]), \\ \text{Min} \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) &= \{(0, -1)\}. \end{aligned} \quad (4.76)$$

Thus,

$$\begin{aligned} \text{Max} \bigcup_{y \in Y} \text{Min}_w \bigcup_{x \in X} F(x, y) &= \{(1, 0)\} \subset \{(0, -1)\} + C \\ &= \text{Min} \bigcup_{x \in X} \text{Max}_w \bigcup_{y \in Y} F(x, y) + C. \end{aligned} \quad (4.77)$$

Therefore, the conclusion of Theorem 4.9 holds.

Remark 4.11. Theorem 3.1 in [5], Theorem 3.1 in [6], or Theorem 4.2 in [7] cannot be applied to Example 4.10 as F is not continuous on $X \times Y$.

If we choose $Z = \mathbb{R}$ and $C = \mathbb{R}_+$ in Theorems 4.5 and 4.9, we always have $C^* = \mathbb{R}_+$ and for every $y \in Y$,

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) \geq \min \bigcup_{x \in X} F(x, y). \quad (4.78)$$

Hence, the condition (iii) of Theorem 4.5 holds. Thus, we have the following corollaries.

Corollary 4.12. *Let X, Y be nonempty compact (not necessarily convex) subsets of real Hausdorff topological vector space $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ has nonempty compact values such that it is lower semi-continuous on X and is upper semi-continuous on $X \times Y$. Assume that the following conditions hold:*

- (i) *for each $x \in X$, $y \mapsto F(x, y)$ is below- $\mathbb{R}_+$ -concave-like on Y ;*
- (ii) *for each $y \in Y$, $x \mapsto F(x, y)$ is above- $\mathbb{R}_+$ -convex-like on X ;*
- (iii) *for every $y \in Y$,*

$$\max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) \geq \min \bigcup_{x \in X} F(x, y). \quad (4.79)$$

Then, for any

$$z_1 \in \max_{y \in Y} \min_{x \in X} \bigcup F(x, y), \quad (4.80)$$

there is a

$$z_2 \in \min \left(\text{co} \left\{ \bigcup_{x \in X} \max_{y \in Y} \bigcup F(x, y) \right\} \right) \quad (4.81)$$

such that

$$z_1 \geq z_2, \quad (4.82)$$

that is,

$$\max_{y \in Y} \min_{x \in X} \bigcup F(x, y) \geq \min \left(\text{co} \left\{ \bigcup_{x \in X} \max_{y \in Y} \bigcup F(x, y) \right\} \right). \quad (4.83)$$

Corollary 4.13. *Under the framework of Corollary 4.12, in addition, let X, Y be two convex subsets, and let F be upper semi-continuous on $X \times Y$. Then,*

$$\max_{y \in Y} \min_{x \in X} \bigcup F(x, y) = \min_{x \in X} \max_{y \in Y} \bigcup F(x, y). \quad (4.84)$$

Proof. By Corollary 4.12, we have

$$\max_{y \in Y} \min_{x \in X} \bigcup F(x, y) \geq \min \left(\text{co} \left\{ \bigcup_{x \in X} \max_{y \in Y} \bigcup F(x, y) \right\} \right). \quad (4.85)$$

Since the set-valued mapping F is upper semi-continuous on $X \times Y$ and Y is compact, by Lemmas 2.4 and 2.5, the set-valued mapping $x \mapsto \max_{y \in Y} \bigcup F(x, y)$ is upper semi-continuous on X . Since X is convex, it is connected. By [16, Theorem 3.1],

$$\bigcup_{x \in X} \max_{y \in Y} \bigcup F(x, y) \quad (4.86)$$

is connected in $\mathbb{R}$, and hence, it is convex. From (4.85),

$$\max_{y \in Y} \min_{x \in X} \bigcup F(x, y) \geq \min \left(\bigcup_{x \in X} \max_{y \in Y} \bigcup F(x, y) \right). \quad (4.87)$$

This completes the proof. $\square$

When $Z = \mathbb{R}$ and $C = \mathbb{R}_+$, from Theorem 4.9, we deduce the following corollary.

Corollary 4.14. *Let X, Y be two nonempty compact convex subsets in real Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that the set-valued mapping $F : X \times Y \rightrightarrows \mathbb{R}$ has nonempty compact values such that it is continuous on Y and is lower semi-continuous on X . Assume that the following conditions hold:*

- (i) *for each $y \in Y$, $x \rightarrow F(x, y)$ is above-naturally $\mathbb{R}_+$ -quasi-convex on X ;*
- (ii) *for each $x \in X$, $y \rightarrow F(x, y)$ is above- $\mathbb{R}_+$ -concave or above-properly $\mathbb{R}_+$ -quasi-concave on Y ;*
- (iii) *for each $y \in Y$, there exists $x_y \in X$ such that*

$$\max F(x_y, y) \leq \max_{y \in Y} \min_{x \in X} \bigcup F(x, y). \quad (4.88)$$

Then,

$$\max_{y \in Y} \min_{x \in X} \bigcup F(x, y) = \min_{x \in X} \max_{y \in Y} \bigcup F(x, y). \quad (4.89)$$

Remark 4.15. Corollary 4.14 includes Proposition 2.1 in [5].

5. Saddle Points for Set-Valued Mappings

In this section, we discuss the existence of several kinds of saddle points for set-valued mappings including the C-loose saddle points, weak C-saddle points, $\mathbb{R}_+$ -loose saddle points, and $\mathbb{R}_+$ -saddle points of F on $X \times Y$.

Definition 5.1. Let $F : X \times Y \rightrightarrows \mathcal{Z}$ be a set-valued mapping. A point $(\bar{x}, \bar{y}) \in X \times Y$ is said to be a

- (a) C-loose saddle point [7] of F on $X \times Y$ if

$$\begin{aligned} F(\bar{x}, \bar{y}) \cap \left(\text{Max} \bigcup_{y \in Y} F(\bar{x}, y) \right) &\neq \emptyset, \\ F(\bar{x}, \bar{y}) \cap \left(\text{Min} \bigcup_{x \in X} F(x, \bar{y}) \right) &\neq \emptyset; \end{aligned} \quad (5.1)$$

- (b) weak C-saddle point [7] of F on $X \times Y$ if

$$F(\bar{x}, \bar{y}) \cap \left(\text{Max}_w \bigcup_{y \in Y} F(\bar{x}, y) \right) \cap \left(\text{Min}_w \bigcup_{x \in X} F(x, \bar{y}) \right) \neq \emptyset; \quad (5.2)$$

(c) $\mathbb{R}_+$ -loose saddle point of F on $X \times Y$ if $Z = \mathbb{R}$ and

$$F(\bar{x}, \bar{y}) = \left[\min \bigcup_{x \in X} F(x, \bar{y}), \max \bigcup_{y \in Y} F(\bar{x}, y) \right]; \quad (5.3)$$

(d) $\mathbb{R}_+$ -saddle point of F on $X \times Y$ if $Z = \mathbb{R}$ and

$$\max \bigcup_{y \in Y} F(\bar{x}, y) = \min \bigcup_{x \in X} F(x, \bar{y}) = F(\bar{x}, \bar{y}). \quad (5.4)$$

It is obvious that every weak C -saddle point is a C -loose saddle point and every $\mathbb{R}_+$ -saddle point is a $\mathbb{R}_+$ -loose saddle point.

Theorem 5.2. *Under the framework of Theorem 4.1, F has $\mathbb{R}_+$ -saddle point if the set-valued mapping $y \mapsto F(x, y)$ is continuous.*

Proof. By Lemmas 2.4 and 2.5, we attained the max and min in Theorem 4.1. By the compactness of X and Y and the lower semi-continuity of F on X and Y , respectively, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$\begin{aligned} \max \bigcup_{y \in Y} \min \bigcup_{x \in X} F(x, y) &= \min \bigcup_{x \in X} F(x, \bar{y}), \\ \min \bigcup_{x \in X} \max \bigcup_{y \in Y} F(x, y) &= \max \bigcup_{y \in Y} F(\bar{x}, y). \end{aligned} \quad (5.5)$$

Combining this with Theorem 4.1, we have

$$\max \bigcup_{y \in Y} F(\bar{x}, y) = \min \bigcup_{x \in X} F(x, \bar{y}) = F(\bar{x}, \bar{y}), \quad (5.6)$$

and hence, F has $\mathbb{R}_+$ -saddle point. $\square$

Theorem 5.3. *Under the framework of Theorem 4.3, F has $\mathbb{R}_+$ -saddle point if the set-valued mapping $y \mapsto F(x, y)$ is continuous.*

Theorem 5.4. *Under the framework of Theorem 4.5 or Theorem 4.9, F has weak C -saddle point if the set-valued mapping $y \mapsto F(x, y)$ is continuous.*

Proof. For any $\xi \in C^*$, the set-valued mapping $\xi \circ F$ satisfies all the conditions of Theorem 5.2 or Theorem 5.3. Hence, $\xi \circ F$ has $\mathbb{R}_+$ -saddle point, that is, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$\max \bigcup_{y \in Y} \xi(F(\bar{x}, y)) = \min \bigcup_{x \in X} \xi(F(x, \bar{y})) = \xi(F(\bar{x}, \bar{y})). \quad (5.7)$$

Then, for any $z \in F(\bar{x}, \bar{y})$,

$$\begin{aligned}\xi(z) &\in \min_{x \in X} \bigcup \xi(F(x, \bar{y})), \\ \xi(z) &\in \max_{y \in Y} \bigcup \xi(F(\bar{x}, y)).\end{aligned}\tag{5.8}$$

Thus, by Proposition 3.14,

$$z \in \text{Min}_w \bigcup_{x \in X} F(x, \bar{y}) \cap \text{Max}_w \bigcup_{y \in Y} F(\bar{x}, y),\tag{5.9}$$

and $(\bar{x}, \bar{y})$ is a weak C -saddle point of F . □

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