Research Article

Computing the Fixed Points of Strictly Pseudocontractive Mappings by the Implicit and Explicit Iterations

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It is known that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. In this paper, we devote to study computing the fixed points of strictly pseudocontractive mappings by the iterations. Two iterative methods (one implicit and another explicit) for finding the fixed point of strictly pseudocontractive mappings have been constructed in Hilbert spaces. As special cases, we can use these two methods to find the minimum norm fixed point of strictly pseudocontractive mappings.

1. Introduction

In this paper, we devote to study computing the fixed points of strictly pseudocontractive mappings by the iterations. Our motivations are mainly in two respects.

Motivation 1

Iterative methods for finding fixed points of nonexpansive mappings have received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing; see [1–35] and the references therein. It is known [36] that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. Therefore it is interesting to develop the algorithms for strictly pseudocontractive mappings.

Motivation 2

In many problems, it is needed to find a solution with minimum norm. In an abstract way, we may formulate such problems as finding a point $x^\dagger$ with the property

$$x^\dagger \in C, \quad \|x^\dagger\| = \min_{x \in C} \|x\|, \quad (1.1)$$

where C is a nonempty closed convex subset of a real Hilbert space H . A typical example is the least-squares solution to the constrained linear inverse problem [37]. Some related works for finding the minimum-norm solution (or fixed point of nonexpansive mappings) have been considered by some authors. The reader can refer to [38–41].

In the present paper, we present two iterative methods (one implicit and another explicit) for finding the fixed point of strictly pseudocontractive mappings in Hilbert spaces. As special cases, we can use these two methods to find the minimum norm fixed point of strictly pseudocontractive mappings.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H .

2.1. Some Concepts

Recall that a mapping $T : C \rightarrow C$ is called nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad (2.1)$$

for all $x, y \in C$. And a mapping $T : C \rightarrow C$ is said to be strictly pseudocontractive if there exists a constant $0 \leq \lambda < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda \|(I - T)x - (I - T)y\|^2, \quad (2.2)$$

for all $x, y \in C$. For such a case, we also say that T is a λ -strictly pseudocontractive mapping. It is clear that, in a real Hilbert space H , (2.2) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \lambda}{2} \|(I - T)x - (I - T)y\|^2, \quad (2.3)$$

for all $x, y \in C$. It is clear that the class of strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings.

Recall that the nearest point (or metric) projection from H onto C is defined as follows: for each point $x \in H$, $P_C[x]$ is the unique point in C with the property:

$$\|x - P_C x\| \leq \|x - y\|, \quad y \in C. \quad (2.4)$$

Note that P_C is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, y - P_C x \rangle \leq 0, \quad y \in C. \quad (2.5)$$

Consequently, P_C is nonexpansive.

2.2. Several Useful Lemmas

Lemma 2.1 (see [42]). *Let H be a real Hilbert space. There holds the following identity:*

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \quad (2.6)$$

for all $x, y \in H$.

Lemma 2.2 (see [43]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a λ -strict pseudocontraction. Then,*

- (i) $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined;
- (ii) $\kappa I + (1 - \kappa)T$ for $\kappa \in [\lambda, 1)$, is nonexpansive.

Lemma 2.3 (see [42]). *Let C be a nonempty closed convex of a real Hilbert space H . Let $T : C \rightarrow C$ be a λ -strictly pseudocontractive mapping. Then $I - T$ is demiclosed at 0 that is if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \rightarrow 0$, then $x = Tx$.*

Lemma 2.4 (see [44]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n \quad (2.7)$$

for all $n \geq 0$ and

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (2.8)$$

Then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 2.5 (see [45]). *Let $\{a_n\}_{n=0}^\infty$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \sigma_n, \quad n \geq 0, \quad (2.9)$$

where $\{\gamma_n\}_{n=0}^\infty \subset (0, 1)$ and $\{\sigma_n\}_{n=0}^\infty$ satisfy

- (i) $\sum_{n=0}^\infty \gamma_n = \infty$,
- (ii) either $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ or $\sum_{n=0}^\infty |\gamma_n \sigma_n| < \infty$.

Then $\{a_n\}_{n=0}^\infty$ converges to 0.

We use the following notation:

- (i) $\text{Fix}(T)$ stands for the set of fixed points of T ;
- (ii) $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x ;
- (iii) $x_n \rightarrow x$ stands for the strong convergence of (x_n) to x .

3. Iterations and Convergence Analysis

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ a λ -strictly pseudocontractive mapping with $\text{Fix}(T) \neq \emptyset$. Let $\kappa \in (0, 1)$ be a constant. For $u \in H$ and any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following implicit manner:*

$$x_n = \kappa T x_n + (1 - \kappa) P_C [\alpha_n u + (1 - \alpha_n) x_n], \quad n \geq 0. \quad (3.1)$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\text{Fix}(T)}(u)$.

Proof. Step 1. The sequence $\{x_n\}$ is well defined.

Set $\beta = \kappa / (1 - (1 - \kappa)\lambda)$. It is easily to check that $\beta \in (0, 1)$. Then, we can rewrite (3.1) as

$$x_n = \frac{\beta(1 - \lambda)}{1 - \beta\lambda} T x_n + \frac{1 - \beta}{1 - \beta\lambda} P_C [\alpha_n u + (1 - \alpha_n) x_n], \quad n \geq 0, \quad (3.2)$$

which is equivalent to the following:

$$x_n = \beta(\lambda x_n + (1 - \lambda) T x_n) + (1 - \beta) P_C [\alpha_n u + (1 - \alpha_n) x_n], \quad n \geq 0. \quad (3.3)$$

Note that $\lambda I + (1 - \lambda)T$ is nonexpansive (see Lemma 2.2). For fix n , we define a mapping $S_n : C \rightarrow C$ by

$$S_n x = \beta(\lambda x + (1 - \lambda) T x) + (1 - \beta) P_C [\alpha_n u + (1 - \alpha_n) x], \quad x \in C. \quad (3.4)$$

For $x, y \in C$, we have

$$\begin{aligned} \|S_n x - S_n y\| &= \|\beta(\lambda I + (1 - \lambda)T)x + (1 - \beta)P_C[\alpha_n u + (1 - \alpha_n)x] \\ &\quad - \beta(\lambda I + (1 - \lambda)T)y - (1 - \beta)P_C[\alpha_n u + (1 - \alpha_n)y]\| \\ &\leq \beta\|x - y\| + (1 - \beta)(1 - \alpha_n)\|x - y\| \\ &= [1 - (1 - \beta)\alpha_n]\|x - y\|, \end{aligned} \quad (3.5)$$

which implies that S_n is a self-contraction of C for every n . Hence S_n has a unique fixed point $x_n \in C$ which is the unique solution of the fixed point equation (3.3).

Step 2. The sequence $\{x_n\}$ is bounded.

Pick up any $x^* \in \text{Fix}(T)$. From (3.3), we have

$$\begin{aligned}
 \|x_n - x^*\| &= \|\beta(\lambda x_n + (1 - \lambda)Tx_n) + (1 - \beta)P_C[\alpha_n u + (1 - \alpha_n)x_n] - x^*\| \\
 &\leq \beta\|\lambda x_n + (1 - \lambda)Tx_n - x^*\| + (1 - \beta)\|P_C[\alpha_n u + (1 - \alpha_n)x_n] - x^*\| \\
 &\leq \beta\|x_n - x^*\| + (1 - \beta)\|\alpha_n(u - x^*) + (1 - \alpha_n)(x_n - x^*)\| \\
 &\leq \beta\|x_n - x^*\| + (1 - \beta)[(1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|u - x^*\|].
 \end{aligned} \tag{3.6}$$

It follows that

$$\|x_n - x^*\| \leq \|u - x^*\|. \tag{3.7}$$

Hence, $\{x_n\}$ is bounded and so is $\{Tx_n\}$.

Step 3. $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.

From (3.3), we have

$$\begin{aligned}
 \|x_n - Tx_n\| &\leq \beta\lambda\|x_n - Tx_n\| + (1 - \beta)\|P_C[\alpha_n u + (1 - \alpha_n)x_n] - P_C[Tx_n]\| \\
 &\leq \beta\lambda\|x_n - Tx_n\| + (1 - \beta)(\alpha_n\|u - Tx_n\| + (1 - \alpha_n)\|x_n - Tx_n\|) \\
 &= [1 - (1 - \alpha_n - \lambda)\beta]\|x_n - Tx_n\| + (1 - \beta)\alpha_n\|u - Tx_n\|.
 \end{aligned} \tag{3.8}$$

It follows that

$$\|x_n - Tx_n\| \leq \frac{1 - \beta}{(1 - \alpha_n - \lambda)\beta} \alpha_n \|u - Tx_n\| \longrightarrow 0. \tag{3.9}$$

Step 4. $x_n \rightarrow \bar{x} \in P_{\text{Fix}(T)}(u)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which converges weakly to a point $\bar{x} \in C$. Noticing (3.9) we can use Lemma 2.3 to get $\bar{x} \in \text{Fix}(T)$.

By using the convexity of the norm and Lemma 2.1, for any $\tilde{x} \in \text{Fix}(T)$, we have

$$\begin{aligned}
 \|x_n - \tilde{x}\|^2 &= \|\beta(\lambda x_n + (1 - \lambda)Tx_n - \tilde{x}) + (1 - \beta)(P_C[\alpha_n u + (1 - \alpha_n)x_n] - \tilde{x})\|^2 \\
 &\leq \beta\|\lambda x_n + (1 - \lambda)Tx_n - \tilde{x}\|^2 + (1 - \beta)\|P_C[\alpha_n u + (1 - \alpha_n)x_n] - \tilde{x}\|^2 \\
 &\leq \beta\|x_n - \tilde{x}\|^2 + (1 - \beta)\|\alpha_n(u - \tilde{x}) + (1 - \alpha_n)(x_n - \tilde{x})\|^2 \\
 &= \beta\|x_n - \tilde{x}\|^2 + (1 - \beta)\left[(1 - \alpha_n)^2\|x_n - \tilde{x}\|^2 + 2\alpha_n(1 - \alpha_n)\langle u - \tilde{x}, x_n - \tilde{x} \rangle + \alpha_n^2\|u - \tilde{x}\|^2\right] \\
 &= \beta\|x_n - \tilde{x}\|^2 + (1 - \beta)\left[\|x_n - \tilde{x}\|^2 - 2\alpha_n\|x_n - \tilde{x}\|^2 + 2\alpha_n\langle u - \tilde{x}, x_n - \tilde{x} \rangle\right. \\
 &\quad \left.+ \alpha_n^2(\|u - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - 2\langle u - \tilde{x}, x_n - \tilde{x} \rangle)\right].
 \end{aligned} \tag{3.10}$$

It turns out that

$$\begin{aligned}\|x_n - \tilde{x}\|^2 &\leq \langle u - \tilde{x}, x_n - \tilde{x} \rangle + \frac{\alpha_n}{2} \left(\|u - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 + 2\|u - \tilde{x}\|\|x_n - \tilde{x}\| \right) \\ &\leq \langle u - \tilde{x}, x_n - \tilde{x} \rangle + \alpha_n M,\end{aligned}\quad (3.11)$$

where $M > 0$ is some constant such that

$$\sup \frac{1}{2} \left\{ \|u - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 + 2\|u - \tilde{x}\|\|x_n - \tilde{x}\| \right\} \leq M. \quad (3.12)$$

Therefore we can substitute $\tilde{x}$ for $\bar{x}$ in (3.11) to get

$$\|x_n - \bar{x}\|^2 \leq \langle u - \bar{x}, x_n - \bar{x} \rangle + \alpha_n M. \quad (3.13)$$

However, $x_n \rightarrow \bar{x}$. This together with (3.13) guarantees that $x_n \rightarrow \bar{x}$. It is clear that $\bar{x} = P_{\text{Fix}(T)}(u)$. As a matter of fact, in (3.11), if we let $n \rightarrow \infty$, then we get

$$\langle u - \tilde{x}, \bar{x} - \tilde{x} \rangle \geq 0, \quad \forall \tilde{x} \in \text{Fix}(T). \quad (3.14)$$

This is equivalent to

$$\langle u - \bar{x}, \bar{x} - \tilde{x} \rangle \geq 0, \quad \forall \tilde{x} \in \text{Fix}(T). \quad (3.15)$$

Hence, $\bar{x} = P_{\text{Fix}(T)}(u)$. Therefore, $x_n \rightarrow \bar{x} = P_{\text{Fix}(T)}(u)$. This completes the proof. $\square$

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Let $\kappa \in (0, 1)$ be a constant. For $u \in H$ and any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following implicit manner:*

$$x_n = \kappa T x_n + (1 - \kappa) P_C[\alpha_n u + (1 - \alpha_n) x_n], \quad n \geq 0. \quad (3.16)$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\text{Fix}(T)}(u)$.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ a λ -strictly pseudocontractive mapping with $\text{Fix}(T) \neq \emptyset$. Let $\kappa \in (0, 1)$ be a constant. For any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following implicit manner:*

$$x_n = \kappa T x_n + (1 - \kappa) P_C[(1 - \alpha_n) x_n], \quad n \geq 0. \quad (3.17)$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\text{Fix}(T)}(0)$ which is the minimum norm fixed point of T .

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Let $\kappa \in (0, 1)$ be a constant. For any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following implicit manner:*

$$x_n = \kappa T x_n + (1 - \kappa) P_C[(1 - \alpha_n)x_n], \quad n \geq 0. \quad (3.18)$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\text{Fix}(T)}(0)$ which is the minimum norm fixed point of T .

Next, we introduce an explicit algorithm for finding the fixed point of T .

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ a λ -strictly pseudocontractive mapping with $\text{Fix}(T) \neq \emptyset$. Let β and δ be two constants in $(0, 1)$ satisfying $\beta + \delta < 1$. For $u \in H$ and any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following explicit manner:*

$$x_{n+1} = (\beta\lambda + \delta)x_n + \beta(1 - \lambda)Tx_n + (1 - \beta - \delta)P_C[\alpha_n u + (1 - \alpha_n)x_n], \quad n \geq 0, \quad (3.19)$$

where $\alpha_n \in (0, 1)$ satisfies the following conditions:

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0,$$

$$(C2) \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\text{Fix}(T)}(u)$.

Proof. Step 1. The sequence $\{x_n\}$ is bounded.

First, we can rewrite (3.19) as

$$x_{n+1} = \beta(\lambda x_n + (1 - \lambda)Tx_n) + \delta x_n + (1 - \beta - \delta)P_C[\alpha_n u + (1 - \alpha_n)x_n], \quad n \geq 0. \quad (3.20)$$

Take $x^* \in \text{Fix}(T)$. From (3.20), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\beta(\lambda x_n + (1 - \lambda)Tx_n - x^*) + \delta(x_n - x^*) + (1 - \beta - \delta)(P_C[\alpha_n u + (1 - \alpha_n)x_n] - x^*)\| \\ &\leq \beta\|\lambda x_n + (1 - \lambda)Tx_n - x^*\| + \delta\|x_n - x^*\| \\ &\quad + (1 - \beta - \delta)\|P_C[\alpha_n u + (1 - \alpha_n)x_n] - x^*\| \\ &\leq \beta\|x_n - p\| + \delta\|x_n - x^*\| + (1 - \beta - \delta)(\alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\|) \\ &= [1 - (1 - \beta - \delta)\alpha_n]\|x_n - x^*\| + (1 - \beta - \delta)\alpha_n\|u - x^*\| \\ &\leq \max\{\|x_n - x^*\|, \|u - x^*\|\}. \end{aligned} \quad (3.21)$$

By induction,

$$\|x_{n+1} - x^*\| \leq \max\{\|x_n - x^*\|, \|u - x^*\|\}. \quad (3.22)$$

Hence, the sequence $\{x_n\}$ is bounded and $\{Tx_n\}$ is also bounded.

Step 2. $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.
We can rewrite (3.20) as

$$x_{n+1} = \delta x_n + (1 - \delta)y_n, \quad (3.23)$$

where

$$y_n = \frac{\beta}{1 - \delta}(\lambda x_n + (1 - \lambda)Tx_n) + \frac{1 - \beta - \delta}{1 - \delta}P_C[\alpha_n u + (1 - \alpha_n)x_n], \quad n \geq 0. \quad (3.24)$$

It follows that

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \frac{\beta}{1 - \delta} \|(\lambda x_{n+1} + (1 - \lambda)Tx_{n+1}) - (\lambda x_n + (1 - \lambda)Tx_n)\| \\ &\quad + \frac{1 - \beta - \delta}{1 - \delta} \|P_C[\alpha_{n+1}u + (1 - \alpha_{n+1})x_{n+1}] - P_C[\alpha_n u + (1 - \alpha_n)x_n]\| \\ &\leq \frac{\beta}{1 - \delta} \|x_{n+1} - x_n\| + \frac{1 - \beta - \delta}{1 - \delta} |\alpha_{n+1} - \alpha_n| (\|u\| + \|x_n\|) \\ &\quad + \frac{1 - \beta - \delta}{(1 - \delta)} (1 - \alpha_{n+1}) \|x_{n+1} - x_n\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1 - \beta - \delta}{1 - \delta} |\alpha_{n+1} - \alpha_n| (\|u\| + \|x_n\|). \end{aligned} \quad (3.25)$$

Thus,

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq \limsup_{n \rightarrow \infty} \frac{1 - \beta - \delta}{1 - \delta} |\alpha_{n+1} - \alpha_n| (\|u\| + \|x_n\|) = 0. \quad (3.26)$$

This together with Lemma 2.4 implies that

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \quad (3.27)$$

Note that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - y_n\| + \|y_n - Tx_n\| \\ &= \|x_n - y_n\| + \left\| \frac{\beta}{1 - \delta}(\lambda x_n + (1 - \lambda)Tx_n) + \frac{1 - \beta - \delta}{1 - \delta}P_C[\alpha_n u + (1 - \alpha_n)x_n] - Tx_n \right\| \\ &\leq \|x_n - y_n\| + \frac{\beta}{1 - \delta} \lambda \|x_n - Tx_n\| + \frac{1 - \beta - \delta}{1 - \delta} \|P_C[\alpha_n u + (1 - \alpha_n)x_n] - Tx_n\| \end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - y_n\| + \frac{\beta}{1-\delta} \lambda \|x_n - Tx_n\| + \frac{1-\beta-\delta}{1-\delta} \alpha_n \|u - Tx_n\| \\
&\quad + \frac{1-\beta-\delta}{1-\delta} (1-\alpha_n) \|x_n - Tx_n\|.
\end{aligned} \tag{3.28}$$

It follows that

$$\|x_n - Tx_n\| \leq \frac{1-\delta}{\beta(1-\lambda)} \|x_n - y_n\| + \frac{1-\beta-\delta}{\beta(1-\lambda)} \alpha_n \|u - Tx_n\|. \tag{3.29}$$

Thus,

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \tag{3.30}$$

Step 3. $\limsup_{n \rightarrow \infty} \langle u - \tilde{x}, x_n - \tilde{x} \rangle \leq 0$, where $\tilde{x} = P_{\text{Fix}(T)}(u)$.

To see this, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying the properties

$$\limsup_{n \rightarrow \infty} \langle u - \tilde{x}, x_n - \tilde{x} \rangle = \lim_{k \rightarrow \infty} \langle u - \tilde{x}, x_{n_k} - \tilde{x} \rangle, \tag{3.31}$$

$$x_{n_k} \rightharpoonup x^* \quad \text{as } k \rightarrow \infty. \tag{3.32}$$

By the demiclosed principle (see Lemma 2.3) and (3.30), we have that $x^* \in \text{Fix}(T)$. So,

$$\limsup_{n \rightarrow \infty} \langle u - \tilde{x}, x_n - \tilde{x} \rangle = \langle u - \tilde{x}, x^* - \tilde{x} \rangle \leq 0. \tag{3.33}$$

Step 4. $x_n \rightarrow \tilde{x}$.

From (3.20), we get

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \beta \|\lambda x_n + (1-\lambda)Tx_n - \tilde{x}\|^2 + \delta \|x_n - \tilde{x}\|^2 \\
&\quad + (1-\beta-\delta) \|P_C[\alpha_n u + (1-\alpha_n)x_n] - \tilde{x}\|^2 \\
&\leq (\beta + \delta) \|x_n - \tilde{x}\|^2 + (1-\beta-\delta) \|\alpha_n(u - \tilde{x}) + (1-\alpha_n)(x_n - \tilde{x})\|^2 \\
&= (\beta + \delta) \|x_n - \tilde{x}\|^2 + (1-\beta-\delta) \left[(1-2\alpha_n) \|x_n - \tilde{x}\|^2 + 2\alpha_n \langle u - \tilde{x}, x_n - \tilde{x} \rangle \right. \\
&\quad \left. + \alpha_n^2 (\|u - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - 2\langle u - \tilde{x}, x_n - \tilde{x} \rangle) \right] \\
&= [1 - 2(1-\beta-\delta)\alpha_n] \|x_n - \tilde{x}\|^2 \\
&\quad + 2(1-\beta-\delta)\alpha_n \left(\langle u - \tilde{x}, x_n - \tilde{x} \rangle + \frac{\alpha_n}{2} (\|u - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - 2\langle u - \tilde{x}, x_n - \tilde{x} \rangle) \right) \\
&= (1-\delta_n) \|x_n - \tilde{x}\|^2 + \delta_n \theta_n,
\end{aligned} \tag{3.34}$$

where $\delta_n = 2(1 - \beta - \delta)\alpha_n$ and

$$\theta_n = \langle u - \tilde{x}, x_n - \tilde{x} \rangle + \frac{\alpha_n}{2} \left(\|u - \tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - 2\langle u - \tilde{x}, x_n - \tilde{x} \rangle \right). \quad (3.35)$$

It is easy to see that $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\limsup_{n \rightarrow \infty} \theta_n \leq 0$. We can therefore apply Lemma 2.5 to (3.34) and conclude that $x_{n+1} \rightarrow \tilde{x}$ as $n \rightarrow \infty$. This completes the proof. $\square$

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Let β and δ be two constants in $(0, 1)$ satisfying $\beta + \delta < 1$. For $u \in H$ and any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following explicit manner:*

$$x_{n+1} = (\beta\lambda + \delta)x_n + \beta(1 - \lambda)Tx_n + (1 - \beta - \delta)P_C[\alpha_n u + (1 - \alpha_n)x_n], \quad n \geq 0, \quad (3.36)$$

where $\alpha_n \in (0, 1)$ satisfies the following conditions:

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0,$$

$$(C2) \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\text{Fix}(T)}(u)$.

Corollary 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ a λ -strictly pseudocontractive mapping with $\text{Fix}(T) \neq \emptyset$. Let β and δ be two constants in $(0, 1)$ satisfying $\beta + \delta < 1$. For any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following explicit manner:*

$$x_{n+1} = (\beta\lambda + \delta)x_n + \beta(1 - \lambda)Tx_n + (1 - \beta - \delta)P_C[(1 - \alpha_n)x_n], \quad n \geq 0, \quad (3.37)$$

where $\alpha_n \in (0, 1)$ satisfies the following conditions:

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0,$$

$$(C2) \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\text{Fix}(T)}(0)$ which is the minimum norm fixed point of T .

Corollary 3.8. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Let β and δ be two constants in $(0, 1)$ satisfying $\beta + \delta < 1$. For any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following explicit manner:*

$$x_{n+1} = (\beta\lambda + \delta)x_n + \beta(1 - \lambda)Tx_n + (1 - \beta - \delta)P_C[(1 - \alpha_n)x_n], \quad n \geq 0, \quad (3.38)$$

where $\alpha_n \in (0, 1)$ satisfies the following conditions:

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0;$$

$$(C2) \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\text{Fix}(T)}(0)$ which is the minimum norm fixed point of T .

4. Conclusion

Finding fixed points of nonlinear mappings (especially, nonexpansive mappings) has received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery and signal processing. It is wellknown that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. In this paper, we devote to construct the methods for computing the fixed points of strictly pseudocontractive mappings. Two iterative methods have been presented. Especially, we can use these two methods to find the minimum norm fixed point of strictly pseudocontractive mappings. The ideas contained in the present paper can help us to solve the minimum norm problems in the applied science.

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References

- [1] F. E. Browder, "Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces," *Archive for Rational Mechanics and Analysis*, vol. 24, pp. 82–90, 1967.
- [2] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.
- [3] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.
- [4] P. L. Lions, "Approximation de points fixes de contractions," vol. 284, no. 21, pp. A1357–A1359, 1977.
- [5] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.
- [6] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.
- [7] S. Reich and A. J. Zaslavski, "Convergence of Krasnoselskii-Mann iterations of nonexpansive operators," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1423–1431, 2000.
- [8] A. T. M. Lau and W. Takahashi, "Fixed point properties for semigroup of nonexpansive mappings on Fréchet spaces," *Nonlinear Analysis*, vol. 70, no. 11, pp. 3837–3841, 2009.
- [9] P. L. Combettes and T. Pennanen, "Generalized Mann iterates for constructing fixed points in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 2, pp. 521–536, 2002.
- [10] H. H. Bauschke, "The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 1, pp. 150–159, 1996.
- [11] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.
- [12] H. K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.
- [13] S. A. Hirstoaga, "Iterative selection methods for common fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1020–1035, 2006.
- [14] A. Petruşel and J.-C. Yao, "Viscosity approximation to common fixed points of families of nonexpansive mappings with generalized contractions mappings," *Nonlinear Analysis*, vol. 69, no. 4, pp. 1100–1111, 2008.
- [15] Y. Yao, R. Chen, and Y. C. Liou, "A unified implicit algorithm for solving the triplehierarchical constrained optimization problem," *Mathematical & Computer Modelling*, vol. 55, no. 3–4, pp. 1506–1515, 2012.
- [16] Y. Yao and N. Shahzad, "Strong convergence of a proximal point algorithm with general errors," *Optimization Letters*, vol. 6, no. 4, pp. 621–628, 2012.
- [17] Y. Yao and N. Shahzad, "New methods with perturbations for non-expansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2011, article 79, 2011.

- [18] S. Reich and H. K. Xu, "An iterative approach to a constrained least squares problem," *Abstract and Applied Analysis*, no. 8, pp. 503–512, 2003.
- [19] Y. Yao, Y. C. Liou, and G. Marino, "Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 279058, 7 pages, 2009.
- [20] N. Shahzad, "Approximating fixed points of non-self nonexpansive mappings in Banach spaces," *Nonlinear Analysis*, vol. 61, no. 6, pp. 1031–1039, 2005.
- [21] H. Zegeye and N. Shahzad, "Viscosity approximation methods for a common fixed point of finite family of nonexpansive mappings," *Applied Mathematics and Computation*, vol. 191, no. 1, pp. 155–163, 2007.
- [22] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.
- [23] N. Shioji and W. Takahashi, "Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 125, no. 12, pp. 3641–3645, 1997.
- [24] W. Takahashi and K. Shimoji, "Convergence theorems for nonexpansive mappings and feasibility problems," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1463–1471, 2000.
- [25] P.-E. Maingé, "Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 469–479, 2007.
- [26] G. Marino and H.-K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.
- [27] Y. Yao, R. Chen, and J.-C. Yao, "Strong convergence and certain control conditions for modified Mann iteration," *Nonlinear Analysis*, vol. 68, no. 6, pp. 1687–1693, 2008.
- [28] L. C. Ceng, P. Cubiotti, and J. C. Yao, "Strong convergence theorems for finitely many nonexpansive mappings and applications," *Nonlinear Analysis*, vol. 67, no. 5, pp. 1464–1473, 2007.
- [29] S. S. Chang, "Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1402–1416, 2006.
- [30] Y. Yao, Y. C. Liou, and G. Marino, "Strong convergence of some algorithms for λ -strict pseudo-contractions in Hilbert spaces," *Bulletin of Australian Mathematical Society*, vol. 85, no. 2, pp. 232–240, 2012.
- [31] Y. Yao and J.-C. Yao, "On modified iterative method for nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1551–1558, 2007.
- [32] F. Cianciaruso, G. Marino, L. Muglia, and Y. Yao, "On a two-step algorithm for hierarchical fixed point problems and variational inequalities," *Journal of Inequalities and Applications*, vol. 2009, Article ID 208692, 13 pages, 2009.
- [33] M. A. Noor, "Some developments in general variational inequalities," *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.
- [34] M. A. Noor, "Extended general variational inequalities," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 182–185, 2009.
- [35] M. A. Noor, "Some aspects of extended general variational inequalities," *Abstract and Applied Analysis*, vol. 2012, Article ID 303569, 16 pages, 2012.
- [36] O. Scherzer, "Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 3, pp. 911–933, 1995.
- [37] A. Sabharwal and L. C. Potter, "Convexly constrained linear inverse problems: iterative least-squares and regularization," *IEEE Transactions on Signal Processing*, vol. 46, no. 9, pp. 2345–2352, 1998.
- [38] Y. Yao, R. Chen, and H.-K. Xu, "Schemes for finding minimum-norm solutions of variational inequalities," *Nonlinear Analysis*, vol. 72, no. 7-8, pp. 3447–3456, 2010.
- [39] Y. L. Cui and X. Liu, "Notes on Browder's and Halpern's methods for nonexpansive mappings," *Fixed Point Theory*, vol. 10, no. 1, pp. 89–98, 2009.
- [40] X. Liu and Y. Cui, "The common minimal-norm fixed point of a finite family of nonexpansive mappings," *Nonlinear Analysis*, vol. 73, no. 1, pp. 76–83, 2010.
- [41] Y. Yao and H. K. Xu, "Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications," *Optimization*, vol. 60, no. 6, pp. 645–658, 2011.
- [42] G. Marino and H.-K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.
- [43] H. Zhou, "Convergence theorems of fixed points for λ -strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis*, vol. 69, no. 2, pp. 456–462, 2008.

- [44] T. Suzuki, "Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces," *Fixed Point Theory and Applications*, no. 1, pp. 103–123, 2005.
- [45] H. K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.