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Research Article

Multiple Solutions for a Class of Multipoint Boundary Value Systems Driven by a One-Dimensional $(p_1, \ldots, p_n)$-Laplacian Operator

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Employing a recent three critical points theorem due to Bonanno and Marano (2010), the existence of at least three solutions for the following multipoint boundary value system $-(|u_i'|^{p_i-2}u_i')' = \lambda F_{u_i}(x, u_1, \ldots, u_n)$ in $(0,1)$, $u_i(0) = \sum_{j=1}^{m} a_j u_i(x_j)$, $u_i(1) = \sum_{j=1}^{m} b_j u_i(x_j)$ for $1 \le i \le n$, is established.

1. Introduction

In this work, we consider the following multipoint boundary value system

$$\begin{aligned} &-\left(\left|u_i'\right|^{p_i-2}u_i'\right)' = \lambda F_{u_i}(x, u_1, \ldots, u_n) \quad \text{in } (0,1), \\ &u_i(0) = \sum_{j=1}^{m} a_j u_i(x_j), \qquad u_i(1) = \sum_{j=1}^{m} b_j u_i(x_j), \end{aligned} \tag{1.1}$$

for $1 \le i \le n$, where $p_i > 1$ for $1 \le i \le n$, $\lambda > 0$, $m, n \ge 1$, $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $F(\cdot, t_1, \ldots, t_n)$ is continuous in $[0,1]$ for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$, $F(x, \cdot, \ldots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in [0,1]$ and $F(x, 0, \ldots, 0) = 0$ for all $x \in [0,1]$, $a_j, b_j \in \mathbb{R}$ for $j = 1, \ldots, m$ and $0 < x_1 < x_2 < x_3 < \cdots < x_m < 1$, and F_{u_i} denotes the partial derivative of F with respect to u_i for $1 \le i \le n$.

The study of multiplicity of solutions is an important mathematical subject which is also interesting from the practical point of view because the physical processes described

by boundary value problems for differential equations exhibit, generally, more than one solution. In [1–3], Ricceri proposed and developed an innovative minimal method for the study of nonlinear eigenvalue problems. Following that, Bonanno [4] gave an application of the method to the two-point problem

$$\begin{gathered} u'' + \lambda f(u) = 0 \quad \text{in } (0,1), \\ u(0) = u(1) = 0. \end{gathered} \tag{1.2}$$

Bonanno also gave more precise versions of the three critical points of Ricceri in [5, 6]. In particular, in [5], an upper bound of the interval of parameters λ for which the functional has three critical points is established. Candito [7] extended the main result of [4] to the nonautonomous case

$$\begin{gathered} u'' + \lambda f(x,u) = 0 \quad \text{in } (a,b), \\ u(a) = u(b) = 0. \end{gathered} \tag{1.3}$$

In [8], He and Ge extended the main results of [4, 7] to the quasilinear differential equation

$$\begin{gathered} \left(\varphi_p(u')\right)' + \lambda f(x,u) = 0 \quad \text{in } (a,b), \\ u(a) = u(b) = 0. \end{gathered} \tag{1.4}$$

In [9], the authors extended the main results of [4, 7, 9] to the quasilinear differential equation with Sturm-Liouville boundary conditions

$$\begin{gathered} \left(\left|u'\right|^{p-2}u'\right)' + \lambda f(x,u) = 0 \quad \text{in } (a,b), \\ \alpha_1 u(a) - \alpha_2 u'(a) = 0, \qquad \beta_1 u(b) - \beta_2 u'(b) = 0, \end{gathered} \tag{1.5}$$

where $p > 1$ is a constant, λ is a positive parameter, $a, b \in \mathbb{R}$; $a < b$. In particular, in [10], the authors motivated by these works, established some criteria for the existence of three classical solutions of the system (1.1), while in [11], based on Ricceri's three critical points theorem [3], the existence of at least three classical solutions to doubly eigenvalue multipoint boundary value systems was established.

In the present paper, based on a three critical points theorem due to Bonanno and Marano [12], we ensure the existence of least three classical solutions for the system (1.1).

Several results are known concerning the existence of multiple solutions for multipoint boundary value problems, and we refer the reader to the papers [13–16] and the references cited therein.

Here and in the sequel, X will denote the Cartesian product of n space

$$X_i = \left\{ \xi \in W^{1,p_i}([0,1]);\ \xi(0) = \sum_{j=1}^{m} a_j \xi(x_j),\ \xi(1) = \sum_{j=1}^{m} b_j \xi(x_j) \right\}, \tag{1.6}$$

for $i = 1, \ldots, n$, that is, $X = X_1 \times \cdots \times X_n$ equipped with the norm

$$\|(u_1, \ldots, u_n)\| = \sum_{i=1}^{n} \|u_i'\|_{p_i}, \tag{1.7}$$

where

$$\|u_i'\|_{p_i} = \left(\int_0^1 |u_i'(x)|^{p_i} dx \right)^{1/p_i} \tag{1.8}$$

for $1 \le i \le n$.

We say that $u = (u_1, \ldots, u_n)$ is a weak solution to (1.1) if $u = (u_1, \ldots, u_n) \in X$ and

$$\int_0^1 \sum_{i=1}^{n} |u_i'(x)|^{p_i - 2} u_i'(x) v_i'(x) dx - \lambda \int_0^1 \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx = 0, \tag{1.9}$$

for every $(v_1, \ldots, v_n) \in X$.

A special case of our main result is the following theorem.

Theorem 1.1. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two positive continuous functions such that the differential 1-form $w := f(\xi, \eta) d\xi + g(\xi, \eta) d\eta$ is integrable, and let F be a primitive of w such that $F(0,0) = 0$. Fix $p, q > 2$, $0 < x_1 < x_2 < 1$, and assume that*

$$\liminf_{(\xi,\eta) \to (0,0)} \frac{F(\xi, \eta)}{|\xi|^p / p + |\eta|^q / q} = \limsup_{|\xi| \to +\infty,\ |\eta| \to +\infty} \frac{F(\xi, \eta)}{|\xi|^p / p + |\eta|^q / q} = 0, \tag{1.10}$$

then there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$, the problem

$$\begin{gathered} -\left(|u_1'|^{p-2} u_1' \right)' = \lambda f(u_1, u_2) \quad \text{in } (0,1), \\ -\left(|u_2'|^{q-2} u_2' \right)' = \lambda g(u_1, u_2) \quad \text{in } (0,1), \\ u_i(0) = a_1 u_i(x_1) + a_2 u_i(x_2), \qquad u_i(1) = b_1 u_i(x_1) + b_2 u_i(x_2) \end{gathered} \tag{1.11}$$

admits at least two positive classical solutions.

The main aim of the present paper is to obtain further applications of [12, Theorem 2.6] (see Theorem 2.1 in the next section) to the system (1.1), and the obtained results are strictly

comparable with those of [9–11], and here we wil give the exact collocation of the interval of positive parameters.

For other basic notations and definitions, we refer the reader to [17–26]. We note that some of the ideas used here were motivated by corresponding ones in [10].

2. Main Results

Our main tool is a three critical points theorem obtained in [12] (see also [1, 2, 5, 27] for related results), which is a more precise version of Theorem 3.2 of [28], to transfer the existence of three solutions of the system (1.1) into the existence of critical points of the Euler functional. We recall it here in a convenient form (see [23]).

Theorem 2.1 (see [12, Theorem 2.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$ such that*

(κ_1) $\sup_{\Phi(x)\le r}\Psi(x)/r < \Psi(\overline{x})/\Phi(\overline{x})$,

(κ_2) *for each $\lambda \in \Lambda_r :=]\Phi(\overline{x})/\Psi(\overline{x}), r/\sup_{\Phi(x)\le r}\Psi(x)[$, the functional $\Phi - \lambda\Psi$ is coercive.*

then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Put

$$k = \max\left\{\sup_{u_i\in X_i\setminus\{0\}} \frac{\max_{x\in[0,1]}|u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}};\ \text{for } 1 \le i \le n\right\}. \tag{2.1}$$

Since $p_i > 1$ for $1 \le i \le n$, and the embedding $X = X_1 \times \cdots \times X_n \hookrightarrow (C^0([0,1]))^n$ is compact, one has $k < +\infty$. Moreover, from [10, Lemma 3.1], one has

$$\sup_{u\in X_i\setminus\{0\}} \frac{\max_{x\in[0,1]}|u(x)|}{\|u'\|_{p_i}} \le \frac{1}{2}\left(1 + \frac{\sum_{j=1}^m |a_j|}{\left|1 - \sum_{j=1}^m a_j\right|} + \frac{\sum_{j=1}^m |b_j|}{\left|1 - \sum_{j=1}^m b_j\right|}\right). \tag{2.2}$$

Put $\phi_{p_i}(s) = |s|^{p_i-1}s$ for $1 \le i \le n$. Let $\phi_{p_i}^{-1}$ denotes the inverse of ϕ_{p_i} for $1 \le i \le n$. then, $\phi_{p_i}^{-1}(t) = \phi_{q_i}(t)$ where $1/p_i + 1/q_i = 1$. It is clear that ϕ_{p_i} is increasing on $\mathbb{R}$,

$$\lim_{t\to-\infty}\phi_{p_i}(t) = -\infty, \qquad \lim_{t\to+\infty}\phi_{p_i}(t) = +\infty. \tag{2.3}$$

Lemma 2.2 (see [10, Lemma 3.3]). *For fixed $\lambda \in \mathbb{R}$ and $u = (u_1, \ldots, u_n) \in (C([0,1]))^n$, define $\alpha_i(t;u) : \mathbb{R} \to \mathbb{R}$ by*

$$\alpha_i(t;u) = \int_0^1 \phi_{p_i}^{-1}\left(t - \lambda\int_0^\delta F_{u_i}(\xi, u_1(\xi), \ldots, u_n(\xi))d\xi\right)d\delta + \sum_{j=1}^m a_j u_i(x_j) - \sum_{j=1}^m b_j u_i(x_j). \tag{2.4}$$

then the equation

$$\alpha_i(t;u) = 0 \tag{2.5}$$

has a unique solution $t_{u,i}$.

Direct computations show the following.

Lemma 2.3 (see [10, Lemma 3.4]). *The function* $u = (u_1, \ldots, u_n)$ *is a solution of the system* (1.1) *if and only if* $u_i(x)$ *is a solution of the equation*

$$u_i(x) = \sum_{j=1}^{m} a_j u_i(x_j) + \int_0^x \phi_{p_i}^{-1}\left(t_{u,i} - \lambda \int_0^\delta F_{u_i}(\xi, u_1(\xi), \ldots, u_n(\xi))d\xi\right)d\delta, \tag{2.6}$$

for $1 \le i \le n$, *where* $t_{u,i}$ *is the unique solution of* (2.5).

Lemma 2.4. *A weak solution to the systems* (1.1) *coincides with classical solution one.*

Proof. Suppose that $u = (u_1, \ldots, u_n) \in X$ is a weak solution to (1.1), so

$$\int_0^1 \sum_{i=1}^{n} \phi_{p_i}(u_i'(x))v_i'(x)dx - \lambda \int_0^1 \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x)dx, \tag{2.7}$$

for every $(v_1, \ldots, v_n) \in X$. Note that, in one dimension, any weakly differentiable function is absolutely continuous, so that its classical derivative exists almost everywhere, and that the classical derivative coincides with the weak derivative. Now, using integration by part, from (2.7), we obtain

$$\sum_{i=1}^{n} \int_0^1 \left[\left(\phi_{p_i}(u_i'(x))\right)' + \lambda F_{u_i}(x, u_1(x), \ldots, u_n(x))\right]v_i(x)dx = 0, \tag{2.8}$$

and so for $1 \le i \le n$,

$$\left(\phi_{p_i}(u_i'(x))\right)' + \lambda F_{u_i}(x, u_1(x), \ldots, u_n(x)) = 0, \tag{2.9}$$

for almost every $x \in (0,1)$. Then, by Lemmas 2.2 and 2.3, we observe

$$u_i(x) = \sum_{j=1}^{m} a_j u_i(x_j) + \int_0^x \phi_{p_i}^{-1}\left(t_{u,i} - \lambda \int_0^\delta F_{u_i}(s, u_1(s), \ldots, u_n(s))ds\right)d\delta, \tag{2.10}$$

for $1 \le i \le n$, where $t_{u,i}$ is the unique solution of (2.5). Hence, $u_i \in C^1([0,1])$ and $\phi_{p_i}(u_i'(x)) \in C^1([0,1])$ for $1 \le i \le n$, namely $u = (u_1, \ldots, u_n)$ is a classical solution to the system (1.1). □

For all $\gamma > 0$, we denote by $K(\gamma)$ the set

$$\left\{(t_1,\dots,t_n) \in R^n : \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \le \gamma\right\}. \tag{2.11}$$

Now, we formulate our main result as follows.

Theorem 2.5. *Assume that there exist $2m$ constants a_j, b_j for $1 \le j \le m$ with $\sum_{j=1}^{m} a_j \ne 1$ and $\sum_{j=1}^{m} b_j \ne 1$, a positive constant r and a function $w = (w_1,\dots,w_n) \in X$ such that*

(A1) $\sum_{i=1}^{n}(\|w_i'\|_{p_i}^{p_i}/p_i) > r$,

(A2) $\int_0^1 \sup_{(t_1,\dots,t_n)\in K(kr)} F(x,t_1,\dots,t_n)dx < (r\prod_{i=1}^{n} p_i)\int_0^1 F(x,w_1(x),\dots,w_n(x))dx/\sum_{i=1}^{n}\prod_{j=1,j\ne i}^{n} p_j\|w_i'\|_{p_i}^{p_i}$ *where* $K(kr) = \{(t_1,\dots,t_n) \mid \sum_{i=1}^{n}(|t_i|^{p_i}/p_i) \le kr\}$ (*see* (2.11)),

(A3) $\limsup_{|t_1|\to+\infty,\dots,|t_n|\to+\infty} F(x,t_1,\dots,t_n)/\sum_{i=1}^{n}(|t_i|^{p_i}/p_i) < \int_0^1 \sup_{(t_1,\dots,t_n)\in K(kr)} \times F(x,t_1,\dots,t_n)dx/kr$ *uniformly with respect to* $x \in [0,1]$.

Then, for each $\lambda \in \Lambda_r :=]\sum_{i=1}^{n}(\|w_{ip_i}'^{p_i}\|/p_i)/\int_0^1 F(x,w_1(x),\dots,w_n(x))dx, r/\int_1^1 \sup_{(t_1,\dots,t_n)\in K(kr)} F(x,t_1,\dots,t_n)dx[$, *the system* (1.1) *admits at least three distinct classical solutions in* X.

Proof. In order to apply Theorem 2.1 to our problem, we introduce the functionals Φ, $\Psi : X \to \mathbb{R}$ for each $u = (u_1,\dots,u_n) \in X$, as follows:

$$\begin{aligned} \Phi(u) &= \sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i}, \\ \Psi(u) &= \int_0^1 F(x,u_1(x),\dots,u_n(x))dx. \end{aligned} \tag{2.12}$$

Since $p_i > 1$ for $1 \le i \le n$, X is compactly embedded in $(C^0([0,1]))^n$ and it is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1,\dots,u_n) \in X$ are the functionals $\Phi'(u)$, $\Psi'(u) \in X^*$, given by

$$\begin{aligned} \Phi'(u)(v) &= \int_0^1 \sum_{i=1}^{n} |u_i'(x)|^{p_i-2} u_i'(x) v_i'(x)dx, \\ \Psi'(u)(v) &= \int_0^1 \sum_{i=1}^{n} F_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)dx \end{aligned} \tag{2.13}$$

for every $v = (v_1,\dots,v_n) \in X$, respectively, as well as Ψ is sequentially weakly upper semicontinuous. Furthermore, Lemma 2.6 of [11] gives that Φ' admits a continuous inverse on X^*, and since Φ' is monotone, we obtain that Φ is sequentially weakly lower semiacontinuous (see [29, Proposition 25.20]). Moreover, $\Psi' : X \to X^*$ is a compact operator. From assumption (A1), we get $0 < r < \Phi(w)$. Since from (2.1) for each $u_i \in X_i$,

$$\sup_{x\in[0,1]} |u_i(x)|^{p_i} \le k\|u_i'\|^{p_i} \tag{2.14}$$

for $i = 1, \ldots, n$, we have

$$\sup_{x\in[0,1]} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq k \sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i}, \tag{2.15}$$

for each $u = (u_1, \ldots, u_n) \in X$, and so using (2.15), we observe

$$\begin{aligned}
\Phi^{-1}(]-\infty, r]) &= \{(u_1, \ldots, u_n) \in X; \Phi(u_1, \ldots, u_n) \leq r\} \\
&= \left\{(u_1, \ldots, u_n) \in X; \sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} \leq r\right\} \\
&\subseteq \left\{(u_1, \ldots, u_n) \in X; \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq kr \ \forall x \in [0,1]\right\},
\end{aligned} \tag{2.16}$$

and it follows that

$$\begin{aligned}
\sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(]-\infty,r])} \Psi(u_1, \ldots, u_n) &= \sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(]-\infty,r])} \int_0^1 F(x, u_1(x), \ldots, u_n(x))dx \\
&\leq \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(kr)} F(x, t_1, \ldots, t_n)dx.
\end{aligned} \tag{2.17}$$

Therefore, owing to assumption (A2), we have

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u_1, \ldots, u_n) &= \sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(]-\infty,r])} \int_0^1 F(x, u_1(x), \ldots, u_n(x))dx \\
&\leq \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(kr)} F(x, t_1, \ldots, t_n)dx \\
&< \left(r\prod_{i=1}^{n} p_i\right) \frac{\int_0^1 F(x, w_1(x), \ldots, w_n(x))dx}{\sum_{i=1}^{n} \prod_{j=1,j\neq i}^{n} p_j \|w_i'\|_{p_i}^{p_i}} \\
&< r\frac{\int_0^1 F(x, w_1(x), \ldots, w_n(x))dx}{\sum_{i=1}^{n} (\|w_i'\|^{p_i}/p_i)} \\
&= r\frac{\Psi(w)}{\Phi(w)}.
\end{aligned} \tag{2.18}$$

Furthermore, from (A3), there exist two constants $\gamma, \upsilon \in \mathbb{R}$ with

$$0 < \gamma < \frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(kr)} F(x, t_1, \ldots, t_n)dx}{r}, \tag{2.19}$$

such that

$$kF(x,t_1,\dots,t_n) \le \gamma \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} + \upsilon, \quad \forall x \in [0,1],\ \forall (t_1,\dots,t_n) \in \mathbb{R}^n. \tag{2.20}$$

Fix $(u_1,\dots,u_n) \in X$, Then

$$F(x,u_1(x),\dots,u_n(x)) \le \frac{1}{k}\left(\gamma \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} + \upsilon\right) \quad \forall x \in [0,1]. \tag{2.21}$$

So, for any fixed $\lambda \in \Lambda_r$, from (2.15) and (2.21), we have

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} - \lambda \int_0^1 F(x,u_1(x),\dots,u_n(x))dx \\
&\ge \sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\gamma}{k}\left(\sum_{i=1}^{n} \frac{1}{p_i}\int_0^1 |u_i(x)|^{p_i}dx\right) - \frac{\lambda\upsilon}{k} \\
&\ge \sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\gamma}{k}\left(k\sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i}\right) - \frac{\lambda\upsilon}{k} \\
&= \sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} - \lambda\gamma\sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\upsilon}{k} \\
&\ge \left(1 - \gamma\frac{r}{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(kr)} F(x,t_1,\dots,t_n)dx}\right)\sum_{i=1}^{n} \frac{\|u_i'\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\upsilon}{k},
\end{aligned} \tag{2.22}$$

and thus,

$$\lim_{\|(u_1,\dots,u_n)\| \to +\infty} (\Phi(u_1,\dots,u_n) - \lambda\Psi(u_1,\dots,u_n)) = +\infty, \tag{2.23}$$

which means that the functional $\Phi - \lambda\Psi$ is coercive. So, all assumptions of Theorem 2.1 are satisfied. Hence, from Theorem 2.1 with $\overline{x} = w$, taking into account that the weak solutions of the system (1.1) are exactly the solutions of the equation $\Phi'(u_1,\dots,u_n) - \lambda\Psi'(u_1,\dots,u_n) = 0$ and using Lemma 2.4, we have the conclusion. □

Now we want to present a verifiable consequence of the main result where the test function w is specified.

Put

$$\sigma_i = \left[2^{p_i-1}\left(x_1^{1-p_i}\left|1 - \sum_{j=1}^{m} a_j\right|^{p_i} + (1-x_m)^{1-p_i}\left|1 - \sum_{j=1}^{m} b_j\right|^{p_i}\right)\right]^{1/p_i} \quad \text{for } 1 \le i \le n. \tag{2.24}$$

Define

$$B_{1,n}(x) = \begin{cases} \left[x\sum_{j=1}^{m} a_j, x\right]^n & \text{if } \sum_{j=1}^{m} a_j < 1, \\ \left[x, x\sum_{j=1}^{m} a_j\right]^n & \text{if } \sum_{j=1}^{m} a_j > 1, \end{cases}$$
$$B_{2,n}(x) = \begin{cases} \left[x\sum_{j=1}^{m} b_j, x\right]^n & \text{if } \sum_{j=1}^{m} b_j < 1, \\ \left[x, x\sum_{j=1}^{m} b_j\right]^n & \text{if } \sum_{j=1}^{m} b_j > 1, \end{cases} \tag{2.25}$$

where $[\cdot,\cdot]^n = [\cdot,\cdot] \times \cdots \times [\cdot,\cdot]$, then we have the following consequence of Theorem 2.5.

Corollary 2.6. *Assume that there exist $2m$ constants a_j, b_j for $1 \le j \le m$ with $\sum_{j=1}^{m} a_j \neq 1$ and $\sum_{j=1}^{m} b_j \neq 1$ and two positive constants θ and τ with $\sum_{i=1}^{n}((\sigma_i\tau)^{p_i}/p_i) > \theta/k\prod_{i=1}^{n} p_i$ such that*

(B1) *$F(x,t_1,\ldots,t_n) \ge 0$ for each $x \in [0,x_1/2] \cup [(1+x_m)/2,1]$ and $(t_1,\ldots,t_n) \in B_{1,n}(\tau) \cup B_{2,n}(\tau)$,*

(B2) *$\sum_{i=1}^{n}((\sigma_i\tau)^{p_i}/p_i)\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\theta/\prod_{i=1}^{n} p_i)} F(x,t_1,\ldots,t_n)dx < (\theta/k\prod_{i=1}^{n} p_i)\int_{x_1/2}^{(1+x_m)/2} F(x,\tau,\ldots,\tau)dx$, where σ_i is given by (2.24) and $K(\theta/\prod_{i=1}^{n} p_i) = \{(t_1,\ldots,t_n) \mid \sum_{i=1}^{n}(|t_i|^{p_i}/p_i) \le c/\prod_{i=1}^{n} p_i\}$ (see (2.11));*

(B3) *$\limsup_{|t_1|\to+\infty,\ldots,|t_n|\to+\infty}(F(x,t_1,\ldots,t_n)/\sum_{i=1}^{n}(|t_i|^{p_i}/p_i)) < \prod_{i=1}^{n} p_i/\theta \times \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\theta/\prod_{i=1}^{n} p_i)} F(x,t_1,\ldots,t_n)dx$ uniformly with respect to $x \in [0,1]$.*

then, for each $\lambda \in]\sum_{i=1}^{n}((\sigma_i\tau)^{p_i}/p_i)/\int_{x_1/2}^{(1+x_m)/2} F(x,\tau,\ldots,\tau)dx, (\theta/k\prod_{i=1}^{n} p_i)/\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\theta/\prod_{i=1}^{n} p_i)} F(x,t_1,\ldots,t_n)dx[$ the systems (1.1) admits at least three distinct classical solutions.

Proof. Set $w(x) = (w_1(x),\ldots,w_n(x))$ such that for $1 \le i \le n$,

$$w_i(x) = \begin{cases} \tau\left(\sum_{j=1}^{m} a_j + \dfrac{2\left(1-\sum_{j=1}^{m} a_j\right)}{x_1}x\right) & \text{if } x \in \left[0, \dfrac{x_1}{2}\right], \\ \tau & \text{if } x \in \left[\dfrac{x_1}{2}, \dfrac{1+x_m}{2}\right], \\ \tau\left(\dfrac{2-\sum_{j=1}^{m} b_j - x_m\sum_{j=1}^{m} b_j}{1-x_m} - \dfrac{2\left(1-\sum_{j=1}^{m} b_j\right)}{1-x_m}x\right) & \text{if } x \in \left[\dfrac{1+x_m}{2}, 1\right], \end{cases} \tag{2.26}$$

and $r = \theta/k\prod_{i=1}^{n} p_i$. It is easy to see that $w = (w_1,\ldots,w_n) \in X$, and, in particular, one has

$$\|w_i'\|_{p_i}^{p_i} = (\sigma_i\tau)^{p_i}, \tag{2.27}$$

for $1 \le i \le n$, which, employing the condition $\sum_{i=1}^{n}((\sigma_i\tau)^{p_i}/p_i) > \theta/k\prod_{i=1}^{n}p_i$, gives

$$\sum_{i=1}^{n}\frac{\|w_i'\|_{p_i}^{p_i}}{p_i} > r. \tag{2.28}$$

Since for $1 \le i \le n$,

$$\begin{aligned}
&\tau\sum_{j=1}^{m}a_j \le w_i(x) \le \tau \quad \text{for each } x \in \left[0, \frac{x_1}{2}\right] \text{ if } \sum_{j=1}^{m}a_j < 1,\\
&\tau \le w_i(x) \le \tau\sum_{j=1}^{m}a_j \quad \text{for each } x \in \left[0, \frac{x_1}{2}\right] \text{ if } \sum_{j=1}^{m}a_j > 1,\\
&\tau\sum_{j=1}^{m}b_j \le w_i(x) \le \tau \quad \text{for each } x \in \left[\frac{1+x_m}{2}, 1\right] \text{ if } \sum_{j=1}^{m}b_j < 1,\\
&\tau \le w_i(x) \le \tau\sum_{j=1}^{m}b_j \quad \text{for each } x \in \left[\frac{1+x_m}{2}, 1\right] \text{ if } \sum_{j=1}^{m}b_j > 1,
\end{aligned} \tag{2.29}$$

the condition (B1) ensures that

$$\int_0^{x_1/2}F(x, w_1(x), \ldots, w_n(x))dx + \int_{(1+x_m)/2}^{1}F(x, w_1(x), \ldots, w_n(x))dx \ge 0. \tag{2.30}$$

Therefore, owing to assumption (B2), we have

$$\begin{aligned}
\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(kr)} F(x, t_1, \ldots, t_n)dx &< \frac{\theta}{\left(\sum_{i=1}^{n}((\sigma_i\tau)^{p_i}/p_i)\right)\left(k\prod_{i=1}^{n}p_i\right)}\int_{x_1/2}^{(1+x_m)/2}F(x, \tau, \ldots, \tau)dx\\
&\le \frac{\theta}{k}\frac{\int_0^1 F(x, w_1(x), \ldots, w_n(x))dx}{\sum_{i=1}^{n}\prod_{j=1,j\ne i}^{n}p_j\|w_i'\|_{p_i}^{p_i}}\\
&= \left(r\prod_{i=1}^{n}p_i\right)\frac{\int_0^1 F(x, w_1(x), \ldots, w_n(x))dx}{\sum_{i=1}^{n}\prod_{j=1,j\ne i}^{n}p_j\|w_i'\|_{p_i}^{p_i}},
\end{aligned} \tag{2.31}$$

where $K(\theta/\prod_{i=1}^{n}p_i) = \{(t_1, \ldots, t_n) \mid \sum_{i=1}^{n}(|t_i|^{p_i}/p_i) \le \theta/\prod_{i=1}^{n}p_i\}$. Moreover, from assumption (B3) it follows that assumption (A3) is fulfilled. Hence, taking into account that

$$\left]\frac{\sum_{i=1}^{n}((\sigma_i\tau)^{p_i}/p_i)}{\int_{x_1/2}^{(1+x_m)/2}F(x, \tau, \ldots, \tau)dx}, \frac{\theta/k\prod_{i=1}^{n}p_i}{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\theta/\prod_{i=1}^{n}p_i)}F(x, t_1, \ldots, t_n)dx}\right[\subseteq \Lambda_r, \tag{2.32}$$

using Theorem 2.5, we have the desired conclusion. □

Let us present an application of Corollary 2.6.

Example 2.7. Let $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be the function defined as

$$F(x,t_1,t_2,t_3) = \begin{cases} 0 & \forall t_i < 0,\ i = 1,2, \\ x^2 t_1^{100} e^{-t_1} & \text{for } t_1 \geq 0,\ t_2 < 0,\ t_3 < 0, \\ x^2 t_2^{100} e^{-t_2} & \text{for } t_1 < 0,\ t_2 \geq 0,\ t_3 < 0, \\ x^2 t_3^{100} e^{-t_3} & \text{for } t_1 < 0,\ t_2 < 0,\ t_3 \geq 0. \\ x^2 \sum_{i=1}^{3} t_i^{100} e^{-t_i} & \text{for } t_i \geq 0,\ i = 1,2,3, \end{cases} \tag{2.33}$$

for each $(x,t_1,t_2,t_3) \in [0,1] \times \mathbb{R}^3$. In fact, by choosing $p_1 = p_2 = p_3 = 3$ and $a_1 = b_1 = x_1 = 1/2$, by a simple calculation, we obtain that $k = 27/8$ and $\sigma_1 = \sigma_2 = \sigma_3 = 4^{1/3}$, and so with $\theta = 9$ and $\tau = 100$, we observe that the assumptions (B1) and (B3) in Corollary 2.6 are satisfied. For (B2),

$$\begin{aligned} &\sum_{i=1}^{3} \frac{(\sigma_i \tau)^{p_i}}{p_i} \int_0^1 \sup_{(t_1,t_2,t_3) \in K\left(\theta/\prod_{i=1}^{3} p_i\right)} F(x,t_1,t_2,t_3)\,dx \\ &\quad = 4(100)^3 \int_0^1 \sup_{(t_1,t_2,t_3) \in K(1/3)} F(x,t_1,t_2,t_3)\,dx \\ &\quad \leq 4(100)^3 \int_0^1 \sup_{(t_1,t_2,t_3) \in K(1/3)} x^2 \sum_{i=1}^{3} t_i^{100} e^{-t_i}\,dx \\ &\quad = 4(100)^3 \max_{(t_1,t_2,t_3) \in K(1/3)} \sum_{i=1}^{3} t_i^{100} e^{-t_i} \int_0^1 x^2\,dx \\ &\quad \leq \frac{4}{3}(100)^3 \left(3 \max_{|t| \leq 1} t^{100} e^{-t}\right) \\ &\quad = 4(100)^3 e \\ &\quad < \frac{13}{4 \times 3^4}(100)^{100} e^{-100} \\ &\quad = \frac{\theta}{k \prod_{i=1}^{3} p_i} \int_{x_1/2}^{(1+x_1)/2} F(x,\tau,\tau,\tau)\,dx. \end{aligned} \tag{2.34}$$

So, for every

$$\lambda \in \left] \frac{4^4 (100)^3}{26(100)^{100} e^{-100}}, \frac{8}{3^5 (100)^3 e} \right[, \tag{2.35}$$

Corollary 2.6 is applicable to the system

$$\begin{aligned}
-\left(\left|u_1'\right|u_1'\right)' &= \lambda x^2\left(u_1^+\right)^{99}e^{-u_1^+}\left(100-u_1^+\right) \quad \text{in } (0,1),\\
-\left(\left|u_2'\right|u_2'\right)' &= \lambda x^2\left(u_2^+\right)^{99}e^{-u_2^+}\left(100-u_2^+\right) \quad \text{in } (0,1),\\
-\left(\left|u_3'\right|u_3'\right)' &= \lambda x^2\left(u_3^+\right)^{99}e^{-u_3^+}\left(100-u_3^+\right) \quad \text{in } (0,1),\\
u_i(0) = u_i(1) &= \frac{1}{2}u_i\left(\frac{1}{2}\right) \quad \text{for } i=1,2,3,
\end{aligned} \tag{2.36}$$

where $t^+ = \max\{t,0\}$.

Here is a remarkable consequence of Corollary 2.6.

Corollary 2.8. *Let $F:\mathbb{R}^n \to \mathbb{R}$ be a C^1 function in R^n such that $F(0,\dots,0)=0$. Assume that there exist $2m$ constants a_j, b_j for $1\le j\le m$ with $\sum_{j=1}^m a_j \ne 1$ and $\sum_{j=1}^m b_j \ne 1$ and two positive constants θ and τ with $\sum_{i=1}^n((\sigma_i\tau)^{p_i}/p_i) > \theta/k\prod_{i=1}^n p_i$ such that*

(C1) *$F(t_1,\dots,t_n)\ge 0$ for each $(t_1,\dots,t_n)\in B_{1,n}(\tau)\cup B_{2,n}(\tau)$,*

(C2) *$\sum_{i=1}^n((\sigma_i\tau)^{p_i}/p_i)\max_{(t_1,\dots,t_n)\in K(\theta/\prod_{i=1}^n p_i)}F(t_1,\dots,t_n) < (\theta(1+x_m-x_1)/2k\prod_{i=1}^n p_i)F(\tau,\dots,\tau)$ where σ_i is given by (2.24),*

(C3) *$\limsup_{|t_1|\to+\infty,\dots,\,|t_n|\to+\infty}(F(t_1,\dots,t_n)/\sum_{i=1}^n(|t_i|^{p_i}/p_i)) < (\prod_{i=1}^n p_i/\theta)\max_{(t_1,\dots,t_n)\in K(\theta/\prod_{i=1}^n p_i)}F(t_1,\dots,t_n)$,*

Then, for each $\lambda \in]\sum_{i=1}^n((\sigma_i\tau)^{p_i}/p_i)/((1+x_m-x_1)/2)F(\tau,\dots,\tau),(\theta/k\prod_{i=1}^n p_i)/\max_{(t_1,\dots,t_n)\in K(\theta/\prod_{i=1}^n p_i)}F(t_1,\dots,t_n)[$, the systems

$$\begin{aligned}
-\left(\left|u_i'\right|^{p_i-2}u_i'\right)' &= \lambda F_{u_i}(u_1,\dots,u_n) \quad \text{in } (0,1),\\
u_i(0) = \sum_{j=1}^m a_j u_i(x_j), &\qquad u_i(1) = \sum_{j=1}^m b_j u_i(x_j),
\end{aligned} \tag{2.37}$$

for $1\le i\le n$, admits at least three distinct classical solutions.

Proof. Set $F(x,t_1,\dots,t_n) = F(t_1,\dots,t_n)$ for all $x\in[0,1]$ and $t_i\in\mathbb{R}$ for $1\le i\le n$. From the hypotheses, we see that all assumptions of Corollary 2.6 are satisfied. So, we have the conclusion by using Corollary 2.6. □

Example 2.9. Let $p_1=p_2=3$, $m=2$, $x_1=1/3$, $x_2=2/3$ and $a_i=b_i=1/3$, $i=1,2$. Consider the system

$$\begin{aligned}
-\left(\left|u_1'\right|u_1'\right)' &= \lambda\left(e^{-u_1}u_1^{11}(12-u_1)\right), \quad \text{in } (0,1),\\
-\left(\left|u_2'\right|u_2'\right)' &= \lambda\left(e^{-u_2}u_2^{9}(10-u_2)\right), \quad \text{in } (0,1),\\
u_1(0) = u_1(1) = \frac{1}{3}u_1\left(\frac{1}{3}\right)+\frac{1}{3}u_1\left(\frac{2}{3}\right), &\qquad u_2(0) = u_2(1) = \frac{1}{3}u_2\left(\frac{1}{3}\right)+\frac{1}{3}u_2\left(\frac{2}{3}\right).
\end{aligned} \tag{2.38}$$

Clearly, (H1) and (H2) hold. A simple calculation shows that $k = 125/8$ and $\sigma_1 = \sigma_2 = (8/3)^{1/3}$. So, by choosing $\theta = 3$ and $\tau = 10$, we observe that the assumptions (C1) and (C3) in Corollary 2.8 hold. For (C3), since $F(t_1, t_2) = t_1^{12}e^{-t_1} + t_2^{10}e^{-t_2}$ for every $(t_1, t_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
\max_{(t_1,t_2)\in K(\theta/\prod_{i=1}^{2}p_i)} F(t_1,t_2) &= \max_{(t_1,t_2)\in K(1/3)} F(t_1,t_2) \\
&= \max_{(t_1,t_2)\in K(1/3)} \left(t_1^{12}e^{-t_1} + t_2^{10}e^{-t_2}\right) \\
&\leq \left[\max_{|t_1|\leq 1} t_1^{12}e^{-t_1} + \max_{|t_2|\leq 1} t_2^{10}e^{-t_2}\right] \\
&= 2e \\
&< \frac{1}{125\cdot 10^3}\left(10^{12}e^{-10} + 10^{10}e^{-10}\right) \\
&= \frac{\theta(1+x_2-x_1)}{\left(\sum_{i=1}^{2}\left((\tau\sigma_i\ )^{p_i}/p_i\right)\right)\left(2k\prod_{i=1}^{2}p_i\right)} F(\tau,\tau).
\end{aligned} \tag{2.39}$$

Note that $\lim_{|t_1|\to\infty,|t_2|\to\infty}(F(t_1,t_2)/\sum_{i=1}^{2}(|t_i|^{p_i}/p_i)) = 0$. We see that for every

$$\lambda \in \left]\frac{8}{3\left(10^9e^{-10} + 10^7e^{-10}\right)}, \frac{4}{375e}\right[, \tag{2.40}$$

Corollary 2.8 is applicable to the system (2.38).

Finally, we prove the theorem in the introduction.

Proof of Theorem 1.1. Since f and g are positive, then F is nonnegative in $\mathbb{R}^2$. Fix $\lambda > \lambda^* := ((\sigma_1\tau)^p/p + (\sigma_2\tau)^q/q)/((1 + x_2 - x_1)/2)F(\tau,\tau)$ for some $\tau > 0$. Note that $\liminf_{(\xi,\eta)\to(0,0)}(F(\xi,\eta)/(|\xi|^p/p + |\eta|^q/q)) = 0$, and there is $\{\theta_n\}_{n\in N} \subseteq]0,+\infty[$such that $\lim_{n\to+\infty}\theta_n = 0$ and

$$\lim_{n\to+\infty} \frac{\max_{(\xi,\eta)\in K(\theta_n/pq)} F(\xi,\eta)}{\theta_n} = 0. \tag{2.41}$$

In fact, one has $\lim_{n\to+\infty}(\max_{(\xi,\eta)\in K(\theta_n/pq)} F(\xi,\eta)/\theta_n) = \lim_{n\to+\infty}(F(\xi_{\theta_n},\eta_{\theta_n})/(|\xi_{\theta_n}|^p/p + |\eta_{\theta_n}|^q/q))\cdot(|\xi_{\theta_n}|^p/p + |\eta_{\theta_n}|^q/q)/\theta_n = 0$, where $F(\xi_{\theta_n},\eta_{\theta_n}) = \sup_{(\xi,\eta)\in K(\theta_n/pq)} F(\xi,\eta)$. Hence, there is $\overline{\theta} > 0$ such that

$$\frac{\max_{(\xi,\eta)\in K(\overline{\theta}/pq)} F(\xi,\eta)}{\overline{\theta}} < \min\left\{\frac{(1+x_2-x_1)}{2\left(q(\sigma_1\tau)^p + p(\sigma_2\tau)^q\right)k} F(\tau,\tau); \frac{1}{\lambda pqk}\right\}, \tag{2.42}$$

and $\overline{\theta} < k(q(\sigma_1\tau)^p + p(\sigma_2\tau)^q)$.

from Corollary 2.8, with $n = 2$ follows the conclusion. □

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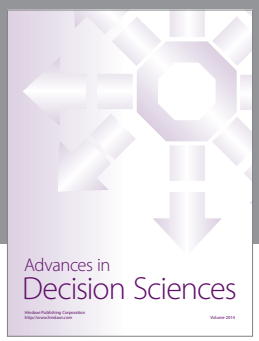

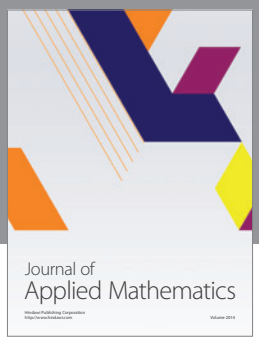

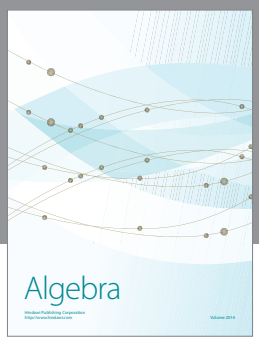

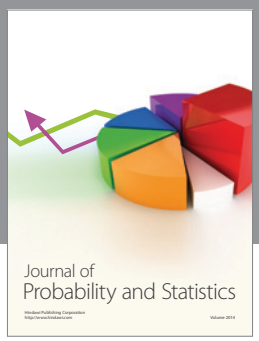

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