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Research Article

On Impulsive Boundary Value Problems of Fractional Differential Equations with Irregular Boundary Conditions

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We study nonlinear impulsive differential equations of fractional order with irregular boundary conditions. Some existence and uniqueness results are obtained by applying standard fixed-point theorems. For illustration of the results, some examples are discussed.

1. Introduction

Boundary value problems of nonlinear fractional differential equations have recently been studied by several researchers. Fractional differential equations appear naturally in various fields of science and engineering and constitute an important field of research. As a matter of fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes [1–4]. Some recent work on boundary value problems of fractional order can be found in [5–23] and the references therein. In [24], some existence and uniqueness results were obtained for an irregular boundary value problem of fractional differential equations.

Dynamical systems with impulse effect are regarded as a class of general hybrid systems. Impulsive hybrid systems are composed of some continuous variable dynamic systems along with certain reset maps that define impulsive switching among them. It is the switching that resets the modes and changes the continuous state of the system. There are three classes of impulsive hybrid systems, namely, impulsive differential systems [25, 26], sampled data or digital control system [27, 28], and impulsive switched system [29, 30].

Applications of such systems include air traffic management [31], automotive control [32, 33], real-time software verification [34], transportation systems [35, 36], manufacturing [37], mobile robotics [38], and process industry [39]. In fact, hybrid systems have a central role in embedded control systems that interact with the physical world. Using hybrid models, one may represent time and event-based behaviors more accurately so as to meet challenging design requirements in the design of control systems for problems such as cut-off control and idle speed control of the engine. For more details, see [40] and the references therein.

The theory of impulsive differential equations of integer order has found its extensive applications in realistic mathematical modelling of a wide variety of practical situations and has emerged as an important area of investigation. The impulsive differential equations of fractional order have also attracted a considerable attention and a variety of results can be found in the papers [41–50].

In this paper, motivated by [24], we study a nonlinear impulsive hybrid system of fractional differential equations with irregular boundary conditions given by

$$\begin{gathered} {}^{C}D^{\alpha}u(t) = f(t,u(t)), \quad 1 < \alpha \le 2,\ t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = I_k^*(u(t_k)), \quad k = 1,2,\ldots,p, \\ u'(0) + (-1)^{\theta}u'(T) + bu(T) = 0, \quad u(0) + (-1)^{\theta+1}u(T) = 0, \quad \theta = 1,2, \end{gathered} \tag{1.1}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $b \in \mathbb{R}, b \neq 0$, $J = [0,T](T > 0), 0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T$, $J' = J \setminus \{t_1, t_2, \ldots, t_p\}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k$ $(k = 1,2,\ldots,p)$, respectively. $\Delta u'(t_k)$ have a similar meaning for $u'(t)$.

Here, we remark that irregular boundary value problems for ordinary and partial differential equations occur in scientific and engineering disciplines and have been addressed by many authors, for instance, see [24] and the references.

The paper is organized as follows. Section 2 deals with some definitions and preliminary results, while the main results are presented in Section 3.

2. Preliminaries

Let us fix $J_0 = [0, t_1]$, $J_{k-1} = (t_{k-1}, t_k]$, $k = 2,\ldots,p+1$ with $t_{p+1} = T$ and introduce the spaces:

$$PC(J,\mathbb{R}) = \left\{u : J \longrightarrow \mathbb{R} \mid u \in C(J_k), k = 0,1,\ldots,p, \text{ and } u(t_k^+) \text{ exist}, k = 1,2,\ldots,p\right\}, \tag{2.1}$$

with the norm $\|u\| = \sup_{t\in J}|u(t)|$, and

$$PC^1(J,\mathbb{R}) = \left\{\begin{matrix} u : J \to \mathbb{R} \mid u \in C^1(J_k), k = 0,1,\ldots,p, \\ \text{and } u(t_k^+), u'(t_k^+) \text{ exist}, k = 1,2,\ldots,p \end{matrix}\right\}, \tag{2.2}$$

with the norm $\|u\|_{PC^1} = \max\{\|u\|, \|u'\|\}$. Obviously, $PC(J,\mathbb{R})$ and $PC^1(J,\mathbb{R})$ are Banach spaces.

Definition 2.1. A function $u \in PC^1(J,\mathbb{R})$ with its Caputo derivative of order α existing on J is a solution of (1.1) if it satisfies (1.1).

To prove the existence of solutions of problem (1.1), we need the following fixed-point theorems.

Theorem 2.2 (see [51]). *Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \overline{\Omega} \to E$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega. \tag{2.3}$$

Then T has a fixed point in $\overline{\Omega}$.

Lemma 2.3 (see [1]). *For $\alpha > 0$, the general solution of fractional differential equation ${}^{C}D^{\alpha}u(t) = 0$ is*

$$u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}, \tag{2.4}$$

where $C_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$ ($[\alpha]$ denotes integer part of α).

Lemma 2.4 (see [1]). *Let $\alpha > 0$. Then*

$$I^{\alpha}\, {}^{C}D^{\alpha}u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1} \tag{2.5}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Lemma 2.5. *For a given $y \in C[0,T]$, a function u is a solution of the following impulsive irregular boundary value problem*

$$\begin{gathered} {}^{C}D^{\alpha}u(t) = y(t), \quad 1 < \alpha \leq 2,\ t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \ldots, p, \\ u'(0) + (-1)^{\theta} u'(T) + bu(T) = 0, \quad u(0) + (-1)^{\theta+1} u(T) = 0, \quad \theta = 1, 2,\ b \neq 0, \end{gathered} \tag{2.6}$$

if and only if u is a solution of the impulsive fractional integral equation

$$u(t)=\begin{cases}\displaystyle\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+\frac{1-(-1)^{\theta+1}}{bT}\int_{t_p}^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\ \displaystyle\quad+\frac{\left[1+(-1)^{\theta+1}\right]t}{bT}\int_{t_p}^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds\\ \displaystyle\quad-\frac{1}{b}\int_{t_p}^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds-\frac{t}{T}\int_{t_p}^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+\mathcal{A},\quad t\in J_0;\\ \displaystyle\int_{t_k}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+\frac{1-(-1)^{\theta+1}}{bT}\int_{t_p}^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\ \displaystyle\quad+\frac{\left[1+(-1)^{\theta+1}\right]t}{bT}\int_{t_p}^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds\\ \displaystyle\quad-\frac{1}{b}\int_{t_p}^T\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds-\frac{t}{T}\int_{t_p}^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\ \displaystyle\quad+\sum_{i=1}^k\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+I_i(u(t_i))\right]\\ \displaystyle\quad+\sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds+I_i^*(u(t_i))\right]\\ \displaystyle\quad+\sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds+I_i^*(u(t_i))\right]+\mathcal{A},\qquad t\in J_k,\ k=1,2,\dots,p,\end{cases}\tag{2.7}$$

where

$$\begin{aligned}\mathcal{A}=&\frac{\left[1+(-1)^{\theta+1}\right]t-T}{bT}\sum_{i=1}^p\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds+I_i^*(u(t_i))\right]\\&+\frac{1-(-1)^{\theta+1}-bt}{bT}\left\{\sum_{i=1}^p\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+I_i(u(t_i))\right]\right.\\&\qquad+\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds+I_i^*(u(t_i))\right]\\&\qquad\left.+\sum_{i=1}^{p}(T-t_p)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds+I_i^*(u(t_i))\right]\right\}.\end{aligned}\tag{2.8}$$

Proof. Let u be a solution of (2.6). Then, by Lemma 2.4, we have

$$u(t)=I^\alpha y(t)-c_1-c_2t=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds-c_1-c_2t,\quad t\in J_0,\tag{2.9}$$

for some $c_1, c_2 \in \mathbb{R}$. Differentiating (2.9), we get

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s)ds - c_2, \quad t \in J_0. \tag{2.10}$$

If $t \in J_1$, then

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} y(s)ds - d_1 - d_2(t-t_1), \\ u'(t) &= \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^t (t-s)^{\alpha-2} y(s)ds - d_2, \end{aligned} \tag{2.11}$$

for some $d_1, d_2 \in \mathbb{R}$. Thus,

$$\begin{aligned} u(t_1^-) &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} y(s)ds - c_1 - c_2 t_1, \quad u(t_1^+) = -d_1, \\ u'(t_1^-) &= \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} y(s)ds - c_2, \quad u'(t_1^+) = -d_2. \end{aligned} \tag{2.12}$$

Using the impulse conditions

$$\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1)), \qquad \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = I_1^*(u(t_1)), \tag{2.13}$$

we find that

$$\begin{aligned} -d_1 &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} y(s)ds - c_1 - c_2 t_1 + I_1(u(t_1)), \\ -d_2 &= \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} y(s)ds - c_2 + I_1^*(u(t_1)). \end{aligned} \tag{2.14}$$

Consequently, we obtain

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} y(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} y(s)ds \\ & + \frac{t-t_1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1-s)^{\alpha-2} y(s)ds + I_1(u(t_1)) + (t-t_1) I_1^*(u(t_1)) - c_1 - c_2 t, \quad t \in J_1. \end{aligned} \tag{2.15}$$

By a similar process, we get

$$\begin{aligned}
u(t) = {} & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + I_i(u(t_i)) \right] \\
& + \sum_{i=1}^{k-1} (t_k - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] \\
& + \sum_{i=1}^{k} (t - t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] - c_1 - c_2 t, \quad t \in J_k,\ k = 1,2,\ldots,p.
\end{aligned} \tag{2.16}$$

Applying the boundary conditions $u'(0) + (-1)^{\theta} u'(T) + bu(T) = 0$ and $u(0) + (-1)^{\theta+1} u(T) = 0$, we find that

$$\begin{aligned}
c_1 = {} & -\frac{1-(-1)^{\theta+1}}{bT} \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{1}{b} \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds \\
& + \sum_{i=1}^{p} \frac{1}{b} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] \\
& - \frac{1-(-1)^{\theta+1}}{bT} \left\{ \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + I_i(u(t_i)) \right] \right. \\
& \qquad + \sum_{i=1}^{p-1} (t_p - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] \\
& \qquad \left. + \sum_{i=1}^{p} (T - t_p) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] \right\}, \\
c_2 = {} & -\frac{1+(-1)^{\theta+1}}{bT} \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + \frac{1}{T} \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\
& - \frac{1+(-1)^{\theta+1}}{bT} \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] \\
& + \frac{1}{T} \left\{ \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + I_i(u(t_i)) \right] \right. \\
& \qquad + \sum_{i=1}^{p-1} (t_p - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] \\
& \qquad \left. + \sum_{i=1}^{p} (T - t_p) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \right] \right\}.
\end{aligned} \tag{2.17}$$

Substituting the value of c_i $(i = 1, 2)$ in (2.9) and (2.16), we obtain (2.7). Conversely, assume that u is a solution of the impulsive fractional integral equation (2.7), then by a direct computation, it follows that the solution given by (2.7) satisfies (2.6). This completes the proof. □

Remark 2.6. With $T = \pi$, the first five terms of the solution (2.7) correspond to the solution for the problem without impulses [24].

3. Main Results

Define an operator $\mathcal{G} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$\begin{aligned}
\mathcal{G}u(t) = {} & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + I_i(u(t_i))\right] \\
& + \sum_{i=1}^{k-1}(t_k - t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\right] \\
& + \sum_{i=1}^{k}(t - t_k)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\right] \\
& + \frac{1-(-1)^{\theta+1}}{bT}\int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \frac{\left[1+(-1)^{\theta+1}\right]t}{bT}\int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds \\
& - \frac{1}{b}\int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds - \frac{t}{T}\int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \\
& + \frac{\left[1+(-1)^{\theta+1}\right]t - T}{bT}\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\right] \\
& + \frac{1-(-1)^{\theta+1} - bt}{bT}\left\{\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + I_i(u(t_i))\right]\right. \\
& \qquad + \sum_{i=1}^{p-1}(t_p - t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\right] \\
& \qquad \left. + \sum_{i=1}^{p}(T - t_p)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\right]\right\}.
\end{aligned} \tag{3.1}$$

Notice that problem (1.1) has a solution if and only if the operator $\mathcal{G}$ has a fixed point.

For the sake of convenience, we set the following notations:

$$\begin{aligned}\mu &= 2(1+p)\left(1+\frac{1}{|b|T}\right)I^{\alpha}a(T)+\left[2(2p-1)T+\frac{5p+1}{|b|}\right]I^{\alpha-1}\\ &\quad+2\left(1+\frac{1}{|b|T}\right)pL_2+\left[2(2p-1)T+\frac{5p-2}{|b|}\right]L_3,\\ \nu &= 2(1+p)\left(1+\frac{1}{|b|T}\right)\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\left[2(2p-1)T+\frac{5p+1}{|b|}\right]\frac{T^{\alpha-1}}{\Gamma(\alpha)}.\end{aligned} \tag{3.2}$$

Theorem 3.1. *Assume that*

($\mathbf{H_1}$) *there exists a nonnegative function* $a(t) \in L(0,T)$ *such that*

$$\left|f(t,u)\right| \le a(t)+\xi|u|^{\rho}, \quad 0<\rho<1, \tag{3.3}$$

where ξ *is a nonnegative constant;*

($\mathbf{H_2}$) *there exist positive constants* L_2 *and* L_3 *such that*

$$|I_k(u)| \le L_2, \quad \left|I_k^*(u)\right| \le L_3, \quad \text{for } t\in J,\ u\in\mathbb{R},\ k=1,2,\ldots,p. \tag{3.4}$$

Then problem (1.1) *has at least one solution.*

Proof. As a first step, we show that the operator $\mathcal{G} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous. Observe that continuity of $\mathcal{G}$ follows from the continuity of f, I_k and I_k^*.

Let $\Omega \subset PC(J,\mathbb{R})$ be bounded. Then, there exist positive constants $L_i > 0$ $(i = 1,2,3)$ such that $|f(t,u)| \le L_1$, $|I_k(u)| \le L_2$, and $|I_k^*(u)| \le L_3$, for all $u \in \Omega$. Thus, for all $u \in \Omega$, we have

$$\begin{aligned}|\mathcal{G}u(t)| \le{}& \int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s))\right|ds+\sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s))\right|ds+|I_i(u(t_i))|\right]\\ &+\sum_{i=1}^{k-1}(t_k-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))\right|ds+\left|I_i^*(u(t_i))\right|\right]\\ &+\sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))\right|ds+\left|I_i^*(u(t_i))\right|\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-(-1)^{\theta+1}}{|b|T}\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s))\right|ds\\
&+\frac{\left[1+(-1)^{\theta+1}\right]t}{|b|T}\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))\right|ds\\
&+\frac{1}{|b|}\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))\right|ds+\frac{t}{T}\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s))\right|ds\\
&+\left|\frac{\left[1+(-1)^{\theta+1}\right]t-T}{bT}\right|\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))\right|ds+\left|I_i^*(u(t_i))\right|\right]\\
&+\left|\frac{1-(-1)^{\theta+1}-bt}{bT}\right|\left\{\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s))\right|ds+\left|I_i(u(t_i))\right|\right]\right.\\
&\qquad+\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))\right|ds+\left|I_i^*(u(t_i))\right|\right]\\
&\qquad\left.+\sum_{i=1}^{p}(T-t_p)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))\right|ds+\left|I_i^*(u(t_i))\right|\right]\right\}\\
\le{}& L_1\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}ds+L_2\right]\\
&+\sum_{i=1}^{p-1}T\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+L_3\right]\\
&+\sum_{i=1}^{p}T\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+L_3\right]+\frac{2L_1}{|b|T}\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\frac{2L_1}{|b|}\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\\
&+\frac{L_1}{|b|}\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+L_1\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\frac{1}{|b|}\sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+L_3\right]\\
&+\frac{2+|b|T}{|b|T}\left\{\sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}ds+L_2\right]+\sum_{i=1}^{p-1}T\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+L_3\right]\right.\\
&\qquad\left.+\sum_{i=1}^{p}T\left[L_1\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+L_3\right]\right\}\\
\le{}&\frac{1}{|b|}\left\{\frac{2(1+p)(|b|T+1)T^{\alpha-1}L_1}{\Gamma(\alpha+1)}+\frac{\left[5p+1+2|b|T(2p-1)\right]T^{\alpha-1}L_1}{\Gamma(\alpha)}\right.\\
&\qquad\left.+\frac{2(1+|b|T)pL_2}{T}+\left[5p-2+2|b|T(2p-1)\right]L_3\right\},
\end{aligned}
\tag{3.5}
$$

which implies

$$\|\mathcal{G}u\| \le \frac{1}{|b|}\left\{\frac{2(1+p)(|b|T+1)T^{\alpha-1}L_1}{\Gamma(\alpha+1)} + \frac{[5p+1+2|b|T(2p-1)]T^{\alpha-1}L_1}{\Gamma(\alpha)} + \frac{2(1+|b|T)pL_2}{T}\right.$$
$$\left. + [5p-2+2|b|T(2p-1)]L_3\right\} := L. \tag{3.6}$$

On the other hand, for any $t \in J_k$, $0 \le k \le p$, we get

$$\begin{aligned}
|(\mathcal{G}u)'(t)| \le{}& \int_{t_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + \sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |I_i^*(u(t_i))|\right] \\
&+ \frac{[1+(-1)^{\theta+1}]}{|b|T}\int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + \frac{1}{T}\int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|ds \\
&+ \frac{[1+(-1)^{\theta+1}]}{|b|T}\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |I_i^*(u(t_i))|\right] \\
&+ \frac{1}{T}\left\{\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))|ds + |I_i(u(t_i))|\right]\right. \\
&\quad + \sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |I_i^*(u(t_i))|\right] \\
&\quad \left. + \sum_{i=1}^{p}(T-t_p)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |I_i^*(u(t_i))|\right]\right\} \\
\le{}& L_1\int_{t_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + \sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + L_3\right] + \frac{2L_1}{|b|T}\int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds \\
&+ \frac{L_1}{T}\int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \frac{2}{|b|T}\sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + L_3\right] \\
&+ \frac{1}{T}\left\{\sum_{i=1}^{p}\left[L_1\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}ds + L_2\right] + \sum_{i=1}^{p-1}T\left[L_1\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + L_3\right]\right. \\
&\quad \left. + \sum_{i=1}^{p}T\left[L_1\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + L_3\right]\right\} \\
\le{}& \frac{(1+p)T^{\alpha-1}L_1}{\Gamma(\alpha+1)} + \frac{3pT^{\alpha-1}L_1}{\Gamma(\alpha)} + \frac{2(1+2p)T^{\alpha-2}L_1}{|b|\Gamma(\alpha)} + \frac{pL_2}{T} + \frac{2pL_3}{|b|T} + (3p-1)L_3 := \overline{L}.
\end{aligned} \tag{3.7}$$

Hence, for $t_1, t_2 \in J_k$ with $t_1 < t_2, 0 \le k \le p$, we have

$$|(\mathcal{G}u)(t_2) - (\mathcal{G}u)(t_1)| \le \int_{t_1}^{t_2} \left|(\mathcal{G}u)'(s)\right| ds \le \overline{L}(t_2 - t_1). \tag{3.8}$$

This implies that G is equicontinuous on all $J_k, k = 0, 1, 2, \ldots, p$ and hence, by the Arzela-Ascoli theorem, the operator $\mathcal{G} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous.

Next, we prove that $\mathcal{G} : \mathcal{B} \to \mathcal{B}$. For that, let us choose $R \ge \max\{2\mu, (2\nu\xi)^{1/(1-\rho)}\}$ and define a ball $\mathcal{B} = \{u \in PC(J, \mathbb{R}) : \|u\| \le R\}$. For any $u \in \mathcal{B}$, by the assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we have

$$\begin{aligned}
|\mathcal{G}u(t)| \le{}& \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds + \sum_{i=1}^{k} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds + |I_i(u(t_i))|\right] \\
&+ \sum_{i=1}^{k-1} (t_k - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds + \left|I_i^*(u(t_i))\right|\right] \\
&+ \sum_{i=1}^{k} (t - t_k) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds + \left|I_i^*(u(t_i))\right|\right] \\
&+ \frac{1 - (-1)^{\theta+1}}{|b|T} \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds \\
&+ \frac{\left[1 + (-1)^{\theta+1}\right] t}{|b|T} \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds \\
&+ \frac{1}{|b|} \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds + \frac{t}{T} \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds \\
&+ \left|\frac{\left[1 + (-1)^{\theta+1}\right] t - T}{bT}\right| \sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds + \left|I_i^*(u(t_i))\right|\right] \\
&+ \left|\frac{1 - (-1)^{\theta+1} - bt}{bT}\right| \left\{\sum_{i=1}^{p} \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds + |I_i(u(t_i))|\right]\right. \\
&\qquad + \sum_{i=1}^{p-1} (t_p - t_i) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds + \left|I_i^*(u(t_i))\right|\right] \\
&\qquad \left. + \sum_{i=1}^{p} (T - t_p) \left[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[a(s) + \xi|u(s)|^{\rho}\right] ds + \left|I_i^*(u(t_i))\right|\right]\right\}
\end{aligned}$$

$$\leq 2(1+p)\left(1+\frac{1}{|b|T}\right)I^{\alpha}a(T)+\left[2(2p-1)T+\frac{5p+1}{|b|}\right]I^{\alpha-1}a(T)+2\left(1+\frac{1}{|b|T}\right)pL_2$$
$$+\left[2(2p-1)T+\frac{5p-2}{|b|}\right]L_3+2(1+p)\left(1+\frac{1}{|b|T}\right)\frac{T^{\alpha}\xi R^{\rho}}{\Gamma(\alpha+1)}$$
$$+\left[2(2p-1)T+\frac{5p+1}{|b|}\right]\frac{T^{\alpha-1}\xi R^{\rho}}{\Gamma(\alpha)}. \tag{3.9}$$

Thus,

$$\|\mathcal{G}u\| \leq \mu+\nu\xi R^{\rho} \leq \frac{R}{2}+\frac{R}{2}=R, \tag{3.10}$$

where μ and ν are given by (3.2). This implies $\mathcal{G}:\mathcal{B}\to\mathcal{B}$. Hence, $\mathcal{G}:\mathcal{B}\to\mathcal{B}$ is completely continuous. Therefore, by the Schauder fixed-point theorem, the operator $\mathcal{G}$ has at least one fixed point. Consequently, problem (1.1) has at least one solution in $\mathcal{B}$. □

Remark 3.2. For $\rho=1$ in $(\mathbf{H_1})$, if $\nu\xi<1$, we can take $R\geq\mu/(1-\nu\xi)$, then the conclusion of Theorem 3.1 holds.

Theorem 3.3. *Suppose that there exist a nonnegative functions $a_1\in L(0,1)$ and a nonnegative number ξ_1 such that $|f(t,u)|\leq a_1(t)+\xi_1|u|^{\rho}$ for $\rho>1$. Furthermore, the assumption $(\mathbf{H_2})$ holds. Then problem* (1.1) *has at least one solution.*

Proof. The proof is similar to that of Theorem 3.1, so we omit it. □

Theorem 3.4. *Suppose that*

$$\lim_{u\to 0}\frac{f(t,u)}{u}=0,\qquad \lim_{u\to 0}\frac{I_k(u)}{u}=0,\qquad \lim_{u\to 0}\frac{I_k^*(u)}{u}=0. \tag{3.11}$$

Then problem (1.1) *has at least one solution.*

Proof. By Theorem 3.1, we know that the operator $\mathcal{G}:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous. In view of (3.11), we can find a constant $r>0$ such that $|f(t,u)|\leq\delta_1|u|$, $|I_k(u)|\leq\delta_2|u|$ and $|I_k^*(u)|\leq\delta_3|u|$ for $0<|u|<r$, where $\delta_i>0$ $(i=1,2,3)$ satisfy

$$\frac{2(1+p)(|b|T+1)T^{\alpha-1}\delta_1}{\Gamma(\alpha+1)}+\frac{[5p+1+2|b|T(2p-1)]T^{\alpha-1}\delta_1}{\Gamma(\alpha)}+\frac{2(1+|b|T)p\delta_2}{T}$$
$$+[5p-2+2|b|T(2p-1)]\delta_3\leq|b|. \tag{3.12}$$

Let $\Omega = \{u \in PC(J,\mathbb{R}) \mid \|u\| < r\}$. Take $u \in PC(J,\mathbb{R})$ such that $\|u\| = r$, which means $u \in \partial\Omega$. Then, as in the proof of Theorem 3.1, we have

$$|\mathcal{G}u(t)| \le \frac{1}{|b|}\left\{\frac{2(1+p)(|b|T+1)T^{\alpha-1}\delta_1}{\Gamma(\alpha+1)} + \frac{[5p+1+2|b|T(2p-1)]T^{\alpha-1}\delta_1}{\Gamma(\alpha)} + \frac{2(1+|b|T)p\delta_2}{T} + [5p-2+2|b|T(2p-1)]\delta_3\right\}\|u\|, \tag{3.13}$$

which, in view of (3.12), implies that $\|\mathcal{G}u\| \le \|u\|$, $u \in \partial\Omega$. Therefore, by Theorem 2.2, the operator $\mathcal{G}$ has at least one fixed point. Thus we conclude that problem (1.1) has at least one solution $u \in \overline{\Omega}$. □

Theorem 3.5. *Assume that*

($\mathbf{H_3}$) *there exist positive constants* K_i $(i = 1,2,3)$ *such that*

$$|f(t,u) - f(t,v)| \le K_1|u-v|, \qquad |I_k(u) - I_k(v)| \le K_2|u-v|, \qquad |I_k^*(u) - I_k^*(v)| \le K_3|u-v|, \tag{3.14}$$

for $t \in J, u, v \in \mathbb{R}$ *and* $k = 1,2,\ldots,p$.

Then problem (1.1) *has a unique solution if*

$$\Lambda = \frac{2(1+p)(|b|T+1)T^{\alpha-1}K_1}{\Gamma(\alpha+1)} + \frac{[5p+1+2|b|T(2p-1)]T^{\alpha-1}K_1}{\Gamma(\alpha)} + \frac{2(1+|b|T)pK_2}{T} + [5p-2+2|b|T(2p-1)]K_3 < |b|. \tag{3.15}$$

Proof. For $u, v \in PC(J,\mathbb{R})$, we have

$$\begin{aligned}|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)| \le{}& \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s)) - f(s,v(s))|ds \\ &+ \sum_{i=1}^{k}\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s)) - f(s,v(s))|ds + |I_i(u(t_i)) - I_i(v(t_i))|\right] \\ &+ \sum_{i=1}^{k-1}(t_k - t_i)\left[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s)) - f(s,v(s))|ds \right. \\ &\left. + |I_i^*(u(t_i)) - I_i^*(v(t_i))|\right]\end{aligned}$$

$$+\sum_{i=1}^{k}(t-t_k)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))-f(s,v(s))\right|ds\right.$$

$$\left.+\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|\right]$$

$$+\frac{1-(-1)^{\theta+1}}{|b|T}\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s))-f(s,v(s))\right|ds$$

$$+\frac{\left[1+(-1)^{\theta+1}\right]t}{|b|T}\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))-f(s,v(s))\right|ds$$

$$+\frac{1}{|b|}\int_{t_p}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))-f(s,v(s))\right|ds$$

$$+\frac{t}{T}\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s))-f(s,v(s))\right|ds$$

$$+\left|\frac{\left[1+(-1)^{\theta+1}\right]t-T}{bT}\right|\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))-f(s,v(s))\right|ds\right.$$

$$\left.+\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|\right]$$

$$+\left|\frac{1-(-1)^{\theta+1}-bt}{bT}\right|$$

$$\times\left\{\sum_{i=1}^{p}\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s))-f(s,v(s))\right|ds+\left|I_i(u(t_i))-I_i(v(t_i))\right|\right]\right.$$

$$+\sum_{i=1}^{p-1}(t_p-t_i)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))-f(s,v(s))\right|ds\right.$$

$$\left.+\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|\right]$$

$$+\sum_{i=1}^{p}(T-t_p)\left[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f(s,u(s))-f(s,v(s))\right|ds\right.$$

$$\left.\left.+\left|I_i^*(u(t_i))-I_i^*(v(t_i))\right|\right]\right\}$$

$$\begin{aligned}
&\leq \frac{1}{|b|}\left\{\frac{2(1+p)(|b|T+1)T^{\alpha-1}K_1}{\Gamma(\alpha+1)}+\frac{[5p+1+2|b|T(2p-1)]T^{\alpha-1}K_1}{\Gamma(\alpha)}\right.\\
&\qquad\left.+\frac{2(1+|b|T)pK_2}{T}+[5p-2+2|b|T(2p-1)]K_3\right\}\|u-v\|\\
&=\frac{\Lambda}{|b|}\|u-v\|,
\end{aligned}\tag{3.16}$$

which, by (3.15), yields $\|Tu-Tv\|<\|u-v\|$. So, $\mathcal{G}$ is a contraction. Therefore, by the Banach contraction mapping principle, problem (1.1) has a unique solution. □

Example 3.6. Consider the following fractional impulsive irregular boundary value problem

$$\begin{gathered}
{}^{C}D^{\alpha}u(t)=\frac{e^{3t}\cos^5[u(t)+e^{u(t)}]}{1+u^4(t)}+\frac{\sin(t+1)}{\sqrt{5+u^2(t)}}|u|^{\rho},\quad 0<t<1,\ t\neq\frac{1}{4},\\
\Delta u\left(\frac{1}{4}\right)=2+3\sin^2\left[\ln\left(1+2u^2\left(\frac{1}{4}\right)\right)\right],\qquad \Delta u'\left(\frac{1}{4}\right)=\frac{7+2u^2(1/4)}{2+u^2(1/4)},\\
u'(0)+(-1)^{\theta}u'(1)+bu(1)=0,\quad u(0)+(-1)^{\theta+1}u(1)=0,\quad \theta=1,2,\ b\neq 0,
\end{gathered}\tag{3.17}$$

where $1<\alpha\leq 2$ and $p=1$.

Observe that

$$|f(t,u)|=\left|\frac{e^{3t}\cos^5[u(t)+e^{u(t)}]}{1+u^4(t)}+\frac{\sin(t+1)}{\sqrt{5+u^2(t)}}|u|^{\rho}\right|\leq e^{3t}+|u|^{\rho}.\tag{3.18}$$

Clearly, $a(t)=e^{3t}, \xi=1, L_2=5, L_3=7/2$, and the conditions of Theorem 3.1 hold for $0<\rho<1$. Thus, by Theorem 3.1, problem (3.17) has at least one solution. In a similar way, for $\rho>1$, the impulsive irregular fractional boundary value problem (3.17) has at least one solution by means of Theorem 3.3.

Example 3.7. Consider the impulsive fractional irregular boundary value problem given by

$$\begin{gathered}
{}^{C}D^{\alpha}u(t)=t^2(1-\cos u(t))+e^{(3+t)}u^4(t),\quad 0<t<1,\ t\neq\frac{1}{5},\\
\Delta u\left(\frac{1}{5}\right)=\frac{\arctan^2 u(1/5)}{5},\qquad \Delta u'\left(\frac{1}{5}\right)=e^{u^3(1/5)}-1,\\
u'(0)+(-1)^{\theta}u'(1)+bu(1)=0,\quad u(0)+(-1)^{\theta+1}u(1)=0,\quad \theta=1,2,\ b\neq 0,
\end{gathered}\tag{3.19}$$

where $1<\alpha\leq 2$ and $p=1$.

It can easily be verified that all the assumptions of Theorem 3.4 are satisfied. Thus, by the conclusion of Theorem 3.4, we deduce that the problem (3.19) has at least one solution.

Example 3.8. Consider

$$\begin{gathered}
{}^{C}D^{7/4}u(t) = 100t^5 + \frac{t^2}{200}e^{-\cos^2 u(t)}, \quad 0 < t < 1,\ t \neq \frac{3}{4}, \\
\Delta u\left(\frac{3}{4}\right) = \frac{1}{9}\cos u\left(\frac{3}{4}\right), \qquad \Delta u'\left(\frac{3}{4}\right) = \frac{|u(3/4)|}{5(1+|u(3/4)|)}, \\
u'(0) + (-1)^{\theta}u'(1) + 8u(1) = 0, \quad u(0) + (-1)^{\theta+1}u(1) = 0, \quad \theta = 1, 2.
\end{gathered} \tag{3.20}$$

Here $q = 7/4, b = 8, T = 1$, and $p = 1$. With

$$K_1 = \frac{1}{200}, \qquad K_2 = \frac{1}{9}, \qquad K_3 = \frac{1}{5}, \tag{3.21}$$

we find that

$$\begin{aligned}
\Lambda = {} & \frac{2(1+p)(|b|T+1)T^{\alpha-1}K_1}{\Gamma(\alpha+1)} + \frac{[5p+1+2|b|T(2p-1)]T^{\alpha-1}K_1}{\Gamma(\alpha)} + \frac{2(1+|b|T)pK_2}{T} \\
& + [5p-2+2|b|T(2p-1)]K_3 < 6.031587 < |b| = 8.
\end{aligned} \tag{3.22}$$

Thus, all the conditions of Theorem 3.5 are satisfied. Consequently, the conclusion of Theorem 3.5 applies and the fractional order impulsive irregular boundary value problem (3.20) has a unique solution on $[0, 1]$.

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