Research Article

On the Convergence of Multistep Iteration for Uniformly Continuous \(\Phi\)-Hemicontractive Mappings

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It is shown that the convergence of the multistep iterative process with errors is obtained for uniformly continuous \(\Phi\)-hemicontractive mappings in real Banach spaces. We also revise the problems of C. E. Chidume and C. O. Chidume \(2005\).

1. Introduction

Let \(E\) be a real Banach space with norm \(\|\cdot\|\) and let \(E^*\) be its dual space. The normalized duality mapping \(J : E \to 2^{E^*}\) is defined by

\[
J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E,
\]

(1.1)

where \(\langle \cdot, \cdot \rangle\) denotes the generalized duality pairing. The single-valued-normalized duality mapping is denoted by \(j\).

A mapping \(T\) with domain \(D(T)\) and range \(R(T)\) in \(E\) is said to be strongly pseudocontractive if there is a constant \(k \in (0, 1)\), and for all \(x, y \in D(T)\), \(\exists j(x - y) \in J(x - y)\) such that

\[
\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2.
\]

(1.2)
The mapping \( T \) is called \( \Phi \)-pseudocontractive if there exists a strictly increasing continuous function \( \Phi : [0, +\infty) \to [0, +\infty) \) with \( \Phi(0) = 0 \) such that

\[
(Tx - Ty, j(x - y)) \leq \|x - y\|^2 - \Phi(\|x - y\|)
\]

holds for all \( x, y \in D(T) \). It is well known that the strongly pseudocontractive mapping must be the \( \Phi \)-pseudocontractive mapping in the special case in which \( \Phi(t) = (1 - k)t^2 \), but the converse is not true in general. That is, the class of strongly pseudocontractive mappings is a proper subclass of the class of \( \Phi \)-pseudocontractive mappings. Let \( F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset \). If the inequalities (1.2) and (1.3) hold for any \( x \in D(T) \) and \( y \in F(T) \), then the corresponding mapping \( T \) is called strongly hemicontractive and \( \Phi \)-hemicontractive, respectively.

Let \( N(T) = \{x \in E : Tx = 0\} \neq \emptyset \). An operator \( T : D(T) \subseteq E \to E \) is called strongly quasiaccetive, \( \Phi \)-quasiaccretive if and only if \( I - T \) is strongly hemicontractive, \( \Phi \)-hemicontractive, respectively, where \( I \) denotes the identity mapping on \( E \). That is, if \( T \) is \( \Phi \)-quasi-accretive, then \( N(T) \neq \emptyset \) and there exists a strictly increasing continuous function \( \Phi : [0, +\infty) \to [0, +\infty) \) with \( \Phi(0) = 0 \) such that

\[
(Tx - Ty, j(x - y)) \geq \Phi(\|x - y\|)
\]

holds for all \( x \in D(T) \) and \( y \in N(T) \). Many authors have studied extensively the strongly convergence problems of the iterative algorithms for the class of operators.

In 2004, Rhoades and Soltuz [1] introduced the multistep iteration as follows.

Let \( D \) be a nonempty closed convex subset of real Banach space \( E \) and let \( T : D \to D \) be a mapping. The multistep iteration \( \{x_n\} \) is defined by

\[
x_0 \in D,
\]

\[
y_n^{p-1} = (1 - b_n^{p-1})x_n + b_n^{p-1}Tx_n, \quad n \geq 0, \quad p \geq 2,
\]

\[
y_n^k = (1 - b_n^k)x_n + b_n^kTy_n^{k+1}, \quad k = p - 2, \quad p - 3, \ldots, 2, 1,
\]

\[
x_{n+1} = (1 - a_n)x_n + a_nTy_n^1, \quad n \geq 0,
\]

where \( \{a_n\}, \{b_n^k\} (k = 1, 2, \ldots, p - 1) \) in \( [0, 1] \) satisfy certain conditions. Obviously, the iteration defined above is generalization of Mann, Ishikawa, and Noor iterations.

Inspired and motivated by the work of Xu [2] and the iteration above, we discuss the following multistep iteration with errors:

\[
x_0 \in D,
\]

\[
y_n^{p-1} = (1 - b_n^{p-1} - d_n^{p-1})x_n + b_n^{p-1}Tx_n + d_n^{p-1}w_n^{p-1}, \quad n \geq 0, \quad p \geq 2,
\]

\[
y_n^k = (1 - b_n^k - d_n^k)x_n + b_n^kTy_n^{k+1} + d_n^kw_n^k, \quad k = p - 2, \quad p - 3, \ldots, 2, 1,
\]

\[
x_{n+1} = (1 - a_n - c_n)x_n + a_nTy_n^1 + c_nw_n, \quad n \geq 0,
\]

where \( \{a_n\}, \{c_n\}, \{b_n^k\}, \{d_n^k\} (k = 1, 2, \ldots, p - 1) \) in \( [0, 1] \) with \( a_n + c_n \leq 1, b_n^k + d_n^k \leq 1, \) \( \{u_n\}, \{w_n^k\} (k = 1, 2, \ldots, p - 1) \) are the bounded sequences of \( D \).
In 2005, C. E. Chidume and C. O. Chidume [3] proved the convergence theorems for fixed points of uniformly continuous generalized $\Phi$-hemicontractive mappings and published in [3]. However, there exists a gap in the proof course of their theorems.

The aim of this paper is to show the convergence of the multistep iteration with errors for fixed points of uniformly continuous $\Phi$-hemicontractive mappings and revise the results of C. E. Chidume and C. O. Chidume [3]. For this, we need the following Lemmas.

**Lemma 1.1** (see [4]). Let $E$ be a real Banach space and let $J : E \to 2^E$ be a normalized duality mapping. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle,$$

for all $x, y \in E$ and for all $j(x + y) \in J(x + y)$.

**Lemma 1.2** (see [5]). Let $\{\delta_n\}_{n=0}^{\infty}, \{\lambda_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be three nonnegative real sequences and let $\Phi : [0, +\infty) \to [0, +\infty)$ be a strictly increasing and continuous function with $\Phi(0) = 0$ satisfying the following inequality:

$$\delta_{n+1}^2 \leq \delta_n^2 - \lambda_n \Phi(\delta_{n+1}) + \gamma_n, \quad n \geq 0,$$

where $\lambda_n \in [0, 1]$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\gamma_n = o(\lambda_n)$. Then $\delta_n \to 0$ as $n \to \infty$.

2. Main Results

**Theorem 2.1.** Let $E$ be an arbitrary real Banach space, $D$ a nonempty closed convex subset of $E$, and $T : D \to D$ a uniformly continuous $\Phi$-hemicontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n^k\}, \{c_n\}, \{d_n^k\}$ be real sequences in $[0, 1]$ and satisfy the conditions:

(i) $a_n + c_n \leq 1, \quad b_n^k + d_n^k \leq 1, \quad k = 1, 2, \ldots, p - 1$;

(ii) $a_n, c_n, b_n^k, d_n^k \to 0$ as $n \to \infty, \quad k = 1, 2, \ldots, p - 1$;

(iii) $c_n = o(a_n), \quad \sum_{n=0}^{\infty} a_n = \infty$.

For some $x_0 \in D$, let $\{u_n\}, \{w_n\}, \{w_n^2\}, \ldots, \{w_n^{p-1}\}$ be any bounded sequences of $D$, and let $\{x_n\}$ be the multistep iterative sequence with errors defined by (1.6). Then (1.6) converges strongly to the fixed point $q$ of $T$.

**Proof.** Since $T : D \to D$ is $\Phi$-hemicontractive mapping, then there exists a strictly increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Tq, j(x - q) \rangle \leq \|x - q\|^2 - \Phi(\|x - q\|),$$

for $x \in D, \quad q \in F(T)$, that is

$$\langle Tx - x, j(x - q) \rangle \leq -\Phi(\|x - q\|).$$

Choose some $x_0 \in D$ and $x_0 \neq Tx_0$ such that $\|x_0 - Tx_0\| \cdot \|x_0 - q\| \in R(\Phi)$ and denote that $r_0 = \|x_0 - Tx_0\| \cdot \|x_0 - q\|$, $R(\Phi)$ is the range of $\Phi$. Indeed, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then
Since \( r_0 \in R(\Phi); \) if \( \sup \{ \Phi(r) : r \in [0, +\infty) \} = r_1 < +\infty \) with \( r_1 < r_0 \) (here, we only give an example. If \( r_0 = 2, \) \( \Phi(t) = t/(1 + t), \) then \( \sup \{ \Phi(r) : r \in [0, +\infty) \} = r_1 = 1 < 2 = r_0, \) then for \( q \in D, \) there exists a sequence \( \{w_n\} \) in \( D \) such that \( w_n \to q \) as \( n \to \infty \) with \( w_n \neq q. \) Furthermore, we obtain that \( Tw_n \to Tq \) as \( n \to \infty. \) So \( \{w_n - Tw_n\} \) is the bounded sequence. Hence, there exists natural number \( n_0 \) such that \( \|w_n - Tw_n\| \cdot \|w_n - q\| < r_1/2 \) for \( n \geq n_0, \) then we redefine \( x_0 = w_{n_0} \) and \( \|x_0 - T x_0\| \cdot \|x_0 - q\| \in R(\Phi). \) This is to ensure that \( \Phi^{-1}(r_0) \) is defined well.

**Step 1.** We show that \( \{x_n\} \) is a bounded sequence.

Set \( R = \Phi^{-1}(r_0), \) then from above formula (\( @ \)), we obtain that \( \|x_0 - q\| \leq R. \) Denote

\[
B_1 = \{ x \in D : \|x - q\| \leq R \}, \quad B_2 = \{ x \in D : \|x - q\| \leq 2R \}. \tag{2.2}
\]

Since \( T \) is the uniformly continuous, so \( T \) is a bounded mapping. We let

\[
M = \sup_{x \in B_2} \{ \|Tx - q\| + 1 \}
+ \max \left\{ \sup_n \{ \|w^1_n - q\| \}, \sup_n \{ \|w^2_n - q\| \}, \ldots, \sup_n \{ \|w^{p-1}_n - q\| \}, \sup_n \{ \|u_n - q\| \} \right\}. \tag{2.3}
\]

Next, we want to prove that \( x_n \in B_1. \) If \( n = 0, \) then \( x_0 \in B_1. \) Now, assume that it holds for some \( n, \) that is, \( x_n \in B_1. \) We prove that \( x_{n+1} \in B_1. \) Suppose that it is not the case, then \( \|x_{n+1} - q\| > R > R/2. \) Since \( T \) is uniformly continuous, then for \( \varepsilon_0 = \Phi(R/2)/8R, \) there exists \( \delta > 0 \) such that \( \|Tx - Ty\| < \varepsilon_0 \) when \( \|x - y\| < \delta. \) Denote

\[
\tau_0 = \min \left\{ 1, \frac{R}{M}, \frac{\Phi(R/2)}{8R(M + 2R)}, \frac{\delta}{2M + 4R} \right\}. \tag{2.4}
\]

Since \( a_n, b^k_n, c_n, d^k_n \to 0 \) as \( n \to \infty \) for \( k = 1, 2, \ldots, p - 1. \) Without loss of generality, we assume that \( 0 \leq a_n, b^k_n, c_n, d^k_n \leq \tau_0 \) for any \( n \geq 0. \) Since \( c_n = o(a_n), \) let \( c_n < a_n \tau_0. \) Now, estimate \( y_n^k - q \) for \( k = 1, 2, \ldots, p - 1. \) By using (1.6), we have

\[
\|y_n^{p-1} - q\| \leq \left( 1 - b_n^{p-1} - d_n^{p-1} \right) \|x_n - q\| + b_n^{p-1} \|Tx_n - q\| + d_n^{p-1} \|w_n^{p-1} - q\|
\]

\[
\leq R + \tau_0 M
\]

\[
\leq 2R, \tag{2.5}
\]

then \( y_n^{p-1} \in B_2. \) Similarly, we have

\[
\|y_n^{p-2} - q\| \leq \left( 1 - b_n^{p-2} - d_n^{p-2} \right) \|x_n - q\| + b_n^{p-2} \|Ty_n^{p-1} - q\| + d_n^{p-2} \|w_n^{p-2} - q\|
\]

\[
\leq R + \tau_0 M
\]

\[
\leq 2R, \tag{2.6}
\]
then \( y_{n}^{p-2} \in B_2 \). We have

\[
\|y_n - q\| \leq \left( 1 - b_n^1 - d_n^1 \right) \|x_n - q\| + b_n^1 \|Ty_n^2 - q\| + d_n^1 \|w_n^1 - q\|
\]

\[
\leq R + \tau_0 M
\]

\[
\leq 2R,
\]

then \( y_n^1 \in B_2 \). Therefore, we get

\[
\|x_{n+1} - q\| \leq (1 - a_n - c_n) \|x_n - q\| + a_n \|Ty_n^1 - q\| + c_n \|u_n - q\|
\]

\[
\leq R + \tau_0 M
\]

\[
\leq 2R.
\]

And we have

\[
\|x_{n+1} - x_n\| \leq a_n \|Ty_n^1 - x_n\| + c_n \|u_n - x_n\|
\]

\[
\leq a_n \left( \|Ty_n^1 - q\| + \|x_n - q\| \right) + c_n \left( \|u_n - q\| + \|x_n - q\| \right)
\]

\[
\leq \tau_0 \left( \|Ty_n^1 - q\| + \|u_n - q\| + 2\|x_n - q\| \right)
\]

\[
\leq \tau_0 (M + 2R)
\]

\[
\leq \frac{\Phi(R/2)}{8R},
\]

\[
\|x_{n+1} - y_n^1\| \leq a_n \|Ty_n^1 - x_n\| + c_n \|u_n - x_n\| + b_n^1 \|Ty_n^2 - x_n\| + d_n^1 \|w_n^1 - x_n\|
\]

\[
\leq a_n \left( \|Ty_n^1 - q\| + \|x_n - q\| \right) + c_n \left( \|u_n - q\| + \|x_n - q\| \right)
\]

\[
+ b_n^1 \left( \|Ty_n^2 - q\| + \|x_n - q\| \right) + d_n^1 \left( \|w_n^1 - q\| + \|x_n - q\| \right)
\]

\[
\leq \tau_0 \left( \|Ty_n^1 - q\| + \|u_n - q\| + 2\|x_n - q\| \right)
\]

\[
+ \left( \|Ty_n^2 - q\| + \|w_n^1 - q\| + 2\|x_n - q\| \right)
\]

\[
\leq \tau_0 (2M + 4R)
\]

\[
\leq \delta.
\]

(2.9)

Further, by using uniform continuity of \( T \), we have

\[
\|Tx_{n+1} - Ty_n^1\| < \frac{\Phi(R/2)}{8R}.
\]

(2.10)
In view of Lemma 1.1 and the above formulas, we obtain

\[ \|x_{n+1} - q\|^2 \]
\[ = \left\| (x_n - q) + a_n \left( Ty_n^1 - x_n \right) + c_n (u_n - x_n) \right\|^2 \]
\[ \leq \|x_n - q\|^2 + 2a_n \left\langle Ty_n^1 - x_n, j(x_{n+1} - q) \right\rangle + 2c_n \left\langle u_n - x_n, j(x_{n+1} - q) \right\rangle \]
\[ \leq \|x_n - q\|^2 + 2a_n \left( Tx_{n+1} - x_{n+1} + x_{n+1} - x_n - Tx_{n+1} + Ty_n^1, j(x_{n+1} - q) \right) \]
\[ + 2c_n \|u_n - x_n\| \cdot \|x_{n+1} - q\| \]
\[ \leq \|x_n - q\|^2 - 2a_n \Phi \left( \|x_{n+1} - q\| \right) + 2a_n \|x_{n+1} - x_n\| \cdot \|x_{n+1} - q\| \]
\[ + 2a_n \left\| Tx_{n+1} - Ty_n^1 \right\| \cdot \|x_{n+1} - q\| + 2c_n \left( \|u_n - q\| + \|x_n - q\| \right) \|x_{n+1} - q\| \]
\[ \leq \|x_n - q\|^2 - 2a_n \Phi \left( \frac{R}{2} \right) + 2a_n \frac{\Phi (R/2)}{8R} \cdot 2R + 2a_n \frac{\Phi (R/2)}{8R} \cdot 2R + 2a_n \tau_0 (R + M) 2R \]
\[ \leq \|x_n - q\|^2 - 2a_n \Phi \left( \frac{R}{2} \right) + 2a_n \frac{\Phi (R/2)}{8R(M + 2R)} (R + M) 2R \]
\[ \leq \|x_n - q\|^2 - \frac{a_n}{2} \Phi \left( \frac{R}{2} \right) \leq R^2, \tag{2.11} \]

which is a contradiction. Hence, \( x_{n+1} \in B_1 \), that is, \( \{x_n\} \) is a bounded sequence; it leads to that \( \{y_1^n\}, \{y_2^n\}, \ldots, \{y_{\nu-1}^n\} \) are all bounded sequences as well.

**Step 2.** We want to prove \( \|x_n - q\| \to 0 \) as \( n \to \infty \).

Since \( a_n, b_n, c_n, d_n^k \to 0 \) as \( n \to \infty \) and \( \{x_n\}, \{y_n^k\} \) are bounded. From (2.9), we obtain

\[ \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0, \quad \lim_{n \to \infty} \|x_{n+1} - y_n^1\| = 0, \quad \lim_{n \to \infty} \left\| Tx_{n+1} - Ty_n^1 \right\| = 0. \tag{2.12} \]

By (2.11), we have

\[ \|x_{n+1} - q\|^2 = \left\| (x_n - q) + a_n \left( Ty_n^1 - x_n \right) + c_n (u_n - x_n) \right\|^2 \]
\[ \leq \|x_n - q\|^2 + 2a_n \left\langle Ty_n^1 - x_n, j(x_{n+1} - q) \right\rangle + 2c_n \left\langle u_n - x_n, j(x_{n+1} - q) \right\rangle \]
\[ \leq \|x_n - q\|^2 + 2a_n \left( Tx_{n+1} - x_{n+1} + x_{n+1} - x_n - Tx_{n+1} + Ty_n^1, j(x_{n+1} - q) \right) \]
\[ + 2c_n \|u_n - x_n\| \cdot \|x_{n+1} - q\| \]
Let $\Phi$ be a uniformly continuous $\Phi$-quasi-accretive operator with $q \in N(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be real sequences in $[0, 1]$ and satisfy the conditions:

(i) $a_n + c_n \leq 1$, $b_n + d_n \leq 1$, $k = 1, 2, \ldots, p - 1$;
(ii) $a_n, c_n, b_n, d_n \to 0$ as $n \to \infty$, $k = 1, 2, \ldots, p - 1$;
(iii) $c_n = o(a_n)$, $\sum_{n=0}^{\infty} a_n = \infty$.

For some $x_0 \in E$, let $\{u_n\}, \{w_n^1\}, \{w_n^2\}, \ldots, \{w_n^{p-1}\}$ be any bounded sequences of $E$, and let $\{x_n\}$ be the multistep iterative sequence with errors defined by

$$x_0 \in D,$$

$$y_n^{p-1} = \left(1 - b_n^{p-1} - d_n^{p-1}\right) x_n + b_n^{p-1} S x_n + d_n^{p-1} w_n^{p-1}, \quad n \geq 0, \quad p \geq 2,$$

$$y_n^k = \left(1 - b_n^k - d_n^k\right) x_n + b_n^k S y_n^{k+1} + d_n^k w_n^k, \quad k = p - 2, p - 3, \ldots, 2, 1,$$

$$x_{n+1} = (1 - a_n - c_n) x_n + a_n S y_n^1 + c_n u_n, \quad n \geq 0,$$

where $S : E \to E$ is defined by $S x = x - T x$ for all $x \in E$. Then (2.14) converges strongly to the fixed point $q$ of $S$.

**Proof.** We find easily that $S$ is a uniformly continuous $\Phi$-hemicontractive. Then the conclusion of Theorem 2.2 is obtained directly by Theorem 2.1. \qed

**Remark 2.3.** In Theorems 2.1 and 2.2, if $b_n^k = d_n^k = 0$ ($k = p - 1, p - 2, \ldots, 2, 1$), then the conclusions are as follows.

**Corollary 2.4.** Let $E$ be an arbitrary real Banach space, $D$ a nonempty closed convex subset of $E$, and $T : D \to D$ a uniformly continuous $\Phi$-hemicontractive mapping with $q \in F(T) \neq \emptyset$. Let $\{a_n\}, \{c_n\}$ be real sequences in $[0, 1]$ and satisfy the conditions (i) $a_n + c_n \leq 1$; (ii) $a_n, c_n \to 0$ as $n \to \infty$; (iii) $c_n = o(a_n)$, and $\sum_{n=0}^{\infty} a_n = \infty$. For some $x_0 \in D$, let $\{u_n\}$ be any bounded sequence of $D$, and let $\{x_n\}$ be Mann iterative sequence with errors defined by $x_{n+1} = (1 - a_n - c_n) x_n + a_n T x_n + c_n u_n$, $n \geq 0$. Then $\{x_n\}$ converges strongly to the fixed point $q$ of $T$.

**Corollary 2.5.** Let $E$ be an arbitrary real Banach space and let $T : E \to E$ be a uniformly continuous $\Phi$-quasi-accretive operator with $q \in N(T) \neq \emptyset$. Let $\{a_n\}, \{c_n\}$ be real sequences in $[0, 1]$ and satisfy the conditions (i) $a_n + c_n \leq 1$; (ii) $a_n, c_n \to 0$ as $n \to \infty$; (iii) $c_n = o(a_n)$, and $\sum_{n=0}^{\infty} a_n = \infty$. For
some $x_0 \in E$, let $\{u_n\}$ be any bounded sequence of $E$, and let $\{x_n\}$ be Mann iterative sequence with errors defined by $x_{n+1} = (1 - a_n - c_n)x_n + a_nSx_n + c_nu_n$, $n \geq 0$, where $S : E \to E$ is defined by $Sx = x - Tx$ for all $x \in E$. Then $\{x_n\}$ converges strongly to the fixed point $q$ of $S$.

Remark 2.6. It is mentioned to notice that there exists a serious shortcoming in the proof process of Theorem 2.3 of [3]. That is, $M_1c_n \leq (\Phi(e)/4)\alpha_n$ does not hold in line 15 of Claim 2 of page 552. The reason is that the conditions $\sum_{n=0}^\infty c_n < +\infty$ and $\sum_{n=0}^\infty b_n = +\infty$, $b_n \to 0$ as $n \to \infty$ can not obtain $c_n = o(b_n)$.

Counterexample, let the iteration parameters be $a_n = 1 - b_n - c_n$, $b_n$, $c_n$ in the following:

$$|b_n| : b_0 = b_1 = 0, \quad b_n = \frac{1}{n}, \quad n \geq 2,$$

$$|c_n| : 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,$$

$$1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,$$

$$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,$$

$$1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,$$

$$1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,$$

$$1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,$$

Then, $\sum_{n=0}^\infty b_n = +\infty$, $\sum_{n=0}^\infty c_n < 2 \sum_{n=0}^\infty (1/n^2) < +\infty$, but $c_n \not= o(b_n)$.

Application 1. Let $E = R$ be a real number space with the usual norm and $D = [0, +\infty)$. Define $T : D \to D$ by

$$Tx = \frac{x^3}{1 + x^2}$$

for all $x \in D$. Then $T$ is uniformly continuous with $F(T) = \{0\}$. Define $\Phi : [0, +\infty) \to [0, +\infty)$ by

$$\Phi(t) = \frac{t^2}{1 + t^2},$$

then $\Phi$ is a strictly increasing function with $\Phi(0) = 0$. For all $x \in D$, $q \in F(T)$, we obtain that

$$\langle Tx - Tq, j(x - q) \rangle = \left\langle \frac{x^3}{1 + x^2} - 0, j(x - 0) \right\rangle$$

$$= \left\langle \frac{x^3}{1 + x^2}, x \right\rangle$$

$$= \frac{x^4}{1 + x^2}$$

$$= |x - q|^2 - \frac{|x - q|^2}{1 + |x - q|^2}$$

$$= |x - q|^2 - \Phi(|x - q|).$$
Therefore, $T$ is a $\Phi$-hemicomtractive mapping. Set
\begin{align*}
a_n &= \frac{1}{n+2}, \quad c_n = \frac{1}{(n+2)^2}, \quad b_n^k = d_n^k = \frac{1}{n+2}, \quad k = 1, 2, \ldots, p - 1 \tag{2.19}
\end{align*}
for all $n \geq 0$.

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**References**

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