Research Article

On Certain Classes of Biharmonic Mappings Defined by Convolution

J. Chen and X. Wang

Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China

Correspondence should be addressed to X. Wang, xtwang@hunnu.edu.cn

Received 1 March 2012; Accepted 7 August 2012

Academic Editor: Saminathan Ponnusamy

Copyright © 2012 J. Chen and X. Wang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We introduce a class of complex-valued biharmonic mappings, denoted by $BH^0(\phi_k; \sigma, a, b)$, together with its subclass $TBH^0(\phi_k; \sigma, a, b)$, and then generalize the discussions in Ali et al. (2010) to the setting of $BH^0(\phi_k; \sigma, a, b)$ and $TBH^0(\phi_k; \sigma, a, b)$ in a unified way.

1. Introduction

A four times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subset \mathbb{C}$ is *biharmonic* if ΔF , the Laplacian of F , is harmonic in D . Note that ΔF is *harmonic* in D if F satisfies the biharmonic equation $\Delta(\Delta F) = 0$ in D , where Δ represents the Laplacian operator

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (1.1)$$

It is known that, when D is simply connected, a mapping F is biharmonic if and only if F has the following representation:

$$F(z) = \sum_{k=1}^2 |z|^{2(k-1)} G_k(z), \quad (1.2)$$

where G_k are complex-valued harmonic mappings in D for $k \in \{1, 2\}$ (cf. [1–6]). Also it is known that G_k can be expressed as the form

$$G_k = h_k + \overline{g_k} \quad (1.3)$$

for $k \in \{1, 2\}$, where all h_k and g_k are analytic in D (cf. [7, 8]).

Biharmonic mappings arise in a lot of physical situations, particularly, in fluid dynamics and elasticity problems, and have many important applications in engineering and biology (cf. [9–11]). However, the investigation of biharmonic mappings in the context of geometric function theory is a recent one (cf. [1–6]).

In this paper, we consider the biharmonic mappings in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $BH^0(\mathbb{D})$ denote the set of all biharmonic mappings F in $\mathbb{D}$ with the following form:

$$\begin{aligned} F(z) &= \sum_{k=1}^2 |z|^{2(k-1)} \left(h_k(z) + \overline{g_k(z)} \right) \\ &= \sum_{k=1}^2 |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} a_{k,j} z^j + \sum_{j=1}^{\infty} \overline{b_{k,j} z^j} \right), \end{aligned} \quad (1.4)$$

with $a_{1,1} = 1$, $a_{2,1} = 0$, $b_{1,1} = 0$, and $b_{2,1} = 0$.

In [12], Qiao and Wang proved that for each $F \in BH^0(\mathbb{D})$, if the coefficients of F satisfy the following inequality:

$$\sum_{k=1}^2 \sum_{j=1}^{\infty} (2(k-1) + j) (|a_{k,j}| + |b_{k,j}|) \leq 2, \quad (1.5)$$

then F is sense preserving, univalent, and starlike in $\mathbb{D}$ (see [12, Theorems 3.1 and 3.2]).

Let S_H denote the set of all univalent harmonic mappings f in $\mathbb{D}$, where

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{j=2}^{\infty} a_j z^j + \sum_{j=1}^{\infty} \overline{b_j z^j}, \quad (1.6)$$

with $|b_1| < 1$. In particular, we use S_H^0 to denote the set of all mappings in S_H with $b_1 = 0$. Obviously, $S_H^0 \subset BH^0(\mathbb{D})$.

In 1984, Clunie and Sheil-Small [7] discussed the class S_H and its geometric subclasses. Since then, there have been many related papers on S_H and its subclasses (see [13, 14] and the references therein). In 1999, Jahangiri [15] studied the class $S_H^*(\alpha)$ consisting of all mappings $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{j=2}^{\infty} |a_j| z^j, \quad g(z) = \sum_{j=1}^{\infty} |b_j| z^j \quad (1.7)$$

and satisfy the condition

$$\frac{\partial}{\partial \theta} \left(\arg f(re^{i\theta}) \right) = \operatorname{Re} \left\{ \frac{zh' - \overline{zg'}}{h + \overline{g}} \right\} > \alpha \quad (1.8)$$

in $\mathbb{D}$, where $0 \leq \alpha < 1$.

For two analytic functions f_1 and f_2 , if

$$f_1(z) = \sum_{j=1}^{\infty} a_j z^j, \quad f_2(z) = \sum_{j=1}^{\infty} A_j z^j, \quad (1.9)$$

then the *convolution* of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = f_1(z) * f_2(z) = \sum_{j=1}^{\infty} a_j A_j z^j. \quad (1.10)$$

By using the convolution, in [16], Ali et al. introduced the class $S_H^0(\phi, \sigma, \alpha)$ of harmonic mappings in the form of (1.6) such that

$$\operatorname{Re} \left\{ \frac{z(h * \phi)'(z) - \overline{\sigma z(g * \phi)'(z)}}{(h * \phi)(z) + \sigma \overline{(g * \phi)(z)}} \right\} > \alpha \quad (1.11)$$

and the class $SP_H^0(\phi, \sigma, \alpha)$ such that

$$\operatorname{Re} \left\{ \left(1 + e^{i\gamma} \right) \frac{z(h * \phi)'(z) - \overline{\sigma z(g * \phi)'(z)}}{(h * \phi)(z) + \sigma \overline{(g * \phi)(z)}} - e^{i\gamma} \right\} > \alpha, \quad (1.12)$$

where $\sigma \in \mathbb{R}$ and $\alpha \in [0, 1)$ are constants, $\gamma \in \mathbb{R}$ and $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ is analytic in $\mathbb{D}$.

Now we consider a class of biharmonic mappings, denoted by $BH^0(\phi_k; \sigma, a, b)$, as follows: $F \in BH^0(\mathbb{D})$ with the form (1.4) is said to be in $BH^0(\phi_k; \sigma, a, b)$ if and only if

$$\operatorname{Re} \left\{ a \frac{\Phi(z)}{\Psi(z)} - b \right\} > 0, \quad (1.13)$$

where

$$\begin{aligned} \Phi(z) &= z \left[\left(\sum_{k=1}^2 |z|^{2(k-1)} (h_k * \phi_k)(z) \right)' + \sigma \left(\sum_{k=1}^2 |z|^{2(k-1)} \overline{(g_k * \phi_k)(z)} \right)' \right], \\ \Psi(z) &= z' \sum_{k=1}^2 |z|^{2(k-1)} \left((h_k * \phi_k)(z) + \sigma \overline{(g_k * \phi_k)(z)} \right), \end{aligned} \quad (1.14)$$

$\phi_k(z) = z + \sum_{j=2}^{\infty} \phi_{k,j} z^j$ are analytic in $\mathbb{D}$ for $k \in \{1, 2\}$, $\sigma \in \mathbb{R}$ is a constant, $a = p + \rho e^{i\gamma}$, $b = q + \rho e^{i\gamma}$, $p, q, \rho \in [0, +\infty)$ are constants with $a - b > 0$, $\gamma \in \mathbb{R}$, and $z = re^{i\theta}$. Here and in what follows, “'” always stands for “ $\partial/\partial\theta$ ”.

Obviously, if $\phi_2 = 0$, $a = 1$ and $b = \alpha$, then $BH^0(\phi_k; \sigma, a, b)$ reduces to $S_H^0(\phi, \sigma, \alpha)$, and if $\phi_2 = 0$, $a = 1 + e^{i\gamma}$ and $b = \alpha + e^{i\gamma}$, then $BH^0(\phi_k; \sigma, a, b)$ reduces to $SP_H^0(\phi, \sigma, \alpha)$.

Further, we use $TBH^0(\phi_k; \sigma, a, b)$ to denote the class consisting of all mappings F in $BH^0(\phi_k; \sigma, a, b)$ with the form

$$F(z) = \sum_{k=1}^2 |z|^{2(k-1)} \left(h_k(z) + \overline{g_k(z)} \right), \quad (1.15)$$

where

$$\begin{aligned} h_k(z) &= a_{k,1}z - \sum_{j=2}^{\infty} a_{k,j}z^j, \quad a_{k,j} \geq 0, \quad a_{1,1} = 1, \quad a_{2,1} = 0, \\ g_k(z) &= \sigma \sum_{j=1}^{\infty} b_{k,j}z^j, \quad b_{k,j} \geq 0, \quad b_{1,1} = b_{2,1} = 0. \end{aligned} \quad (1.16)$$

The object of this paper is to generalize the discussions in [16] to the setting of $BH^0(\phi_k; \sigma, a, b)$ and $TBH^0(\phi_k; \sigma, a, b)$ in a unified way. The organization of this paper is as follows. In Section 2, we get a convolution characterization for $BH^0(\phi_k; \sigma, a, b)$. As a corollary, we derive a sufficient coefficient condition for mappings in $BH^0(\mathbb{D})$ to belong to $BH^0(\phi_k; \sigma, a, b)$. The main results are Theorems 2.1 and 2.3. In Section 3, first, we get a coefficient characterization for $TBH^0(\phi_k; \sigma, a, b)$, and then find the extreme points of $TBH^0(\phi_k; \sigma, a, b)$. The corresponding results are Theorems 3.1 and 3.6.

2. A Convolution Characterization

We begin with a convolution characterization for $BH^0(\phi_k; \sigma, a, b)$.

Theorem 2.1. *Let $F \in BH^0(\mathbb{D})$. Then $F \in BH^0(\phi_k; \sigma, a, b)$ if and only if*

$$\begin{aligned} & \sum_{k=1}^2 |z|^{2(k-1)} (h_k * \phi_k)(z) * \left(\frac{z + ((ax - a + 2b)/(2a - 2b))z^2}{(1 - z)^2} \right) \\ & - \sigma \sum_{k=1}^2 |z|^{2(k-1)} \overline{(g_k * \phi_k)(z)} * \left(\frac{((ax + b)/(a - b))\bar{z} - ((ax - a + 2b)/(2a - 2b))\bar{z}^2}{(1 - \bar{z})^2} \right) \neq 0, \end{aligned} \quad (2.1)$$

for all $z \in \mathbb{D} \setminus \{0\}$ and all $x \in \mathbb{C}$ with $|x| = 1$.

Proof. By definition, a necessary and sufficient condition for a mapping F in $BH^0(\mathbb{D})$ to be in $BH^0(\phi_k; \sigma, a, b)$ is given by (1.13). Let

$$G(z) = \frac{1}{a-b} \left(a \frac{\Phi(z)}{\Psi(z)} - b \right). \quad (2.2)$$

Then $G(0) = 1$, and so the condition (1.13) is equivalent to

$$G(z) \neq \frac{x-1}{x+1}, \quad (2.3)$$

for all $z \in \mathbb{D} \setminus \{0\}$ and all $x \in \mathbb{C}$ with $|x| = 1$ and $x \neq -1$. Obviously, (2.3) holds if and only if

$$a(x+1)\Phi(z) - b(x+1)\Psi(z) - (a-b)(x-1)\Psi(z) \neq 0. \quad (2.4)$$

Straightforward computations show that

$$\begin{aligned} & a(x+1)\Phi(z) - b(x+1)\Psi(z) - (a-b)(x-1)\Psi(z) \\ &= a(x+1)z' \sum_{k=1}^2 |z|^{2(k-1)} \left(z + \sum_{j=2}^{\infty} j a_{k,j} \phi_{k,j} z^j - \sigma \sum_{j=2}^{\infty} j \overline{b_{k,j} \phi_{k,j} z^j} \right) \\ &\quad - (ax - a + 2b)z' \sum_{k=1}^2 |z|^{2(k-1)} \left(z + \sum_{j=2}^{\infty} a_{k,j} \phi_{k,j} z^j + \sigma \sum_{j=2}^{\infty} \overline{b_{k,j} \phi_{k,j} z^j} \right) \\ &= z' \sum_{k=1}^2 |z|^{2(k-1)} (h_k * \phi_k)(z) * \left(\frac{2(a-b)z + (ax - a + 2b)z^2}{(1-z)^2} \right) \\ &\quad - \sigma z' \sum_{k=1}^2 |z|^{2(k-1)} \overline{(g_k * \phi_k)(z)} * \left(\frac{2(ax+b)\bar{z} - (ax - a + 2b)\bar{z}^2}{(1-\bar{z})^2} \right), \end{aligned} \quad (2.5)$$

from which we see that (2.3) is true if and only if so is (2.1). The proof is complete. $\square$

Remark 2.2. If $h_2 = g_2 = 0$, $a = 1$ and $b = \alpha$, then Theorem 2.1 coincides with Theorem 2.1 in [16], and if $h_2 = g_2 = 0$, $a = 1 + e^{i\gamma}$, and $b = \alpha + e^{i\gamma}$, then Theorem 2.1 coincides with Theorem 2.3 in [16].

As an application of Theorem 2.1, we derive a sufficient condition for mappings in $BH^0(\mathbb{D})$ to be in $BH^0(\phi_k; \sigma, a, b)$ in terms of their coefficients.

Theorem 2.3. *Let $F \in BH^0(\mathbb{D})$. Then $F \in BH^0(\phi_k; \sigma, a, b)$ if*

$$\sum_{k=1}^2 \sum_{j=2}^{\infty} \frac{j \|a\|_{\max} - \|b\|_{\max}}{a-b} |\phi_{k,j} a_{k,j}| + |\sigma| \sum_{k=1}^2 \sum_{j=2}^{\infty} \frac{j \|a\|_{\max} + \|b\|_{\max}}{a-b} |\phi_{k,j} b_{k,j}| \leq 1, \quad (2.6)$$

here and in the following, $\|z\|_{\max} = \max_{\gamma \in R} \{|x + ye^{i\gamma}|\} = x + y$, where $z = x + ye^{i\gamma}$, x and $y \in [0, +\infty)$ are constants.

Proof. For F given by (1.4), we see that

$$\begin{aligned}
 L(z) &\triangleq \left| \sum_{k=1}^2 |z|^{2(k-1)} (h_k * \phi_k)(z) * \left(\frac{z + ((ax - a + 2b)/(2a - 2b))z^2}{(1 - z)^2} \right) \right. \\
 &\quad \left. - \sigma \sum_{k=1}^2 |z|^{2(k-1)} \overline{(g_k * \phi_k)(z)} * \left(\frac{((ax + b)/(a - b))\bar{z} - ((ax - a + 2b)/(2a - 2b))\bar{z}^2}{(1 - \bar{z})^2} \right) \right| \\
 &= \left| z + \sum_{k=1}^2 |z|^{2(k-1)} \sum_{j=2}^{\infty} \left(j + (j-1) \frac{ax - a + 2b}{2a - 2b} \right) \phi_{k,j} a_{k,j} z^j \right. \\
 &\quad \left. - \sigma \sum_{k=1}^2 |z|^{2(k-1)} \sum_{j=2}^{\infty} \left(j \frac{ax + b}{a - b} - (j-1) \frac{ax - a + 2b}{2a - 2b} \right) \overline{\phi_{k,j} b_{k,j} z^j} \right|.
 \end{aligned} \tag{2.7}$$

If F is the identity, obviously, $L(z) = |z|$.

If F is not the identity, then

$$L(z) > |z| \left(1 - \sum_{k=1}^2 \sum_{j=2}^{\infty} \frac{j \|a\|_{\max} - \|b\|_{\max}}{a - b} |\phi_{k,j} a_{k,j}| - |\sigma| \sum_{k=1}^2 \sum_{j=2}^{\infty} \frac{j \|a\|_{\max} + \|b\|_{\max}}{a - b} |\phi_{k,j} b_{k,j}| \right). \tag{2.8}$$

Hence the assumption implies that $L(z) > 0$ for all $z \in \mathbb{D} \setminus \{0\}$ and all $x \in \mathbb{C}$ with $|x| = 1$. It follows from Theorem 2.1 that $F \in BH^0(\phi_k; \sigma, a, b)$. $\square$

Remark 2.4. If $h_2 = g_2 = 0$, $a = 1$ and $b = \alpha$, then Theorem 2.3 coincides with Theorem 2.2 in [16], and if $h_2 = g_2 = 0$, $a = 1 + e^{i\gamma}$ and $b = \alpha + e^{i\gamma}$, then Theorem 2.3 coincides with Theorem 2.4 in [16].

3. A Coefficient Characterization and Extreme Points

We start with a coefficient characterization for $TBH^0(\phi_k; \sigma, a, b)$.

Theorem 3.1. Let $\phi_k(z) = z + \sum_{j=2}^{\infty} \phi_{k,j} z^j$ with $\phi_{k,j} \geq 0$, and let F be of the form (1.15). Then $F \in TBH^0(\phi_k; \sigma, a, b)$ if and only if

$$\sum_{k=1}^2 \sum_{j=2}^{\infty} \frac{j \|a\|_{\max} - \|b\|_{\max}}{a - b} \phi_{k,j} a_{k,j} + \sigma^2 \sum_{k=1}^2 \sum_{j=2}^{\infty} \frac{j \|a\|_{\max} + \|b\|_{\max}}{a - b} \phi_{k,j} b_{k,j} \leq 1. \tag{3.1}$$

Proof. By similar arguments as in the proof of Theorem 2.3, we see that it suffices to prove the “only if” part. For $F \in TBH^0(\phi_k; \sigma, a, b)$, obviously, (1.13) is equivalent to

$$\operatorname{Re} \left\{ \frac{P(z) - Q(z)}{z - \sum_{k=1}^2 |z|^{2(k-1)} \left(\sum_{j=2}^{\infty} a_{k,j} \phi_{k,j} z^j - \sigma^2 \sum_{j=2}^{\infty} b_{k,j} \phi_{k,j} \bar{z}^j \right)} \right\} > 0 \quad (3.2)$$

in $\mathbb{D}$, where

$$\begin{aligned} P(z) &= (a - b)z - \sum_{k=1}^2 |z|^{2(k-1)} \sum_{j=2}^{\infty} (aj - b) a_{k,j} \phi_{k,j} z^j, \\ Q(z) &= \sigma^2 \sum_{k=1}^2 |z|^{2(k-1)} \sum_{j=2}^{\infty} (aj + b) b_{k,j} \phi_{k,j} \bar{z}^j. \end{aligned} \quad (3.3)$$

Letting $z \rightarrow 1^-$ through real values leads to the desired inequality. So the proof is complete. $\square$

Remark 3.2. If $h_2 = g_2 = 0$, $a = 1$, and $b = \alpha$, then Theorem 3.1 coincides with Theorem 3.1 in [16].

It follows from Theorem 3.1 that we have the following.

Corollary 3.3. Let $\phi_k(z) = z + \sum_{j=2}^{\infty} \phi_{k,j} z^j$ with $\phi_{k,j} \geq \phi_{1,2} > 0$ ($k \in \{1, 2\}$, $j \geq 2$) and $|\sigma| \geq (2\|a\|_{\max} - \|b\|_{\max}) / (2\|a\|_{\max} + \|b\|_{\max})$. If $F \in TBH^0(\phi_k; \sigma, a, b)$, then for $|z| = r < 1$, one has

$$r - \frac{a - b}{(2\|a\|_{\max} - \|b\|_{\max})\phi_{1,2}} r^2 \leq |F(z)| \leq r + \frac{a - b}{(2\|a\|_{\max} - \|b\|_{\max})\phi_{1,2}} r^2. \quad (3.4)$$

The result is sharp with equality for mappings

$$F(z) = z - \frac{a - b}{(2\|a\|_{\max} - \|b\|_{\max})\phi_{1,2}} z^2. \quad (3.5)$$

Theorem 3.1 and Corollary 3.3 imply the following

Corollary 3.4. Under the hypotheses of Corollary 3.3, one has that $TBH^0(\phi_k; \sigma, a, b)$ is closed under the convex combination.

Definition 3.5. Let X be a topological vector space over the field of complex numbers, and let E be a subset of X . A point $x \in E$ is called an extreme point of E if it has no representation of the form $x = ty + (1 - t)z$ ($0 < t < 1$) as a proper convex combination of two distinct points y and z in E (cf. [17]).

We now determine the extreme points of $TBH^0(\phi_k; \sigma, a, b)$.

Theorem 3.6. *Let*

- (1) $h_{11}(z) = z$,
- (2) $h_{21}(z) = g_{11}(z) = g_{21}(z) = 0$,
- (3) $h_{kj}(z) = z - |z|^{2(k-1)}((a-b)/(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j})z^j$ for $k \in \{1, 2\}$ and all $j \geq 2$,
- (4) $g_{kj}(z) = z + |z|^{2(k-1)}((a-b)/(\sigma(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j})\bar{z}^j$ for $k \in \{1, 2\}$ and all $j \geq 2$.

Under the hypotheses of Corollary 3.3, one has that $F \in TBH^0(\phi_k; \sigma, a, b)$ if and only if it can be expressed as

$$F(z) = \sum_{k=1}^2 \sum_{j=1}^{\infty} (x_{kj}h_{kj}(z) + y_{kj}g_{kj}(z)), \quad (3.6)$$

where $x_{21} = y_{11} = y_{21} = 0$, all other x_{kj} and y_{kj} are nonnegative, and $\sum_{k=1}^2 \sum_{j=1}^{\infty} (x_{kj} + y_{kj}) = 1$.

In particular, the extreme points of $TBH^0(\phi_k; \sigma, a, b)$ are all mappings h_{kj} and g_{kj} listed in (1), (3), and (4) above.

Proof. It follows from the assumptions that

$$\begin{aligned} F(z) &= \sum_{k=1}^2 \sum_{j=1}^{\infty} (x_{kj}h_{kj}(z) + y_{kj}g_{kj}(z)) \\ &= z - \sum_{k=1}^2 \sum_{j=2}^{\infty} |z|^{2(k-1)} \frac{a-b}{(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j}} x_{kj} z^j \\ &\quad + \sigma \sum_{k=1}^2 \sum_{j=2}^{\infty} |z|^{2(k-1)} \frac{a-b}{\sigma^2(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j}} y_{kj} \bar{z}^j, \end{aligned} \quad (3.7)$$

whence

$$\begin{aligned} &\sum_{k=1}^2 \sum_{j=2}^{\infty} \frac{(j\|a\|_{\max} - \|b\|_{\max})}{a-b} \phi_{k,j} \cdot \frac{a-b}{(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j}} x_{kj} \\ &\quad + \sigma^2 \sum_{k=1}^2 \sum_{j=2}^{\infty} \frac{(j\|a\|_{\max} + \|b\|_{\max})}{a-b} \phi_{k,j} \cdot \frac{a-b}{\sigma^2(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j}} y_{kj} \\ &= \sum_{k=1}^2 \sum_{j=2}^{\infty} x_{kj} + \sum_{k=1}^2 \sum_{j=2}^{\infty} y_{kj} \\ &\leq 1, \end{aligned} \quad (3.8)$$

and so Theorem 3.1 implies that $F \in TBH^0(\phi_k; \sigma, a, b)$.

Conversely, assume $F \in TBH^0(\phi_k; \sigma, a, b)$, and let

$$\begin{aligned} x_{21} = y_{11} = y_{21} &= 0, & x_{11} &= 1 - \sum_{k=1}^2 \sum_{j=2}^{\infty} x_{kj} - \sum_{k=1}^2 \sum_{j=2}^{\infty} y_{kj}, \\ x_{kj} &= \frac{(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j}a_{k,j}}{a-b}, \\ y_{kj} &= \frac{\sigma^2(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j}b_{k,j}}{a-b}, \end{aligned} \quad (3.9)$$

for $k \in \{1, 2\}$ and all $j \geq 2$. Then

$$F(z) = z - \sum_{k=1}^2 |z|^{2(k-1)} \sum_{j=2}^{\infty} a_{k,j} z^j + \sigma \sum_{k=1}^2 |z|^{2(k-1)} \sum_{j=2}^{\infty} b_{k,j} \bar{z}^j. \quad (3.10)$$

The proof of the theorem is complete. $\square$

Remark 3.7. If $h_2 = g_2 = 0$, $a = 1$ and $b = \alpha$, then Theorem 3.6 coincides with Theorem 3.2 in [16].

Acknowledgment

The research was partly supported by NSFs of China (No. 11071063).

References

- [1] Z. Abdulhadi and Y. Abu Muhanna, "Landau's theorem for biharmonic mappings," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 1, pp. 705–709, 2008.
- [2] Z. Abdulhadi, Y. Abu Muhanna, and S. Khuri, "On univalent solutions of the biharmonic equation," *Journal of Inequalities and Applications*, no. 5, pp. 469–478, 2005.
- [3] Z. Abdulhadi, Y. Abu Muhanna, and S. Khuri, "On some properties of solutions of the biharmonic equation," *Applied Mathematics and Computation*, vol. 177, no. 1, pp. 346–351, 2006.
- [4] S. Chen, S. Ponnusamy, and X. Wang, "Landau's theorem for certain biharmonic mappings," *Applied Mathematics and Computation*, vol. 208, no. 2, pp. 427–433, 2009.
- [5] S. Chen, S. Ponnusamy, and X. Wang, "Compositions of harmonic mappings and biharmonic mappings," *Bulletin of the Belgian Mathematical Society*, vol. 17, no. 4, pp. 693–704, 2010.
- [6] S. Chen, S. Ponnusamy, and X. Wang, "Bloch constant and Landau's theorem for planar p -harmonic mappings," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 1, pp. 102–110, 2011.
- [7] J. G. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiæ Scientiarum Fennicæ. Series A I*, vol. 9, pp. 3–25, 1984.
- [8] P. Duren, *Harmonic Mappings in the Plane*, vol. 156 of *Cambridge Tracts in Mathematics*, Cambridge University Press, New York, NY, USA, 2004.
- [9] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1965.
- [10] S. A. Khuri, "Biorthogonal series solution of Stokes flow problems in sectorial regions," *SIAM Journal on Applied Mathematics*, vol. 56, no. 1, pp. 19–39, 1996.
- [11] W. E. Langlois, *Slow Viscous Flow*, Macmillan, New York, NY, USA, 1964.
- [12] J. Qiao and X. Wang, "On p -harmonic univalent mappings," *Acta Mathematica Scientia*, vol. 32, pp. 588–600, 2012 (Chinese).

- [13] A. Ganczar, "On harmonic univalent mappings with small coefficients," *Demonstratio Mathematica*, vol. 34, no. 3, pp. 549–558, 2001.
- [14] H. Silverman, "Harmonic univalent functions with negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 1, pp. 283–289, 1998.
- [15] J. M. Jahangiri, "Harmonic functions starlike in the unit disk," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 2, pp. 470–477, 1999.
- [16] R. M. Ali, B. A. Stephen, and K. G. Subramanian, "Subclasses of harmonic mappings defined by convolution," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1243–1247, 2010.
- [17] P. L. Duren, *Univalent Functions*, vol. 259, Springer, New York, NY, USA, 1983.