Research Article

Argument Property for Certain Analytic Functions

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Let P be the class of functions $p(z)$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$. The object of the present paper is to derive certain argument inequalities of analytic functions $p(z)$ in P .

1. Introduction

Let P be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (1.1)$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. For functions p and g in the class P , we say that p is subordinate to g if there exists an analytic function w in U with $w(0) = 0$, $|w(z)| < 1$ ($z \in U$), and such that $p(z) = g(w(z))$ ($z \in U$). We denote this subordination by

$$p < g \quad (z \in U). \quad (1.2)$$

If g is univalent in U , then this subordination $p < g$ is equivalent to $p(0) = g(0)$ and $p(U) \subset g(U)$.

Recently, several authors investigated various argument properties of analytic functions (see, e.g., [1–6]). The object of the present paper is to discuss some argument inequalities for p in the class P .

Throughout this paper, we let

$$0 < \alpha_1 \leq 1, \quad 0 < \alpha_2 \leq 1, \quad \beta = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}, \quad c = e^{\beta\pi i}. \quad (1.3)$$

In order to prove our main result, we will need the following lemma.

Lemma 1.1 (see [6]). *Let $\lambda_0, \lambda, a, b \in \mathbb{R}$ and $\mu \in \mathbb{C}$. Also let*

$$\begin{aligned} \lambda_0 a \geq 0, \quad \lambda(b+2) \geq 0, \quad (b+1) \operatorname{Re}(\mu) \geq 0, \\ |b+1| \leq \frac{2}{\alpha_1 + \alpha_2}, \quad |a-b-1| \leq \frac{1}{\max\{\alpha_1, \alpha_2\}}. \end{aligned} \quad (1.4)$$

If $q \in P$ satisfies

$$\lambda_0(q(z))^a + \lambda(q(z))^{b+2} + \mu(q(z))^{b+1} + zq'(z)(q(z))^b < h(z) \quad (z \in U), \quad (1.5)$$

where

$$\begin{aligned} h(z) = \lambda_0 \left(\frac{1+cZ}{1-z} \right)^{a((\alpha_1+\alpha_2)/2)} + \left(\frac{1+cZ}{1-z} \right)^{(1/2)(b+1)(\alpha_1+\alpha_2)} \\ \times \left(\mu + \lambda \left(\frac{1+cZ}{1-z} \right)^{(\alpha_1+\alpha_2)/2} + \frac{\alpha_1 + \alpha_2}{2} \left(\frac{z}{1-z} + \frac{cZ}{1+cZ} \right) \right) \end{aligned} \quad (1.6)$$

is (close to convex) univalent, then

$$-\frac{\alpha_2\pi}{2} < \arg(q(z)) < \frac{\alpha_1\pi}{2} \quad (z \in U). \quad (1.7)$$

The bounds α_1 and α_2 in (1.7) are sharp for the function q defined by

$$q(z) = \left(\frac{1+cZ}{1-z} \right)^{(\alpha_1+\alpha_2)/2}. \quad (1.8)$$

Remark 1.2 (see [6]). The function q defined by (1.8) is analytic and univalently convex in U and

$$q(U) = \left\{ w : w \in \mathbb{C}, -\frac{\alpha_2\pi}{2} < \arg w < \frac{\alpha_1\pi}{2} \right\}. \quad (1.9)$$

2. Main Result

Our main theorem is given by the following.

Theorem 2.1. *Let*

$$\lambda_0 > 0, \quad 0 < a \leq \frac{1}{\max\{\alpha_1, \alpha_2\}}, \quad |b+1| \leq \frac{2}{\alpha_1 + \alpha_2}, \quad 0 \leq a-b-1 \leq \frac{1}{\max\{\alpha_1, \alpha_2\}}. \quad (2.1)$$

If $p \in P$ satisfies

$$-\frac{\gamma_2 \pi}{2} < \arg\left(\lambda_0(p(z))^a + zp'(z)(p(z))^b\right) < \frac{\gamma_1 \pi}{2} \quad (z \in U), \quad (2.2)$$

where

$$\begin{aligned} \gamma_j &= \gamma_j(a, b, \alpha_1, \alpha_2) \\ &= a\alpha_j + \frac{2}{\pi} \tan^{-1} \left(\frac{((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \cos((a-b-1)\alpha_j\pi/2)}{2\lambda_0\delta_j(a, b, \alpha_1, \alpha_2) + ((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \sin((a-b-1)\alpha_j\pi/2)} \right) \\ &\quad (j = 1, 2), \\ \delta_j(a, b, \alpha_1, \alpha_2) &= \frac{\left[\left(1 - ((\mathcal{A}) \cos(\beta\pi/2))^2\right)^{1/2} + (-1)^j \sin(\beta\pi/2) \right]^{1+\mathcal{A}}}{2 \left[1 - (\mathcal{A}) (\cos(\beta\pi/2))^2 + (-1)^j \left(1 - ((\mathcal{A}) \cos(\beta\pi/2))^2\right)^{1/2} \sin(\beta\pi/2) \right]^{\mathcal{A}}} \quad (j = 1, 2), \end{aligned} \quad (2.3)$$

where $\mathcal{A}$ denotes $(a-b-1)(\alpha_1 + \alpha_2)/2$, then

$$-\frac{\alpha_2 \pi}{2} < \arg(p(z)) < \frac{\alpha_1 \pi}{2} \quad (z \in U). \quad (2.4)$$

The bounds γ_1 and γ_2 in (2.2) are the largest numbers such that (2.4) holds true.

Proof. By taking $\lambda = \mu = 0$ in Lemma 1.1, we find that if $p \in P$ satisfies

$$\lambda_0(p(z))^a + zp'(z)(p(z))^b < h(z) \quad (z \in U), \quad (2.5)$$

where

$$h(z) = \left(\frac{1+cZ}{1-z} \right)^{(b+1)(\alpha_1+\alpha_2)/2} \left(\lambda_0 \left(\frac{1+cZ}{1-z} \right)^{(a-b-1)(\alpha_1+\alpha_2)/2} + \frac{\alpha_1 + \alpha_2}{2} \left(\frac{z}{1-z} + \frac{cZ}{1+cZ} \right) \right), \quad (2.6)$$

then (2.4) holds true.

For $z = e^{i\theta}$, $z \neq 1$ and $z \neq -1/c$, we get

$$\begin{aligned} \frac{z}{1-z} &= -\frac{1}{2} + \frac{i}{2} \cot \frac{\theta}{2}, & \frac{cz}{1+cz} &= \frac{1}{2} + \frac{i}{2} \tan \frac{\theta + \beta\pi}{2}, \\ \frac{1+cz}{1-z} &= \frac{1+e^{i(\theta+\beta\pi)}}{1-e^{i\theta}} = \frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)} e^{\alpha_1\pi i/(\alpha_1+\alpha_2)} \neq 0. \end{aligned} \quad (2.7)$$

We consider the following two cases.

(i) If

$$k(\theta) = \cos \frac{\theta + \beta\pi}{2} \sin \frac{\theta}{2} > 0, \quad (2.8)$$

then from (2.7), and (2.6), we have

$$\begin{aligned} h(e^{i\theta}) &= \left(\frac{\cos((\theta + \beta\pi)/2)}{\sin(\theta/2)} \right)^{(b+1)(\alpha_1+\alpha_2)/2} \cdot e^{(b+1)\alpha_1\pi i/2} \\ &\quad \times \left(\lambda_0 \left(\frac{\cos((\theta + \beta\pi)/2)}{\sin(\theta/2)} \right)^{(a-b-1)(\alpha_1+\alpha_2)/2} \cdot e^{(a-b-1)\alpha_1\pi i/2} + i \frac{(\alpha_1 + \alpha_2) \cos(\beta\pi/2)}{4k(\theta)} \right), \end{aligned} \quad (2.9)$$

and so

$$\arg(h(e^{i\theta})) = \frac{1}{2}a\alpha_1\pi + \tan^{-1} \left(\frac{((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \cos((a-b-1)\alpha_1\pi/2)}{2\lambda_0 k_1(\theta) + ((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \sin((a-b-1)\alpha_1\pi/2)} \right), \quad (2.10)$$

where $\lambda_0 > 0$, $0 \leq (a-b-1)(\alpha_1 + \alpha_2) \leq 2$, $e^{i\theta} \neq 1$, $e^{i\theta} \neq -1/c$,

$$k_1(\theta) = k(\theta) \left(\frac{\cos((\theta + \beta\pi)/2)}{\sin(\theta/2)} \right)^{(a-b-1)(\alpha_1+\alpha_2)/2} > 0. \quad (2.11)$$

We now calculate the maximum value of $k_1(\theta)$. It is easy to verify that

$$\lim_{\theta \rightarrow 0} k_1(\theta) = \lim_{e^{i\theta} \rightarrow -1/c} k_1(\theta) = 0 \quad (2.12)$$

and that

$$\begin{aligned}
 k'_1(\theta) &= -\frac{(a-b-1)(\alpha_1+\alpha_2)}{4} \left(\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)} \right)^{(a-b-1)(\alpha_1+\alpha_2)/2-1} \cdot \frac{\cos(\beta\pi/2)}{(\sin(\theta/2))^2} k(\theta) \\
 &\quad + \frac{1}{2} \left(\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)} \right)^{(a-b-1)(\alpha_1+\alpha_2)/2} \cdot \cos\left(\theta + \frac{\beta\pi}{2}\right) \\
 &= \frac{1}{2} \left(\frac{\cos((\theta+\beta\pi)/2)}{\sin(\theta/2)} \right)^{(a-b-1)(\alpha_1+\alpha_2)/2} \cdot \\
 &\quad \times \left(\cos\left(\theta + \frac{\beta\pi}{2}\right) - \frac{(a-b-1)(\alpha_1+\alpha_2)}{2} \cos \frac{\beta\pi}{2} \right).
 \end{aligned} \tag{2.13}$$

Set

$$\theta_1 = \cos^{-1} \left(\frac{(a-b-1)(\alpha_1+\alpha_2)}{2} \cos \frac{\beta\pi}{2} \right) - \frac{\beta\pi}{2}, \tag{2.14}$$

then $k'_1(\theta_1) = 0$. Noting that

$$\begin{aligned}
 0 &\leq (a-b-1)(\alpha_1+\alpha_2) \leq 2, \quad -1 < \beta < 1, \\
 \frac{|\beta|\pi}{2} &< \cos^{-1} \left(\frac{(a-b-1)(\alpha_1+\alpha_2)}{2} \cos \frac{\beta\pi}{2} \right) < \frac{\pi}{2},
 \end{aligned} \tag{2.15}$$

we easily have

$$0 < \theta_1 < \pi, \quad 0 < \theta_1 + \frac{\beta\pi}{2} < \frac{\pi}{2}, \quad 0 < \frac{\theta_1 + \beta\pi}{2} < \frac{\pi}{2}. \tag{2.16}$$

Hence, $k(\theta_1) > 0$, and it follows from (2.11) to (2.16) that

$$\begin{aligned}
 0 &< k_1(\theta) \leq k_1(\theta_1) \\
 &= \left(\sin \frac{\theta_1}{2} \right)^{-(a-b-1)(\alpha_1+\alpha_2)} \cdot \left(\cos \frac{\theta_1 + \beta\pi}{2} \sin \frac{\theta_1}{2} \right)^{1+\mathcal{A}} \\
 &= \left(\frac{1 - \cos \theta_1}{2} \right)^{-\mathcal{A}} \cdot \left(\frac{1}{2} \left(\sin \left(\theta_1 + \frac{\beta\pi}{2} \right) - \sin \frac{\beta\pi}{2} \right) \right)^{1+\mathcal{A}} \\
 &= \frac{\left[\left(1 - ((\mathcal{A}) \cos(\beta\pi/2))^2 \right)^{1/2} - \sin(\beta\pi/2) \right]^{1+\mathcal{A}}}{2 \left[1 - ((\mathcal{A}) \cos(\beta\pi/2))^2 - \left(1 - ((\mathcal{A}) \cos(\beta\pi/2))^2 \right)^{1/2} \sin(\beta\pi/2) \right]^{\mathcal{A}}} \\
 &= \delta_1(a, b, \alpha_1, \alpha_2),
 \end{aligned} \tag{2.17}$$

where $\mathcal{A}$ denotes $(a - b - 1)(\alpha_1 + \alpha_2)/2$. Thus, by using (2.1), (2.10), and (2.17), we arrive at

$$\begin{aligned} \pi &> \arg\left(h\left(e^{i\theta}\right)\right) \geq \arg\left(h\left(e^{i\theta_1}\right)\right) \\ &= \frac{1}{2}a\alpha_1\pi + \tan^{-1}\left(\frac{((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \cos((a - b - 1)\alpha_1\pi/2)}{2\lambda_0\delta_1(a, b, \alpha_1, \alpha_2) + ((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \sin((a - b - 1)\alpha_1\pi/2)}\right) \\ &= \frac{\gamma_1\pi}{2} > 0. \end{aligned} \quad (2.18)$$

(ii) If $k(\theta) < 0$, then we obtain

$$\begin{aligned} h\left(e^{i\theta}\right) &= \left(-\frac{\cos((\theta + \beta\pi)/2)}{\sin(\theta/2)}\right)^{(b+1)(\alpha_1+\alpha_2)/2} \cdot e^{-(b+1)\alpha_2\pi i/2} \\ &\quad \times \left(\lambda_0\left(-\frac{\cos((\theta + \beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} \cdot e^{-(a-b-1)\alpha_2\pi i/2} + i\frac{(\alpha_1 + \alpha_2) \cos(\beta\pi/2)}{4k(\theta)}\right), \end{aligned} \quad (2.19)$$

which leads to

$$\arg\left(h\left(e^{i\theta}\right)\right) = -\frac{1}{2}a\alpha_2\pi - \tan^{-1}\left(\frac{((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \cos((a - b - 1)\alpha_2\pi/2)}{2\lambda_0k_2(\theta) + ((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \cos((a - b - 1)\alpha_2\pi/2)}\right), \quad (2.20)$$

where $\lambda_0 > 0$, $0 \leq (a - b - 1)(\alpha_1 + \alpha_2) \leq 2$, $e^{i\theta} \neq 1$, $e^{i\theta} \neq -1/c$,

$$k_2(\theta) = -k(\theta)\left(-\frac{\cos((\theta + \beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} > 0. \quad (2.21)$$

Now, we have

$$\begin{aligned} \lim_{\theta \rightarrow 0} k_2(\theta) &= \lim_{e^{i\theta} \rightarrow -1/c} k_2(\theta) = 0, \\ k'_2(\theta) &= \frac{1}{2}\left(-\frac{\cos((\theta + \beta\pi)/2)}{\sin(\theta/2)}\right)^{(a-b-1)(\alpha_1+\alpha_2)/2} \left(\frac{(a - b - 1)(\alpha_1 + \alpha_2)}{2} \cos \frac{\beta\pi}{2} - \cos\left(-\theta - \frac{\beta\pi}{2}\right)\right). \end{aligned} \quad (2.22)$$

Let

$$\theta_2 = -\cos^{-1}\left(\frac{(a - b - 1)(\alpha_1 + \alpha_2)}{2} \cos \frac{\beta\pi}{2}\right) - \frac{\beta\pi}{2}, \quad (2.23)$$

then $k'_2(\theta_2) = 0$, $\theta_1 + \theta_2 = -\beta\pi$,

$$-\pi < \theta_2 < 0, \quad -\frac{\pi}{2} < \theta_2 + \frac{\beta\pi}{2} < 0, \quad -\frac{\pi}{2} < \frac{\theta_2 + \beta\pi}{2} < 0. \quad (2.24)$$

Hence, we deduce that $k(\theta_2) < 0$ and

$$\begin{aligned} 0 &< k_2(\theta) \leq k_2(\theta_2) \\ &= \left(-\sin \frac{\theta_2}{2}\right)^{-(a-b-1)(\alpha_1+\alpha_2)} \cdot \left(-\cos \frac{\theta_2 + \beta\pi}{2} \sin \frac{\theta_2}{2}\right)^{1+\mathcal{A}} \\ &= \left(\frac{1 - \cos \theta_2}{2}\right)^{-\mathcal{A}} \cdot \left(\frac{1}{2} \left(\sin \frac{\beta\pi}{2} - \sin\left(\theta_2 + \frac{\beta\pi}{2}\right)\right)\right)^{1+\mathcal{A}} \\ &= \frac{\left[\left(1 - ((\mathcal{A}) \cos(\beta\pi/2))^2\right)^{1/2} + \sin(\beta\pi/2)\right]^{1+\mathcal{A}}}{2 \left[1 - (\mathcal{A})(\cos(\beta\pi/2))^2 + \left(1 - ((\mathcal{A}) \cos(\beta\pi/2))^2\right)^{1/2} \sin(\beta\pi/2)\right]^{\mathcal{A}}} \\ &= \delta_2(a, b, \alpha_1, \alpha_2), \end{aligned} \quad (2.25)$$

where $\mathcal{A} = (a - b - 1)(\alpha_1 + \alpha_2)/2$. Further, by using (2.1), (2.20), and (2.25), we find that

$$\begin{aligned} -\pi &< \arg\left(h\left(e^{i\theta}\right)\right) \leq \arg\left(h\left(e^{i\theta_2}\right)\right) \\ &= -\frac{1}{2}a\alpha_2\pi - \tan^{-1}\left(\frac{((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \cos((a - b - 1)\alpha_2\pi/2)}{2\lambda_0\delta_2(a, b, \alpha_1, \alpha_2) + ((\alpha_1 + \alpha_2)/2) \cos(\beta\pi/2) \cos((a - b - 1)\alpha_2\pi/2)}\right) \\ &= -\frac{\gamma_2\pi}{2} < 0. \end{aligned} \quad (2.26)$$

In view of $h(0) = 1 > 0$, we conclude from (2.18) and (2.26) that $h(U)$ properly contains the angular region $-\gamma_2\pi/2 < \arg w < \gamma_1\pi/2$ in the complex w -plane. Therefore, if $p \in P$ satisfies (2.2), then the subordination relation (2.5) holds true, and thus we arrive at (2.4).

Furthermore, for the function q defined by (1.8), we have

$$\begin{aligned} -\frac{\alpha_2\pi}{2} &< \arg(q(z)) < \frac{\alpha_1\pi}{2} \quad (z \in U), \\ \lambda_0(q(z))^a + zq'(z)(q(z))^b &= h(z). \end{aligned} \quad (2.27)$$

Hence, by using (2.18) and (2.25), we see that the bounds γ_1 and γ_2 in (2.2) are best possible. $\square$

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