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Research Article

On the Global Well-Posedness of the Viscous Two-Component Camassa-Holm System

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We establish the local well-posedness for the viscous two-component Camassa-Holm system. Moreover, applying the energy identity, we obtain a global existence result for the system with $(u_0, \eta_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

1. Introduction

We are interested in the global well-pose dness of the initial value problem associated to the viscous version of the two-component Camassa-Holm shallow water system [1–3], namely,

$$\begin{gathered} m_t + um_x + 2u_x m - Au_x + \rho\rho_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ m = u - u_{xx,}, \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where the variable $u(t,x)$ represents the horizontal velocity of the fluid or the radial stretch related to a prestressed state, and $\rho(t,x)$ is related to the free surface elevation from equilibrium or scalar density with the boundary assumptions, $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. The parameter $A > 0$ characterizes a linear underlying shear flow, so that (1.1) models wave-current interactions [4–6]. All of those are measured in dimensionless units.

Set $p(x) := (1/2)e^{-|x|}$, $x \in \mathbb{R}$. Then $(1-\partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$, where $*$ denotes the spatial convolution. Let $\eta = \rho - 1$, (1.1) can be rewritten as a quasilinear nonlocal evolution system of the type

$$\begin{gathered} u_t + uu_x = -\partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\eta^2 + \eta\right), \quad t > 0,\ x \in \mathbb{R}, \\ \eta_t + u\eta_x + \eta u_x + u_x = 0, \quad t > 0,\ x \in \mathbb{R}. \end{gathered} \tag{1.2}$$

The system (1.1) without vorticity, that is, $A = 0$, was also rigorously justified by Constantin and Ivanov [1] to approximate the governing equations for shallow water waves. The multipeakon solutions of the same system have been constructed by Popivanov and Slavova [7], and the corresponding integral surface is partially ruled. Chen et al. [8] established a reciprocal transformation between the two-component Camassa-Holm system and the first negative flow of AKNS hierarchy. More recently, Holm et al. [9] proposed a modified two-component Camassa-Holm system which possesses singular solutions in component ρ. Mathematical properties of (1.1) with $A = 0$ have been also studied further in many works. For example, Escher et al. [10] investigated local well-posedness for the two-component Camassa-Holm system with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$ with $s \geq 2$ and derived some precise blow-up scenarios for strong solutions to the system. Constantin and Ivanov [1] provided some conditions of wave breaking and small global solutions. Gui and Liu [11] recently obtained results of local well-posedness in the Besov spaces and wave breaking for certain initial profiles. More recently, Gui and Liu [12] studied global existence and wave-breaking criteria for the system (1.2) with initial data $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$ with $s > (3/2)$.

In this paper, we consider the global well-posedness of the viscous two-component Camassa-Holm system

$$\begin{gathered} u_t + uu_x - u_{xx} = -\partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\eta^2 + \eta\right), \quad t > 0,\ x \in \mathbb{R}, \\ \eta_t + u\eta_x + \eta u_x + u_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \\ \eta(0,x) = \eta_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.3}$$

The goal of the present paper is to study global existence of solutions for (1.3) to better understand the properties of the two-component Camassa-Holm system (1.2). We state the main result as follows.

Theorem 1.1. *For $(u_0, \eta_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, there exists a unique global solution (u, η) of (1.3) such that*

$$\begin{gathered} u(t,x) \in C\left([0,\infty); L_x^2(\mathbb{R})\right) \cap C\left((0,\infty); H_x^1(\mathbb{R})\right), \\ \eta(t,x) \in C\left([0,\infty); L_x^2(\mathbb{R})\right). \end{gathered} \tag{1.4}$$

To proof Theorem 1.1, we will first establish global well-posedness of the following regularized two-component system with $\varepsilon > 0$ given:

$$\begin{gathered} u_t + uu_x - u_{xx} = -\partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2 + \frac{1}{2}u_x^2 - Au + \frac{1}{2}\eta^2 + \eta\right), \quad t > 0,\ x \in \mathbb{R}, \\ \eta_t - \varepsilon\eta_{xx} + u\eta_x + \eta u_x + u_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \\ \eta(0,x) = \eta_0(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.5}$$

that is,

$$\begin{gathered} m_t - m_{xx} + um_x + 2u_x m - Au_x + \eta\eta_x + \eta_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ \eta_t - \varepsilon\eta_{xx} + u\eta_x + \eta u_x + u_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ m = u - u_{xx}, \quad t \geq 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad \eta(0,x) = \eta_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.6}$$

Due to the Duhamel's principle, we can also rewrite (1.6) as an integral equation

$$\begin{gathered} u(t,x) = e^{t\partial_x^2}u_0 + \int_0^t e^{(t-\tau)\partial_x^2} f\left(u, \partial_x u, \eta\right) d\tau, \\ \eta(t,x) = e^{\varepsilon t\partial_x^2}\eta_0 + \int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g\left(u, \partial_x u, \eta, \partial_x \eta\right) d\tau, \end{gathered} \tag{1.7}$$

where $g(u, \partial_x u, \eta, \partial_x \eta) = -u\partial_x\eta - \eta\partial_x u - \partial_x u$,

$$f\left(u, \partial_x u, \eta\right) = -\partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2 + \frac{1}{2}(\partial_x u)^2 - Au + \frac{1}{2}\eta^2 + \eta\right) - \frac{1}{2}\partial_x\left(u^2\right), \tag{1.8}$$

$e^{t\partial_x^2}u_0 = \left(e^{-4\pi^2 t\xi}\widehat{u_0}(\xi)\right)^\vee$, $e^{\varepsilon t\partial_x^2}\eta_0 = \left(e^{-4\pi^2\varepsilon t\xi}\widehat{\eta_0}(\xi)\right)^\vee$, here and in what follows, we denote the Fourier (or inverse Fourier) transform of a function f by $\widehat{f}$ (or $f^\vee$).

The remainder of the paper is organized as follows. In Section 2, we will set up and introduce some estimates for the nonlinear part of (1.5). In Section 3, we will get the local well-posedness of (1.3) by constructing the global well-posedness of (1.5) using the contraction argument and energy identity. The last section is devoted to the proof of Theorem 1.1.

2. Preliminaries

We will list some lemmas needed in Section 3. First, we state the following lemma which consists of the crucial inequality involving the operator $\partial_x(1-\partial_x^2)^{-1}$.

Lemma 2.1 (see [13]). *For $g, h \in L^2(\mathbb{R})$,*

$$\left\| \partial_x \left(1 - \partial_x^2\right)^{-1} (gh) \right\|_{L_x^2} \leq c \|g\|_{L_x^2} \|h\|_{L_x^2}, \tag{2.1}$$

or more generally

$$\left\| |\partial_x|^s \left(1 - \partial_x^2\right)^{-1} (gh) \right\|_{L_x^2} \leq c \|g\|_{L_x^2} \|h\|_{L_x^2}, \tag{2.2}$$

for all $s < 3/2$.

The next two lemmas are regarding the nonlinear part of (1.5).

Lemma 2.2. *Consider the following:*

$$\begin{aligned} &\left\| \int_0^t e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x) d\tau \right\|_{L_T^\infty L_x^2} \\ &\quad \leq C\left(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\eta\|_{L_T^2 L_x^2}^2 + T^{1/2}\left(\|u\|_{L_T^2 L_x^2} + \|\eta\|_{L_T^2 L_x^2} \right) \right), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\left\| \int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u, \partial_x u, \eta, \partial_x \eta)(\tau, x) d\tau \right\|_{L_T^\infty L_x^2} \\ &\quad \leq C\left(\|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\partial_x \eta\|_{L_T^2 L_x^2}^2 + T^{1/2} \|\partial_x u\|_{L_T^2 L_x^2} \right). \end{aligned} \tag{2.4}$$

Proof. Let us prove (2.3) firstly. Thanks to Lemma 2.1, the Sobolev embedding theorem $H_x^1(\mathbb{R}) \hookrightarrow L_x^\infty(\mathbb{R})$, and the Hölder's inequality, we have

$$\begin{aligned} &\left\| \int_0^t e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x) d\tau \right\|_{L_T^\infty L_x^2} \\ &\quad \leq \sup_{t \in [0,T]} \int_0^t \left\| e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x) \right\|_{L_x^2} d\tau \\ &\quad \leq \int_0^T \|u\|_{L_x^2}^2 dt + \frac{1}{2} \int_0^T \|\partial_x u\|_{L_x^2}^2 dt + A \int_0^T \|u\|_{L_x^2} dt + \frac{1}{2} \int_0^T \|\eta\|_{L_x^2}^2 dt \\ &\qquad + \int_0^T \|\eta\|_{L_x^2} dt + \int_0^T \|u \partial_x u\|_{L_x^2} dt, \end{aligned} \tag{2.5}$$

which yields that

$$\begin{aligned}
&\left\|\int_0^t e^{(t-\tau)\partial_x^2} f(u,\partial_x u,\eta)(\tau,x)d\tau\right\|_{L_T^2L_x^2} \\
&\quad\le \int_0^T \|u\|_{L_x^2}^2 dt + \frac{1}{2}\int_0^T \|\partial_x u\|_{L_x^2}^2 dt + AT^{1/2}\left(\int_0^T \|u\|_{L_x^2}^2 dt\right)^{1/2} + \frac{1}{2}\int_0^T \|\eta\|_{L_x^2}^2 dt \\
&\qquad + T^{1/2}\left(\int_0^T \|\eta\|_{L_x^2}^2 dt\right)^{1/2} + \int_0^T \|u\|_{H_x^1}\|\partial_x u\|_{L_x^2} dt \\
&\quad\le C\left(\|u\|_{L_T^2L_x^2}^2 + \|\partial_x u\|_{L_T^2L_x^2}^2 + \|\eta\|_{L_T^2L_x^2}^2 + T^{1/2}\left(\|u\|_{L_T^2L_x^2} + \|\eta\|_{L_T^2L_x^2}\right)\right).
\end{aligned} \tag{2.6}$$

Similarly, we can get

$$\begin{aligned}
&\left\|\int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u,\partial_x u,\eta,\partial_x\eta)(\tau,x)d\tau\right\|_{L_T^\infty L_x^2} \\
&\quad\le \sup_{t\in[0,T]}\int_0^t \left\|e^{\varepsilon(t-\tau)\partial_x^2} g(u,\partial_x u,\eta,\partial_x\eta)(\tau,x)\right\|_{L_x^2} d\tau \\
&\quad\le \int_0^T \|u\partial_x\eta\|_{L_x^2} dt + \int_0^T \|\eta\partial_x u\|_{L_x^2} dt + \int_0^T \|\partial_x u\|_{L_x^2} dt \\
&\quad\le \int_0^T \|u\|_{L_x^\infty}\|\partial_x\eta\|_{L_x^2} dt + \int_0^T \|\eta\|_{L_x^\infty}\|\partial_x u\|_{L_x^2} dt + T^{1/2}\left(\int_0^T \|\partial_x u\|_{L_x^2}^2 dt\right)^{1/2},
\end{aligned} \tag{2.7}$$

and then

$$\begin{aligned}
&\left\|\int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u,\partial_x u,\eta,\partial_x\eta)(\tau,x)\ d\tau\right\|_{L_T^\infty L_x^2} \\
&\quad\le \int_0^T \|u\|_{H_x^1}\|\partial_x\eta\|_{L_x^2} dt + \int_0^T \|\eta\|_{H_x^1}\|\partial_x u\|_{L_x^2} dt + T^{1/2}\left(\int_0^T \|\partial_x u\|_{L_x^2}^2 dt\right)^{1/2} \\
&\quad\le \int_0^T \|\partial_x u\|_{L_x^2}\|\partial_x\eta\|_{L_x^2} dt + \int_0^T \|\partial_x\eta\|_{L_x^2}\|\partial_x u\|_{L_x^2} dt + T^{1/2}\left(\int_0^T \|\partial_x u\|_{L_x^2}^2 dt\right)^{1/2} \\
&\quad\le 2\int_0^T \|\partial_x u\|_{L_x^2}^2 dt + 2\int_0^T \|\partial_x\eta\|_{L_x^2}^2 dt + T^{1/2}\left(\int_0^T \|\partial_x u\|_{L_x^2}^2 dt\right)^{1/2},
\end{aligned} \tag{2.8}$$

which implies (2.4). □

Lemma 2.3. *Consider the following:*

$$\begin{aligned}&\left\|\int_0^t e^{(t-\tau)\partial_x^2} f(u,\partial_x u,\eta)(\tau,x)d\tau\right\|_{L_T^2L_x^2}\\&\quad\le CT^{1/2}\left(\|u\|_{L_T^2L_x^2}^2+\|\partial_x u\|_{L_T^2L_x^2}^2+\|\eta\|_{L_T^2L_x^2}^2+T^{1/2}\left(\|u\|_{L_T^2L_x^2}+\|\eta\|_{L_T^2L_x^2}\right)\right),\end{aligned}\tag{2.9}$$

$$\begin{aligned}&\left\|\int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u,\partial_x u,\eta,\partial_x\eta)(\tau,x)d\tau\right\|_{L_T^2L_x^2}\\&\quad\le CT^{1/2}\left(\|\partial_x u\|_{L_T^2L_x^2}^2+\|\partial_x\eta\|_{L_T^2L_x^2}^2+T^{1/2}\|\partial_x u\|_{L_T^2L_x^2}\right).\end{aligned}\tag{2.10}$$

Proof. We mainly prove (2.9). For this, we have that

$$\begin{aligned}\left\|\int_0^t e^{(t-\tau)\partial_x^2} f(u,\partial_x u,\eta)(\tau,x)d\tau\right\|_{L_T^2L_x^2}&\le\left(\int_0^T\left\|\int_{\mathbb{R}}\left(\int_0^t e^{(t-\tau)\partial_x^2}f(\tau,x)d\tau\right)^2dx\right\|_{L_T^\infty}dt\right)^{1/2}\\&\le T^{1/2}\left\|\int_0^t e^{(t-\tau)\partial_x^2}f(\tau,x)d\tau\right\|_{L_T^\infty L_x^2}.\end{aligned}\tag{2.11}$$

Therefore, applying Lemma 2.2, we can easily obtain (2.9).

Similarly, we can also obtain (2.10). □

Let us state the following lemma, which was obtained in [13] (up to a slight modification).

Lemma 2.4. *For any $u_0\in H^1(\mathbb{R})$ and $\delta>0$, there exists $T_1=T_1(u_0)>0$ such that*

$$\left\|\partial_x e^{t\partial_x^2}u_0\right\|_{L_T^2L_x^2}=\left(\int_0^{T_1}\int_{\mathbb{R}}\left|\partial_x e^{t\partial_x^2}u_0\right|^2dx\,dt\right)^{1/2}\le\delta.\tag{2.12}$$

For any $\eta_0\in L^2(\mathbb{R})$, $\varepsilon>0$, and $\delta>0$, there exists $T_2=T_2(\eta_0,\varepsilon)>0$ such that

$$\left\|\partial_x e^{\varepsilon t\partial_x^2}\eta_0\right\|_{L_T^2L_x^2}=\left(\int_0^{T_2}\int_{\mathbb{R}}\left|\partial_x e^{\varepsilon t\partial_x^2}\eta_0\right|^2dx\,dt\right)^{1/2}\le\delta.\tag{2.13}$$

Next, we consider the nonlinear part of (1.7). When written as $v = u - e^{t\partial_x^2}u_0$, $\mu = \eta - e^{\varepsilon t\partial_x^2}\eta_0$, then we have

$$\begin{aligned}
\partial_t v &= \partial_x^2 v + f(u, \partial_x u, \eta),\\
\partial_t \mu &= \varepsilon\partial_x^2 \mu + g(u, \partial_x u, \eta, \partial_x \eta),\\
v(x,0) &= 0,\\
\mu(x,0) &= 0,
\end{aligned} \tag{2.14}$$

that is,

$$\begin{aligned}
v(x,t) &= \int_0^t e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x)\,d\tau,\\
\mu(x,t) &= \int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u, \partial_x u, \eta, \partial_x \eta)(\tau, x)\,d\tau.
\end{aligned} \tag{2.15}$$

First, we have the following basic estimates.

Lemma 2.5. *Consider the following:*

$$\|f\|_{L_T^1 L_x^2} \le C\left(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\eta\|_{L_T^2 L_x^2}^2 + T^{1/2}\left(\|u\|_{L_T^2 L_x^2} + \|\eta\|_{L_T^2 L_x^2}\right)\right), \tag{2.16}$$

$$\|g\|_{L_T^1 L_x^2} \le C\left(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\eta\|_{L_T^2 L_x^2}^2 + \|\partial_x \eta\|_{L_T^2 L_x^2}^2 + T^{1/2}\|\partial_x u\|_{L_T^2 L_x^2}\right). \tag{2.17}$$

Proof. First, let us prove (2.16),

$$\begin{aligned}
\|f\|_{L_T^1 L_x^2} &= \left\|\frac{1}{2}\partial_x\left(u^2\right) + \partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2 + \frac{1}{2}(\partial_x u)^2 - Au + \frac{1}{2}\eta^2 + \eta\right)\right\|_{L_T^1 L_x^2}\\
&\le \int_0^T \left\|\frac{1}{2}\partial_x(u^2) + \partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2 + \frac{1}{2}(\partial_x u)^2 - Au + \frac{1}{2}\eta^2 + \eta\right)\right\|_{L_x^2} dt\\
&\le \frac{1}{2}\int_0^T \left\|\partial_x\left(1-\partial_x^2\right)^{-1}(\partial_x u)^2\right\|_{L_x^2} dt + \int_0^T \left\|\partial_x(1-\partial_x^2)^{-1}(u^2)\right\|_{L_x^2} dt\\
&\quad + \int_0^T \left\|\partial_x\left(1-\partial_x^2\right)^{-1}(Au)\right\|_{L_x^2} dt + \frac{1}{2}\int_0^T \left\|\partial_x\left(1-\partial_x^2\right)^{-1}\left(\eta^2\right)\right\|_{L_x^2} dt\\
&\quad + \int_0^T \|u\partial_x u\|_{L_x^2} dt + \int_0^T \left\|\partial_x\left(1-\partial_x^2\right)^{-1}(\eta)\right\|_{L_x^2} dt,
\end{aligned} \tag{2.18}$$

which implies

$$\begin{aligned}\|f\|_{L_T^1L_x^2} \le{}& \frac{1}{2}\int_0^T \|\partial_x u\|_{L_x^2}^2 dt + \int_0^T \|u\|_{L_x^2}^2 dt + \int_0^T \|u\|_{L_x^\infty}\|\partial_x u\|_{L_x^2} dt + A\int_0^T \|u\|_{L_x^2} dt \\ &+ \frac{1}{2}\int_0^T \|\eta\|_{L_x^2}^2 dt + \int_0^T \|\eta\|_{L_x^2} dt,\end{aligned} \tag{2.19}$$

then we can get that

$$\begin{aligned}\|f\|_{L_T^1L_x^2} \le{}& \frac{1}{2}\int_0^T \|\partial_x u\|_{L_x^2}^2 dt + C\int_0^T \|u\|_{L_x^2}^2 dt + \int_0^T \|u\|_{H_x^1}\|\partial_x u\|_{L_x^2} dt \\ &+ \frac{1}{2}\int_0^T \|\eta\|_{L_x^2}^2 dt + T^{1/2}\left(\int_0^T \|\eta\|_{L_x^2}^2 dt\right)^{1/2} + AT^{1/2}\left(\int_0^T \|u\|_{L_x^2}^2 dt\right)^{1/2} \\ \le{}& C\left(\|u\|_{L_T^2L_x^2}^2 + \|\partial_x u\|_{L_T^2L_x^2}^2 + \|\eta\|_{L_T^2L_x^2}^2 + T^{1/2}\left(\|u\|_{L_T^2L_x^2} + \|\eta\|_{L_T^2L_x^2}\right)\right),\end{aligned} \tag{2.20}$$

where we applied Lemma 2.1, Sobolev embedding theorem $H_x^1(\mathbb{R}) \hookrightarrow L_x^\infty(\mathbb{R})$, and Hölder's inequality. This proves (2.16).

Next, we prove (2.17),

$$\begin{aligned}\|g\|_{L_T^1L_x^2} &= \|-u\partial_x\eta - \eta\partial_x u - \partial_x u\|_{L_T^1L_x^2} \\ &\le \int_0^T \|u\partial_x\eta + \eta\partial_x u + \partial_x u\|_{L_x^2} dt \\ &\le \int_0^T \|u\partial_x\eta\|_{L_x^2} dt + \int_0^T \|\eta\partial_x u\|_{L_x^2} dt + \int_0^T \|\partial_x u\|_{L_x^2} dt \\ &\le \int_0^T \|u\|_{L_x^\infty}\|\partial_x\eta\|_{L_x^2} dt + \int_0^T \|\eta\|_{L_x^\infty}\|\partial_x u\|_{L_x^2} dt + \int_0^T \|\partial_x u\|_{L_x^2} dt,\end{aligned} \tag{2.21}$$

which yields that

$$\begin{aligned}\|g\|_{L_T^1L_x^2} \le{}& \int_0^T \|u\|_{H_x^1}\|\partial_x\eta\|_{L_x^2} dt + \int_0^T \|\eta\|_{H_x^1}\|\partial_x u\|_{L_x^2} dt + \int_0^T \|\partial_x u\|_{L_x^2} dt \\ \le{}& C\left(\int_0^T \|u\|_{L_x^2}^2 dt + \int_0^T \|\partial_x u\|_{L_x^2}^2 dt + \int_0^T \|\eta\|_{L_x^2}^2 dt + \int_0^T \|\partial_x\eta\|_{L_x^2}^2 dt + \int_0^T \|\partial_x u\|_{L_x^2} dt\right) \\ \le{}& C\left(\|u\|_{L_T^2L_x^2}^2 + \|\partial_x u\|_{L_T^2L_x^2}^2 + \|\eta\|_{L_T^2L_x^2}^2 + \|\partial_x\eta\|_{L_T^2L_x^2}^2 + T^{1/2}\|\partial_x u\|_{L_T^2L_x^2}\right),\end{aligned} \tag{2.22}$$

and this ends the proof of (2.17). □

Then we have some estimates for $\partial_x v$ and $\partial_x \mu$.

Lemma 2.6. *Consider the following:*

$$\|\partial_x v\|_{L_T^2 L_x^2} \le C\Big(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\eta\|_{L_T^2 L_x^2}^2 + T^{1/2}\Big(\|u\|_{L_T^2 L_x^2} + \|\eta\|_{L_T^2 L_x^2}\Big)\Big), \tag{2.23}$$

$$\|\partial_x \mu\|_{L_T^2 L_x^2} \le C\varepsilon^{-1/2}\Big(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\eta\|_{L_T^2 L_x^2}^2 + \|\partial_x \eta\|_{L_T^2 L_x^2}^2 + T^{1/2}\|\partial_x u\|_{L_T^2 L_x^2}\Big). \tag{2.24}$$

Proof. We mainly prove (2.24). We have that

$$\begin{aligned}\|\mu\|_{L_T^\infty L_x^2} &= \left\|\int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u, \partial_x u, \eta, \partial_x \eta)(\tau, x)\, d\tau\right\|_{L_T^\infty L_x^2} \\ &\le \int_0^T \|g\|_{L_x^2} dt = \|g\|_{L_T^1 L_x^2}.\end{aligned} \tag{2.25}$$

Multiply μ to the second equation of (2.14) and integrate with respect to x over $\mathbb{R}$. After integration by parts, we have

$$\begin{gathered}\frac{d}{dt}\int_{\mathbb{R}} \frac{\mu^2}{2} dx + \varepsilon \int_{\mathbb{R}} (\partial_x \mu)^2 dx = \int_{\mathbb{R}} \mu g dx, \\ \frac{1}{2}\int_{\mathbb{R}} \mu^2(T) dx - \frac{1}{2}\int_{\mathbb{R}} \mu^2(0) dx + \varepsilon \int_0^T \int_{\mathbb{R}} (\partial_x \mu)^2 dx\, dt = \int_0^T \int_{\mathbb{R}} \mu g dx\, dt,\end{gathered} \tag{2.26}$$

for any $\varepsilon > 0$, which implies

$$\begin{gathered}\varepsilon \int_0^T \int_{\mathbb{R}} (\partial_x \mu)^2 dx\, dt \le \int_0^T \int_{\mathbb{R}} \mu g dx\, dt, \\ \|\partial_x \mu\|_{L_T^2 L_x^2}^2 \le \frac{1}{\varepsilon}\int_0^T \|\mu\|_{L_x^2} \|g\|_{L_x^2} dt \le \frac{1}{\varepsilon}\|\mu\|_{L_T^\infty L_x^2} \|g\|_{L_T^1 L_x^2}.\end{gathered} \tag{2.27}$$

By (2.25), together with (2.27), we have that

$$\|\partial_x \mu\|_{L_T^2 L_x^2} \le \frac{1}{\varepsilon^{1/2}} \|g\|_{L_T^1 L_x^2}. \tag{2.28}$$

By Lemma 2.5, (2.24) follows.

Similarly, we can also obtain (2.23). □

3. Local Well-Posedness

Let $z := \binom{u}{\eta}$, $A(z) := \begin{pmatrix} e^{t\partial_x^2} u_0 \\ e^{\varepsilon t\partial_x^2} \eta_0 \end{pmatrix}$,

$$
\begin{aligned}
B(z) &:= \begin{pmatrix} \displaystyle\int_0^t e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x) d\tau \\ \displaystyle\int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u, \partial_x u, \eta, \partial_x \eta)(\tau, x) d\tau \end{pmatrix}, \\
D_1 &:= C\Big([0,T); L_x^2(\mathbb{R})\Big) \cap C\Big((0,T); H_x^1(\mathbb{R})\Big), \qquad D_2 := C\Big([0,T); L_x^2(\mathbb{R})\Big), \\
X_a^T &:= \Big\{ z \in D_1 \times D_2 : [\![z]\!] = \|z - A(z)\|_{L_T^\infty L_x^2} + \|\partial_x z\|_{L_T^2 L_x^2} + \|z\|_{L_T^2 L_x^2} \le a \Big\},
\end{aligned}
\tag{3.1}
$$

and define the mapping $\Phi : X_a^T \to X_a^T$ by

$$
\Phi(z) = A(z) + B(z). \tag{3.2}
$$

Theorem 3.1. *For any $\varepsilon > 0$, there exist $T = T_\varepsilon > 0$ and $a > 0$ such that $\Phi(X_a^T) \subseteq X_a^T$. In addition, $\Phi : X_a^T \to X_a^T$ is a contraction mapping.*

Proof. We first need to show that the map is well defined for some appropriate a and T. Let $z \in X_a^T$, then we have

$$
[\![\Phi z]\!] = \|\Phi z - A(z)\|_{L_T^\infty L_x^2} + \|\Phi z\|_{L_T^2 L_x^2} + \|\partial_x(\Phi z)\|_{L_T^2 L_x^2}. \tag{3.3}
$$

Considering the terms in (3.3) one by one, from Lemma 2.2, the first term in (3.3) can be estimated as follows:

$$
\begin{aligned}
\|\Phi z - A(z)\|_{L_T^\infty L_x^2} &= \left\| \int_0^t e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x) d\tau \right\|_{L_T^\infty L_x^2} \\
&\quad + \left\| \int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u, \partial_x u, \eta, \partial_x \eta)(\tau, x) d\tau \right\|_{L_T^\infty L_x^2} \\
&\le C\Big(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\eta\|_{L_T^2 L_x^2}^2 + \|\partial_x \eta\|_{L_T^2 L_x^2}^2 \Big) \\
&\quad + CT^{1/2} \Big(\|u\|_{L_T^2 L_x^2} + \|\partial_x u\|_{L_T^2 L_x^2} + \|\eta\|_{L_T^2 L_x^2} \Big) \\
&\le C[\![z]\!]^2 + CT^{1/2} [\![z]\!].
\end{aligned}
\tag{3.4}
$$

From Lemma 2.3, the second term in (3.3) can be estimated as follows:

$$\begin{aligned}
\|\Phi z\|_{L_T^2 L_x^2} &= \left\| e^{t\partial_x^2} u_0 + \int_0^t e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x) d\tau \right\|_{L_T^2 L_x^2} \\
&\quad + \left\| e^{\varepsilon t \partial_x^2} \eta_0 + \int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u, \partial_x u, \eta, \partial_x \eta)(\tau, x) d\tau \right\|_{L_T^2 L_x^2} \\
&\le \left\| e^{t\partial_x^2} u_0 \right\|_{L_T^2 L_x^2} + \left\| \int_0^t e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x) d\tau \right\|_{L_T^2 L_x^2} \\
&\quad + \left\| e^{\varepsilon t \partial_x^2} \eta_0 \right\|_{L_T^2 L_x^2} + \left\| \int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u, \partial_x u, \eta, \partial_x \eta)(\tau, x) d\tau \right\|_{L_T^2 L_x^2},
\end{aligned} \tag{3.5}$$

which implies

$$\begin{aligned}
\|\Phi z\|_{L_T^2 L_x^2} &\le T^{1/2} \|u_0\|_{L_x^2} + \left\| \int_0^t e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x) d\tau \right\|_{L_T^2 L_x^2} \\
&\quad + T^{1/2} \|\eta_0\|_{L_x^2} + \left\| \int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u, \partial_x u, \eta, \partial_x \eta)(\tau, x) d\tau \right\|_{L_T^2 L_x^2} \\
&\le T^{1/2} \|z_0\|_{L_x^2} + C T^{1/2} [\![z]\!]^2 + C T [\![z]\!].
\end{aligned} \tag{3.6}$$

From Lemma 2.6, the third term in (3.3) can be estimated as follows:

$$\begin{aligned}
\|\partial_x(\Phi z)\|_{L_T^2 L_x^2} &= \left\| \partial_x e^{t\partial_x^2} u_0 + \partial_x \int_0^t e^{(t-\tau)\partial_x^2} f(u, \partial_x u, \eta)(\tau, x) d\tau \right\|_{L_T^2 L_x^2} \\
&\quad + \left\| \partial_x e^{\varepsilon t \partial_x^2} \eta_0 + \partial_x \int_0^t e^{\varepsilon(t-\tau)\partial_x^2} g(u, \partial_x u, \eta, \partial_x \eta)(\tau, x) d\tau \right\|_{L_T^2 L_x^2} \\
&\le \left\| \partial_x e^{t\partial_x^2} u_0 \right\|_{L_T^2 L_x^2} + \|\partial_x v\|_{L_T^2 L_x^2} + \left\| \partial_x e^{\varepsilon t \partial_x^2} \eta_0 \right\|_{L_T^2 L_x^2} + \|\partial_x \mu\|_{L_T^2 L_x^2} \\
&\le 2\delta + C\left(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\eta\|_{L_T^2 L_x^2}^2 + T^{1/2}\left(\|u\|_{L_T^2 L_x^2} + \|\eta\|_{L_T^2 L_x^2} \right) \right) \\
&\quad + \frac{C}{\varepsilon^{1/2}} \left(\|u\|_{L_T^2 L_x^2}^2 + \|\partial_x u\|_{L_T^2 L_x^2}^2 + \|\eta\|_{L_T^2 L_x^2}^2 + \|\partial_x \eta\|_{L_T^2 L_x^2}^2 + T^{1/2} \|\partial_x u\|_{L_T^2 L_x^2} \right) \\
&\le 2\delta + C\left(1 + \frac{1}{\varepsilon^{1/2}}\right) [\![z]\!]^2 + C T^{1/2} \left(1 + \frac{1}{\varepsilon^{1/2}}\right) [\![z]\!].
\end{aligned} \tag{3.7}$$

Combining (3.4)–(3.7), we have that

$$\begin{aligned}
[\![\Phi z]\!] &\le 2\delta + T^{1/2}\|z_0\|_{L_x^2} + C[\![z]\!]^2 + CT^{1/2}[\![z]\!]^2 + C\left(1+\frac{1}{\varepsilon^{1/2}}\right)[\![z]\!]^2 \\
&\quad + CT[\![z]\!] + CT^{1/2}\left(2+\frac{1}{\varepsilon^{1/2}}\right)[\![z]\!] \\
&\le 2\delta + T^{1/2}\|z_0\|_{L_x^2} + C\left(2+T^{1/2}\right)\left(1+\frac{1}{\varepsilon^{1/2}}\right)a^2 \\
&\quad + CTa + CT^{1/2}\left(2+\frac{1}{\varepsilon^{1/2}}\right)a.
\end{aligned} \tag{3.8}$$

With appropriate values of δ, a, and T, we are able to have that $[\![\Phi z]\!] \le a$, that is, $\Phi : X_a^T \to X_a^T$ is well defined.

Similar to the above argument, we can show that $\Phi : X_a^T \to X_a^T$ is a contraction mapping,

$$[\![\Phi z_1 - \Phi z_2]\!] \le C'[\![z_1 - z_2]\!], \tag{3.9}$$

where $C' = C'(T, a, \varepsilon, \|z_1\|_{L_T^2L_x^2}, \|z_2\|_{L_T^2L_x^2}, \|\partial_x z_1\|_{L_T^2L_x^2}, \|\partial_x z_2\|_{L_T^2L_x^2})$ can be chosen as $0 < C' < 1$ with appropriate values of T and a. □

Theorem 3.2. *For any $\varepsilon > 0$ and $(u_0, \eta_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, there exist a $T = T(u_0, \eta_0, \varepsilon) > 0$ and a unique solution $(u_\varepsilon, \eta_\varepsilon)$ of* (1.5) *such that*

$$\begin{aligned}
&u_\varepsilon(x,t) \in C\left([0,T); L_x^2(\mathbb{R})\right) \cap C\left((0,T); H_x^1(\mathbb{R})\right), \\
&\eta_\varepsilon(x,t) \in C\left([0,T); L_x^2(\mathbb{R})\right).
\end{aligned} \tag{3.10}$$

Proof. Theorem 3.2 is merely Theorem 3.1 with a standard uniqueness argument. □

Theorem 3.3. *For any $\varepsilon > 0$ and $(u_0, \eta_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, there exists a unique global solution $(u_\varepsilon, \eta_\varepsilon)$ of* (1.5) *such that*

$$\begin{aligned}
&u_\varepsilon(x,t) \in C\left([0,\infty); L_x^2(\mathbb{R})\right) \cap C\left((0,\infty); H_x^1(\mathbb{R})\right), \\
&\eta_\varepsilon(x,t) \in C\left([0,\infty); L_x^2(\mathbb{R})\right).
\end{aligned} \tag{3.11}$$

Proof. To prove Theorem 3.3, we need only to establish the *a priori* energy identity. Multiplying the first equation in (1.6) by u and integrating by parts (with respect to x over $\mathbb{R}$), we have that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(u^2+u_x^2\right)dx + \int_{\mathbb{R}}\left(u_x^2+u_{xx}^2\right)dx - \frac{1}{2}\int_{\mathbb{R}}\eta^2 u_x dx + \int_{\mathbb{R}} u\eta_x dx = 0, \tag{3.12}$$

where we used the relation $m = u - u_{xx}$ and $\int_{\mathbb{R}} u^2 u_x dx = 0$. Multiplying the first equation in (1.6) by η and integrating by parts (with respect to x over $\mathbb{R}$), we have that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \eta^2 dx + \varepsilon \int_{\mathbb{R}} \eta_x^2 dx + \frac{1}{2}\int_{\mathbb{R}} \eta^2 u_x dx - \int_{\mathbb{R}} u\eta_x dx = 0. \tag{3.13}$$

From (3.12) and (3.13), we obtain the energy identity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left(u^2 + u_x^2 + \eta^2\right) dx + \int_{\mathbb{R}} \left(u_x^2 + u_{xx}^2\right) dx + \varepsilon \int_{\mathbb{R}} \eta_x^2 dx = 0, \tag{3.14}$$

which gives rise to the following inequality independent of ε and T:

$$\sup_{t\in[0,T)} \left(\|u(t,\cdot)\|_{H^1}^2 + \|\eta(t,\cdot)\|_{L^2}^2\right) + 2\|u_x\|_{L_T^2(H^1)}^2 \le \|u_0\|_{H^1}^2 + \|\eta_0\|_{L^2}^2. \tag{3.15}$$

According to Theorem 3.2 and the energy inequality (3.15), one can extend the local solution to the global one by a standard contradiction argument, which completes the proof of Theorem 3.3. □

From Theorem 3.3, one has the local well-posedness of system (1.3).

Theorem 3.4. *For $(u_0, \eta_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, there exist a $T = T(u_0, \eta_0) > 0$ and a unique local solution (u, η) of* (1.3) *such that*

$$\begin{gathered} u(x,t) \in C\left([0,T); L_x^2(\mathbb{R})\right) \cap C\left((0,T); H_x^1(\mathbb{R})\right), \\ \eta(x,t) \in C\left([0,T); L_x^2(\mathbb{R})\right). \end{gathered} \tag{3.16}$$

Similar to the proof of Theorem 4.1 in [12] (up to a slight modification), we may get the following.

Theorem 3.5. *Let $z_0 = (u_0, \eta_0) \in H^s \times H^{s-1}$, $s \ge 1$, and let $z = (u, \eta)$ be the corresponding solution to* (1.3). *Assume that $T_{z_0}^* > 0$ is the maximal time of existence, then*

$$T_{z_0}^* < \infty \implies \int_0^{T_{z_0}^*} \|\partial_x u(\tau)\|_{L^\infty} d\tau = \infty. \tag{3.17}$$

4. Proof of Theorem 1.1

Proof of Theorem 1.1. From Theorem 3.4, we have got the local solution of (1.3). On the other hand, according to the second equation of (1.3), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\eta\|_{L_x^2}^2 &\le 2\|\partial_x u\|_{L_x^\infty}\|\eta\|_{L_x^2}^2 + \|\partial_x u\|_{L_x^2}\|\eta\|_{L_x^2} \\ &\le 4\|\partial_x u\|_{L_x^\infty}\|\eta\|_{L_x^2}^2 + \|\partial_x u\|_{L_x^2}\|\eta\|_{L_x^2} \\ &\le 4\|\partial_x u\|_{L_x^\infty}\|\eta\|_{L_x^2}^2 + \|\eta\|_{L_x^2}^2 + \frac{1}{4}\|\partial_x u\|_{L_x^2}^2 \\ &= \left(4\|\partial_x u\|_{L_x^\infty} + 1\right)\|\eta\|_{L_x^2}^2 + \frac{1}{4}\|\partial_x u\|_{L_x^2}^2.\end{aligned} \tag{4.1}$$

An application of Gronwall's inequality yields

$$\|\eta\|_{L_x^2}^2 \le \left(\|\eta_0\|_{L_x^2}^2 + \frac{1}{4}\|\partial_x u\|_{L_T^2 L_x^2}^2\right) e^{\int_0^t (1+4\|\partial_x u\|_{L_x^\infty})d\tau}. \tag{4.2}$$

Similar to the proof of Theorem 3.3, we may get the energy identity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(u^2 + u_x^2 + \eta^2\right)dx + \frac{d}{dt}\int_{\mathbb{R}}\left(u_x^2 + u_{xx}^2\right)dx = 0, \tag{4.3}$$

which implies

$$\sup_{t\in[0,T)}\left(\|u(t,\cdot)\|_{H^1}^2 + \|\eta(t,\cdot)\|_{L^2}^2\right) + 2\|u_x\|_{L_T^2(H^1)}^2 \le \|u_0\|_{H^1}^2 + \|\eta_0\|_{L^2}^2. \tag{4.4}$$

Due to the Sobolev embedding theorem $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (4.4), we obtain that for any $T < +\infty$,

$$\int_0^T \|\partial_x u\|_{L_x^\infty} dt \le \int_0^T \|\partial_x u\|_{H_x^1} dt \le T^{1/2}\|\partial_x u\|_{L_T^2(H_x^1)} < +\infty. \tag{4.5}$$

Therefore, from Theorem 3.5, we deduce that the maximal existence time $T = +\infty$. This proves Theorem 1.1. □

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