Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 401217, 14 pages
doi:10.1155/2012/401217

Research Article

The Optimal Homotopy Asymptotic Method for the Solution of Higher-Order Boundary Value Problems in Finite Domains

Javed Ali,[1] S. Islam,[2] Hamid Khan,[3] and Syed Inayat Ali Shah[1]

[1] *Islamia College Peshawar (Chartered University), Khyber Pakhtunkhawa, Peshawar 25120, Pakistan*
[2] *Department of Mathematics, CIIT, H-8, Islamabad 44000, Pakistan*
[3] *FAST NU, Peshawar 25100, Pakistan*

Correspondence should be addressed to S. Islam, saeed@comsats.edu.pk

Received 28 July 2011; Accepted 12 October 2011

Academic Editor: Muhammad Aslam Noor

We solve some higher-order boundary value problems by the optimal homotopy asymptotic method (OHAM). The proposed method is capable to handle a wide variety of linear and nonlinear problems effectively. The numerical results given by OHAM are compared with the exact solutions and the solutions obtained by Adomian decomposition (ADM), variational iteration (VIM), homotopy perturbation (HPM), and variational iteration decomposition method (VIDM). The results show that the proposed method is more effective and reliable.

1. Introduction

In this paper, we consider a well-posed nth-order problem of the form

$$u^{(n)} = \varphi\left(u, u', \ldots, u^{(n-1)}\right) + \phi(r), \quad a < r < b, \tag{1.1}$$

with boundary conditions: $u^{(k)}(a) = \alpha_i$ and $u^{(k)}(b) = \beta_i$, where $k(< n)$ is a nonnegative integer, α_i and β_i are real finite constants, and $\phi(r)$ is a continuous function on $[a, b]$.

Such types of problems have been investigated by many authors [1, 2] due to their mathematical importance and the potential for applications in hydrodynamic and hydromagnetic stability. Fifth-order boundary value problems arise in the mathematical modeling of viscoelastic flows. Sixth- and eighth-order differential equation govern physics of some hydrodynamic stability problems. When an infinite horizontal layer of fluid is heated

from below and is subject to the action of rotation, instability sets in. When this instability is as ordinary convection, the ordinary differential equation is sixth order; when the instability sets in as overstability, it is modeled by an eighth-order ordinary differential equation. If an infinite horizontal layer of fluid is heated from below, with the supposition that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modeled by tenth-order boundary value problem.

So for the solution of these problems is concerned, many methods appeared in literature. The recent analytic methods are Adomian decomposition method (ADM) [3–5], variational iteration method (VIM) [6], homotopy perturbation method (HPM) [7–9], homotopy analysis method (HAM) [10, 11], differential transform method (DTM) [12], and so forth. Classical perturbation methods are based on the assumptions of small or large parameters, and they cannot produce a general form of an approximate solution. The nonperturbation methods like ADM and DTM can deal strongly with nonlinear problems, but the convergence region of their series solution is generally small. The HPM, which is an elegant combination of homotopy and perturbation technique, overcomes the restrictions of small or large parameters in the problems. It deals with a wide variety of nonlinear problems effectively. Recently, Marinca et al. [13–17] introduced OHAM for approximate solution of nonlinear problems of thin film flow of a fourth-grade fluid down a vertical cylinder. In their work, they have used this method to understand the behavior of nonlinear mechanical vibration of electrical machine. They also used the same method for the solution of nonlinear equations arising in the steady-state flow of a fourth-grade fluid past a porous plate and for the solution of nonlinear equation arising in heat transfer. This method is straight forward, reliable, and explicitly defined. It provides a convenient way to control the convergence of the series solution and allows adjustment of convergence region where it is needed.

Fifth- and sixth-order linear and nonlinear problems were solved by Wazwaz [18, 19], while using decomposition method. Noor et al. [20–25] investigated these type of problems using variational iteration method (VIM), homotopy perturbation method (HPM), and variational iteration decomposition method (VIDM). Modified variational iteration method (MVIM) and iterative method (ITM) were used by Mohyud-Din et al. [26, 27] for such type of problems. Kasi Viswanadham and Murali Krishna [28] used Quintic B-Spline Galerkin method for fifth-order boundary value problems. Siraj-ul-Islam et al. [29, 30] used numerical scheme for the solution of fifth- and sixth-order boundary value problems.

Recently, Ali et al. [31, 32] used OHAM for the solution of multipoint boundary value problems and twelfth-order boundary value problems. We use OHAM to find the approximate analytic solution of some higher-order BVPs. The results of OHAM are compared with those of exact solution, and the errors are compared with the existing results. This paper is organized as follows: Section 2 is devoted to the analysis of the proposed method. Some numerical examples are presented in Section 3. In Section 4, we concluded by discussing results of the numerical simulation using Mathematica.

2. Method Analysis for Two-Point Boundary Value Problems

Consider the differential equation

$$\mathcal{L}u = \mathcal{N}u + \phi, \tag{2.1}$$

along with boundary conditions:

$$\mathcal{B}\left(u, \frac{\partial u}{\partial r}\right) = 0, \tag{2.2}$$

where $\mathcal{L}$ is linear, $\mathcal{N}$ is a nonlinear, and $\mathcal{B}$ is a boundary operator. ϕ is a known function which is continues for $r : r \in \Omega$. According to OHAM, we can construct a homotopy defined by

$$(1-p)\mathcal{L}(u(r;p)) - \hbar(p)\left[\mathcal{L}(u(r;p)) - \mathcal{N}(u(r;p)) - \phi(r)\right] = 0, \tag{2.3}$$

where $p \in [0,1]$ is an embedding parameter, and $\hbar(p)$ is a nonzero auxiliary function for $p \neq 0$ and $h(0) = 0$. Equation (2.3) satisfies

$$\mathcal{L}u = 0, \quad \text{for } p = 0, \qquad \mathcal{L}u = \mathcal{N}u + \phi, \quad \text{for } p = 1. \tag{2.4}$$

The solution, $u(r,0) = v_0(r)$, of $\mathcal{L}u = 0$ traces the solution curve $u(r)$ of (2.1), continuously as p approaches to 1, where v_0 is the solution of the zeroth-order problem, that will come in the next few lines.

The auxiliary function $\hbar(p)$ is chosen in the form (it is a commonly used form)

$$\hbar(p) = \sum_{i=1}^{m} p^i C_i, \tag{2.5}$$

where C_i: $i = 1, 2, \ldots, m$ are the convergence controlling constants which are to be determined. We will use this function unless otherwise stated. The auxiliary function can be chosen in a variety of ways, as reported by Marinca et al. [13–17]. We will use some other forms of $\hbar(p)$ as well.

To get an approximate solution, we expand $u(r,p)$ in Taylor's series about p in the following manner:

$$u(r;p) = v_0(r) + \sum_{m=1}^{\infty} v_m(r, C_1, C_2, \ldots, C_m) p^m. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.3) and equating the coefficient of like powers of p, we obtain the following linear equations which can be integrated directly.

Zeroth-order problem:

$$\mathcal{L}v_0 = 0, \qquad \mathcal{B}\left(v_0, \frac{\partial v_0}{\partial n}\right) = 0. \tag{2.7}$$

First-order problem:

$$\mathcal{L}v_1 = (1 + C_1)\mathcal{L}v_0 - C_1(\mathcal{N}_0 v_0 - \phi), \qquad \mathcal{B}\left(v_1, \frac{\partial v_1}{\partial n}\right) = 0. \tag{2.8}$$

Second-order problem:

$$\mathcal{L}v_2 = (1 + C_1)\mathcal{L}v_1 - C_1\mathcal{N}_1(v_0, v_1) + C_2(\mathcal{L}v_0 - \mathcal{N}_0 v_0 - \phi), \qquad \mathcal{B}\left(v_2, \frac{\partial v_2}{\partial n}\right) = 0. \tag{2.9}$$

Though we can construct higher-order problems easily, solutions upto the second-order problems are enough to produce excellent results.

If the series (2.6) is convergent at $p = 1$, then the approximate solution in our case is,

$$\tilde{u}(r) = v(r) = v_0(r) + v_1(r, C_1) + v_2(r, C_1, C_2). \tag{2.10}$$

By substituting (2.10) into (2.1), the resulting residual is

$$\mathcal{R}(r, C_1, C_2) = \mathcal{L}(\tilde{u}(r)) - \mathcal{N}(\tilde{u}(r)) - \phi(r). \tag{2.11}$$

If $\mathcal{R} = 0$, $\tilde{u}$ will be the exact solution. Otherwise, we minimize $\mathcal{R}$ over domain of the problem. To find the optimal values of C_i which minimizes $\mathcal{R}$, many methods can be applied [13–17]. We follow two methods: the method of least squares and the Galerekin's method. According to the method of least squares, we first construct the functional

$$\mathcal{J}(C_1, C_2) = \int_a^b \mathcal{R}^2 dr, \tag{2.12}$$

and then minimizing it, we have

$$\frac{\partial \mathcal{J}}{\partial C_1} = \frac{\partial \mathcal{J}}{\partial C_2} = 0. \tag{2.13}$$

According to the Galerekin's method, we solve the following system for C_1 and C_2:

$$\int_a^b \mathcal{R}\frac{\partial \tilde{u}}{\partial C_1} dr = 0, \qquad \int_a^b \mathcal{R}\frac{\partial \tilde{u}}{\partial C_2} dr = 0. \tag{2.14}$$

Knowing C_1 and C_2, the approximate solution is well determined.

3. Some Numerical Examples

Example 3.1 (fifth-order linear). Consider the following problem:

$$y^{(v)}(x) = y - 15e^x - 10xe^x, \quad 0 < x < 1, \tag{3.1}$$

with boundary conditions

$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0, \qquad y(1) = 0, \qquad y'(1) = -e. \tag{3.2}$$

The exact solution of this problem is $y(x) = x(1 - x)e^x$.

We choose the auxiliary function as $h(p) = p(C_1 + C_2 x)$. Plugging in this value in (2.3) of Section 2, we obtain the following linear problems which can be integrated directly.

Zeroth-order problem:

$$\begin{gathered} y_0^{(5)}(x) = 0, \\ y_0(0) = 0, \qquad y_0'(0) = 0, \qquad y_0''(0) = 0, \qquad y_0(1) = 0, \qquad y_0'(1) = -e. \end{gathered} \tag{3.3}$$

First-order problem:

$$\begin{gathered} y_1^{(5)}(x, C_1, C_2) = 5e^x(3 + 2x)(C_1 + C_2 x) - (C_1 + C_2 x) y_0(x), \\ y_1(0) = 0, \qquad y_1'(0) = 0, \qquad y_1''(0) = 0, \qquad y_1(1) = 0, \qquad y_1'(1) = 0. \end{gathered} \tag{3.4}$$

Second-order problem:

$$\begin{gathered} y_2^{(5)}(x, C_1, C_2) = (1 + C_1 + C_2 x) y_1^{(5)}(x, C_1, C_2) - (C_1 + C_2 x) y_1(x, C_1, C_2), \\ y_2(0) = 0, \qquad y_2'(0) = 0, \qquad y_2''(0) = 0, \qquad y_2(1) = 0, \qquad y_2'(1) = 0. \end{gathered} \tag{3.5}$$

Adding up the solutions of these problems, the second-order approximate solution,

$$\tilde{y}(x) = y_0(x) + y_1(x, C_1, C_2) + y_2(x, C_1, C_2) + O\left(x^{15}\right), \tag{3.6}$$

is determined by knowing the optimal values of the auxiliary constants, C_1 and C_2. Using Galerkin's method, we obtain $C_1 = -1.000245451$, $C_2 = 0.000124615$.

By considering these values, (3.6) becomes

$$\begin{aligned} \tilde{y}(x) = {} & x - 0.5x^3 - 0.333333x^4 - 0.125x^5 - 0.0333333x^6 - 0.00694444x^7 \\ & - 0.00119049x^8 - 0.000173601x^9 - 0.0000220495x^{10} - 2.48013 \times 10^{-6} x^{11} \\ & - 2.50501 \times 10^{-7} x^{12} - 2.27501 \times 10^{-8} x^{13} - 2.0326 \times 10^{-9} x^{14} + O\left(x^{15}\right). \end{aligned} \tag{3.7}$$

Numerical results of the solution (3.7) are displayed in Table 1.

Example 3.2 (another fifth-order linear). Consider the following problem:

$$y^{(5)}(x) + x y(x) = 19x \operatorname{Cos}(x) - 2x^3 \operatorname{Cos}(x) - 41 \operatorname{Sin}(x) + 2x^2 \operatorname{Sin}(x), \tag{3.8}$$

Table 1: It shows comparison of the solutions obtained by OHAM (3.7), ADM [18], HPM [20], VIM [22], ITM [24], and VIHPM [21]. From the numerical results, it is clear that OHAM is more efficient and accurate.

x	Exact sol.	OHAM sol.	E^* (3.7)	E^* (ADM)	E^* (HPM)	E^* (VIM)	E^* (ITM)	E^* (VIHPM)
0.0	0.00000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.099465383	0.099465383	$-1E-12$	$-3E-11$	$-3E-11$	$-3E-11$	$-3E-11$	$-3E-11$
0.2	0.195424441	0.195424441	$-7E-12$	$-2E-10$	$-2E-10$	$-2E-10$	$-2E-10$	$-2E-10$
0.3	0.28347035	0.28347035	$-1E-11$	$-4E-10$	$-4E-10$	$-4E-10$	$-4E-10$	$-4E-10$
0.4	0.358037927	0.358037927	$-1E-11$	$-8E-10$	$-8E-10$	$-8E-10$	$-8E-10$	$-8E-10$
0.5	0.412180318	0.412180318	$-8E-12$	$-1E-9$	$-1E-9$	$-1E-9$	$-1E-9$	$-1E-9$
0.6	0.437308512	0.437308512	$3E-14$	$-2E-9$	$-2E-9$	$-2E-9$	$-2E-9$	$-2E-9$
0.7	0.422888068	0.422888068	$6E-12$	$-2E-9$	$-2E-9$	$-2E-9$	$-2E-9$	$-2E-9$
0.8	0.356086548	0.356086548	$2E-12$	$-2E-9$	$-2E-9$	$-2E-9$	$-2E-9$	$-2E-9$
0.9	0.221364280	0.221364280	$-3E-11$	$-1E-9$	$-1E-9$	$-1E-9$	$-1E-9$	$-1E-9$
1.0	0.0000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

E^* = exact − approximate.

with boundary conditions

$$\begin{aligned} y(-1) &= y(1) = \operatorname{Cos}(1), \\ y'(-1) &= y'(1) = -4\operatorname{Cos}(1) + \operatorname{Sin}(1), \\ y''(-1) &= 3\operatorname{Cos}(1) - 8\operatorname{Sin}(1). \end{aligned} \tag{3.9}$$

Exact solution of this problem is $y(x) = (2x^2 - 1)\operatorname{Cos}(x)$.

Considering the second-order solution $\tilde{y}(x) = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2) + O(x^{13})$, we use the method of least squares to obtain $C_1 = 0.9940605306$, $C_2 = -3.9762851376$.

Having these values, our solution in this case is

$$\begin{aligned} \tilde{y}(x) = {} & -0.999978 + 2.49992x^2 - 1.04155x^4 + 0.0846365x^6 - 0.00276866x^8 \\ & + 0.0000420239x^{10} + 3.38286 \times 10^{-7}x^{12} + O\left(x^{13}\right). \end{aligned} \tag{3.10}$$

Numerical results of the solution (3.10) are displayed in Table 2.

Example 3.3 ([33] fifth-order nonlinear). Consider the following problem:

$$y^{(v)}(x) = y^3(x)e^{-x}, \quad 0 < x < 1, \tag{3.11}$$

with boundary conditions

$$y(0) = 1, \qquad y'(0) = 1/2, \qquad y''(0) = 1/4, \qquad y(1) = e^{1/2}, \qquad y'(1) = \frac{1}{2e^{1/2}}. \tag{3.12}$$

The exact solution for this problem is $y(x) = e^{x/2}$.

Table 2: The maximum absolute error as reported in [28] is 1.8775×10^{-5}, while in our case, it is 8.5757×10^{-8}.

x	Exact sol.	OHAM sol. (3.10)	E^* (3.10)
−1.0	0.540302306	0.540302301	$3.90E-9$
−0.8	0.195077879	0.195077879	$8.44E-9$
−0.6	−0.231093972	−0.231094010	$3.74E-8$
−0.4	−0.626321476	−0.626321543	$6.70E-8$
−0.2	−0.901661252	−0.901661334	$8.22E-8$
0.0	−1.000000000	−1.000000086	$8.58E-8$
0.2	−0.901661252	−0.901661334	$8.22E-8$
0.4	−0.626321476	−0.626321543	$6.70E-8$
0.6	−0.231093972	−0.231094010	$3.74E-8$
0.8	0.195077879	0.195077879	$8.44E-9$
1.0	0.540302306	0.540302301	$3.90E-9$

E^* = exact − approximate.

We consider the second-order solution, $\tilde{y}(x) = y_0(x)+y_1(x,C_1)+y_2(x,C_1,C_2)+O(x^{15})$. Using Galerkin's procedure in Section 2, we obtain the following values:

$$C_1 = 0.010868466, \qquad C_2 = -0.029423113. \tag{3.13}$$

The second-order approximate solution is

$$\begin{aligned}\tilde{y}(x) = 1 &+ \frac{x}{2} + \frac{x^2}{8} + 0.0205993x^3 + 0.0030533x^4 + 0.0000630671x^5 + 5.2556\times10^{-6}x^6 \\ &+ 3.754\times10^{-7}x^7 + 2.29401\times10^{-8}x^8 + 1.74892\times10^{-9}x^9 - 3.2692\times10^{-11}x^{10} \\ &- 1.48596\times10^{-12}x^{11} - 6.20243\times10^{-14}x^{12} - 2.58495\times10^{-15}x^{13} \\ &+ 2.75287\times10^{-17}x^{14} + O\left(x^{15}\right).\end{aligned} \tag{3.14}$$

Numerical results of the solution (3.14) are displayed in Table 3.

Example 3.4 (sixth-order nonlinear). Consider the following problem:

$$y^{(vi)}(x) = e^x y^2(x), \quad 0 < x < 1, \tag{3.15}$$

with boundary conditions

$$y(0) = 1, \qquad y'(0) = 1, \qquad y''(0) = 1, \qquad y(1) = e^{-1}, \qquad y'(1) = -e^{-1}, \qquad y''(1) = e^{-1}. \tag{3.16}$$

The exact solution is $y(x) = e^{-x}$.

Table 3

x	Exact solution	OHAM sol. (3.14)	E^* (3.14)	E^* (DTM [33])
0.0	0.00000	0.0000	0.0000	0.0000
0.1	1.105170918	1.105170919	$-9.2E-10$	$1.0E-9$
0.2	1.221402758	1.221402763	$-5.0E-9$	$2.0E-9$
0.3	1.349858808	1.349858818	$-1.1E-8$	$1.0E-8$
0.4	1.491824698	1.491824713	$-1.5E-8$	$2.0E-8$
0.5	1.648721271	1.648721287	$-1.6E-8$	$3.1E-8$
0.6	1.822118800	1.822118814	$-1.4E-8$	$3.7E-8$
0.7	2.013752707	2.013752717	$-9.9E-9$	$4.1E-8$
0.8	2.225540928	2.225540934	$-5.6E-9$	$3.1E-8$
0.9	2.459603111	2.459603112	$-1.1E-9$	$1.4E-8$
1.0	2.718281828	2.718281824	0.000	0.000

E^* = exact − approximate.

For this problem, we take the auxiliary function $h(p) = p(C_1 + C_2e^{-x})$,

$$\tilde{y}(x) = y_0(x) + y_1(x, C_1, C_2) + y_2(x, C_1, C_2) + O\left(x^{13}\right). \tag{3.17}$$

Using Galerkin's method, we obtain $C_1 = 0.41243798998$, $C_2 = 0.0014069149$.

OHAM solution in this case is

$$\begin{aligned}\tilde{y}(x) = {} & 1 - 0.999999994x + \frac{x^2}{2} - 0.166666775x^3 + \frac{x^4}{24} - 0.008332465x^5 \\ & + 0.001387441x^6 - 0.000197784x^7 + 0.000025071x^8 - 3.002 \times 10^{-6}x^9 \\ & + 2.918 \times 10^{-7}x^{10} - 6.7 \times 10^{-9}x^{11} - 2.257 \times 10^{-9}x^{12} + 1.806 \times 10^{-10}x^{13} \\ & - 1.974 \times 10^{-11}x^{14} + O\left(x^{15}\right).\end{aligned} \tag{3.18}$$

Numerical results of the solution (3.18) are displayed in Table 4.

Example 3.5 (eighth-order nonlinear). Consider the following problem:

$$y^{(viii)}(x) = e^{-x}y^2(x), \quad 0 < x < 1, \tag{3.19}$$

with boundary conditions

$$\begin{gathered} y'(0) = 1, \qquad y'(0) = 1, \qquad y''(0) = 1, \qquad y'''(0) = 1, \\ y(1) = e, \qquad y'(1) = e, \qquad y''(1) = e. \end{gathered} \tag{3.20}$$

Table 4: It shows comparison of the OHAM solution (3.18) with the exact solution and the errors obtained by decomposition method (ADM) [19], homotopy perturbation method (HPM) [24], and the variational iteration method [22]. It is clear from the results that the method we applied is more efficient and accurate than the other methods.

x	Exact solution	OHAM sol (3.18)	E^* (3.18)	E^* (ADM)	E^* (HPM)	E^* (VIM)
0.0	1	1	0	0	0	0
0.1	0.904837418	0.904837418	$-4.82E-10$	$-1.2E-4$	$-1.2E-4$	$-1.2E-4$
0.2	0.818730753	0.818730753	$-4.92E-10$	$-2.3E-4$	$-2.3E-4$	$-2.3E-4$
0.3	0.740818221	0.740818221	$-2.37E-11$	$-3.2E-4$	$-3.2E-4$	$-3.2E-4$
0.4	0.670320046	0.670320046	$5.11E-10$	$-3.8E-4$	$-3.8E-4$	$-3.8E-4$
0.5	0.60653066	0.60653066	$6.42E-10$	$-4.0E-4$	$-4.0E-4$	$-4.0E-4$
0.6	0.548811636	0.548811636	$2.02E-10$	$-3.9E-4$	$-3.9E-4$	$-3.9E-4$
0.7	0.496585304	0.496585304	$-5.37E-10$	$-3.3E-4$	$-3.3E-4$	$-3.3E-4$
0.8	0.449328964	0.449328965	$-1.02E-09$	$-2.4E-4$	$-2.4E-4$	$-2.4E-4$
0.9	0.40656966	0.40656966	$-8.23E-10$	$-1.2E-4$	$-1.2E-4$	$-1.2E-4$
1.0	0.367879441	0.367879441	$-2.05E-12$	$2.0E-9$	$2.0E-9$	$2.0E-9$

E^* = exact − approximate.

Considering the second-order solution $\tilde{y}(x) = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2) + O(x^{13})$, the following values of the convergence controlling constants are obtained by using Galerkin's method:

$$C_1 = -1.451894673 \times 10^{-11}, \qquad C_2 = -0.000647581. \tag{3.21}$$

The approximate solution in this case is

$$\begin{aligned}\tilde{y}(x) = {} & 1 + x + 0.5x^2 + 0.166667x^3 + 0.0416275x^4 + 0.00884857x^5 + 0.00117027x^6 \\ & + 0.00033161x^7 + 1.6061 \times 10^{-8}x^8 + 1.78456 \times 10^{-9}x^9 + 1.78456 \times 10^{-10}x^{10} \\ & + 1.62233 \times 10^{-11}x^{11} + 1.3494 \times 10^{-12}x^{12} + O\left(x^{13}\right).\end{aligned} \tag{3.22}$$

If the method of least squares is used to determine C's, we have then

$$C_1 = 1.793 \times 10^{-8}, \qquad C_2 = -1.001347284. \tag{3.23}$$

The approximate solution in this case is

$$\begin{aligned}\tilde{y}(x) = {} & 1 + x + 0.5x^2 + 0.166667x^3 + 0.0416667x^4 + 0.00833313x^5 + 0.00138918x^6 \\ & + 0.000198233x^7 + 0.000024835x^8 + 2.75944 \times 10^{-6}x^9 + 2.75944 \times 10^{-7}x^{10} \\ & + 2.50859 \times 10^{-8}x^{11} + 2.08656 \times 10^{-9}x^{12} + O\left(x^{13}\right).\end{aligned} \tag{3.24}$$

Table 5

x	Exact	OHAM sol.	E^* (3.22)	E^* (3.24)	E^* (3.26)	E^* ([25])
0.0	1	1	0	0	0	0
0.1	1.105170918	1.105170915	$2.60E-9$	$-3.5E-12$	$1.39E-16$	$1.27E-5$
0.2	1.221402758	1.221402732	$2.62E-8$	$-3.5E-11$	$5.32E-16$	$2.43E-5$
0.3	1.349858808	1.349858729	$7.86E-8$	$-1.1E-10$	$1.48E-15$	$3.35E-5$
0.4	1.491824698	1.491824562	$1.36E-7$	$-1.8E-10$	$3.36E-15$	$3.94E-5$
0.5	1.648721271	1.648721109	$1.61E-7$	$-2.2E-10$	$2.28E-14$	$4.16E-5$
0.6	1.822118800	1.822118662	$1.39E-7$	$-1.9E-10$	$2.21E-13$	$3.96E-5$
0.7	2.013752707	2.013752625	$8.22E-8$	$-1.1E-10$	$1.64E-12$	$3.38E-5$
0.8	2.225540928	2.225540900	$2.80E-8$	$-2.8E-11$	$9.36E-12$	$2.45E-5$
0.9	2.459603111	2.459603108	$2.84E-9$	$4.1E-11$	$4.36E-11$	$1.29E-5$
1.0	2.718281828	2.718281828	$1.14E-13$	$1.8E-10$	$1.73E-10$	$1.00E-9$

E^* = exact − approximate.

Let us use the auxiliary function $h(p) = p(C_1 + C_2e^{-x})$ and consider the second-order solution

$$\tilde{y}(x) = y_0(x) + y_1(x, C_1, C_2) + y_2(x, C_1, C_2) + O\left(x^{13}\right). \tag{3.25}$$

Using Galerkin's method, we obtain $C_1 = -0.9993171458$, $C_2 = -0.0012314995$.

The OHAM solution in this case is

$$\begin{aligned}\tilde{y}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} &+ 0.041666667x^4 + 0.008333333x^5 + 0.001388889x^6 \\ &+ 1.984 \times 10^{-4}x^7 + 2.480 \times 10^{-5}x^8 + 2.756 \times 10^{-6}x^9 + 2.756 \times 10^{-7}x^{10} \\ &+ 2.505 \times 10^{-8}x^{11} + 2.088 \times 10^{-9}x^{12} + O\left(x^{13}\right).\end{aligned} \tag{3.26}$$

Numerical results of the solutions (3.22), (3.24), and (3.26) are displayed in Table 5.

Example 3.6 (nineth-order linear). Consider the following problem:

$$y^{(9)}(x) = y(x) - 9e^x, \tag{3.27}$$

with boundary conditions

$$\begin{gathered} y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = -2, \quad y''''(0) = -3, \\ y(1) = 0, \quad y'(1) = -e, \quad y''(1) = -2e, \quad y'''(1) = -3e. \end{gathered} \tag{3.28}$$

Exact solution is $y(x) = (1 - x)e^x$.

For this linear problem, we take $h(p) = p(C_1 + C_2x)$, and according to the rest of the procedure of OHAM, the second-order solution, $\tilde{y}(x) = y_0(x) + y_1(x, C_1, C_2) + y_2(x, C_1, C_2) + O(x^{13})$, is determined by the values of C_i: $i = 1, 2$. Following the Galerkin's method, we obtain $C_1 = -1$, $C_2 = 0$, for $a = 0$ and $b = 1$.

Table 6

x	Exact	OHAM sol.	E^* (3.29)	E^* ([27])
0.0	1	1	0.0000	0.0000
0.1	0.994653826	0.994653826	$1.03E-16$	$-2.0E-10$
0.2	0.977122206	0.977122206	$1.33E-16$	$-2.0E-10$
0.3	0.944901165	0.944901165	$-2.12E-16$	$-2.0E-10$
0.4	0.895094818	0.895094818	$-1.30E-14$	$-2.0E-10$
0.5	0.824360635	0.824360635	$-2.44E-13$	$-2.0E-10$
0.6	0.728847520	0.728847520	$-2.64E-12$	$-6.0E-10$
0.7	0.604125812	0.604125812	$-1.97E-11$	$-1.0E-09$
0.8	0.445108186	0.445108186	$-1.13E-10$	$-2.0E-09$
0.9	0.245960311	0.245960311	$-5.26E-10$	$-3.4E-09$
1.0	0.0000	$2.09E-09$	$-2.09E-09$	0.000

E^* = exact − approximate.

The second-order approximate solution is

$$\begin{aligned} y(x) = 1 &- \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - 0.03333333333x^5 - 0.0069444444x^6 - 0.0011904762x^7 \\ &- 0.0001736111x^8 - 0.00002204586x^9 - 2.48016\times 10^{-6}x^{10} - 2.50521\times 10^{-7}x^{11} \\ &- 2.29644\times 10^{-8}x^{12} + O\left(x^{13}\right). \end{aligned} \tag{3.29}$$

Numerical results of the solution (3.29) are displayed in Table 6.

Example 3.7 (tenth-order nonlinear). Consider the following problem:

$$\begin{gathered} y^{(X)}(x) = e^{-x}y^2(x), \quad 0 < x < 1, \\ y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 1, \quad y'''(0) = 1, \quad y^{(iv)}(0) = 1, \\ y(1) = e, \quad y'(1) = e, \quad y''(1) = e, \quad y'''(1) = e, \quad y^{(iv)}(1) = e. \end{gathered} \tag{3.30}$$

We consider the second-order solution $\tilde{y}(x) = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2) + O(x^{13})$.

To find the values of C_i, we apply the Galarkin's method. So solving the system

$$\int_a^b R\frac{\partial \tilde{v}}{\partial C_1}dr = 0, \qquad \int_a^b R\frac{\partial \tilde{v}}{\partial C_2}dr = 0, \tag{3.31}$$

we have $C_1 = 0$, $C_2 = -1.023966086$.

Table 7

x	Exact	OHAM sol.	E^* (3.32)	E^* ([27])
0.0	1	1	0	0
0.1	1.105170918	1.105170918	$8.12E-17$	$-1.41E-6$
0.2	1.221402758	1.221402758	$2.18E-18$	$-2.69E-6$
0.3	1.349858808	1.349858808	$-2.84E-18$	$-3.70E-6$
0.4	1.491824698	1.491824698	$7.95E-16$	$-4.35E-6$
0.5	1.648721271	1.648721271	$1.91E-14$	$-4.58E-6$
0.6	1.822118800	1.822118800	$2.15E-13$	$-4.36E-6$
0.7	2.013752707	2.013752707	$1.63E-12$	$-3.71E-6$
0.8	2.225540928	2.225540928	$9.33E-12$	$-2.69E-6$
0.9	2.459603111	2.459603111	$4.36E-11$	$-1.42E-6$
1.0	2.718281828	2.718281828	$1.73E-10$	$2.00E-9$

E^* = exact − approximate.

In this case, the approximate solution is

$$\begin{aligned}\tilde{y}(x) = {} & 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + 0.008333323x^5 + 0.00138894x^6 + 0.000198312x^7 \\ & + 0.000024898x^8 + 2.712 \times 10^{-6}x^9 + 2.8218 \times 10^{-7}x^{10} + 2.5652 \times 10^{-8}x^{11} \\ & + 2.1377 \times 10^{-9}x^{12} + O\left(x^{13}\right).\end{aligned} \tag{3.32}$$

Numerical results of the solution (3.32) are displayed in Table 7.

4. Conclusions

In this paper, we have used OHAM to find the approximate analytic solution to higher-order two-point boundary value problems in finite domain. It is observed that the method is explicit, effective, and reliable. It works well for higher-order problems and represents the fastest convergence as well as a remarkable low error. The OHAM also provides us with a very simple way to control and adjust the convergence of the series solution using the auxiliary constants C_i's which are optimally determined. Furthermore, by using different forms of the auxiliary function, more accuracy can be obtained. It has been also observed that for determining the optimal values of C's, the performance of both the least squares and the Galerkin's method is problem dependent. One can select one of these two which best suits the problem solution.

References

[1] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, The International Series of Monographs on Physics, Clarendon Press, Oxford, UK, 1961.

[2] R. P. Agarwal, *Boundary Value Problems for Higher Order Differential Equations*, World Scientific, Teaneck, NJ, USA, 1986.

[3] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, vol. 60 of *Fundamental Theories of Physics*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.
[4] G. Adomian, "A review of the decomposition method and some recent results for nonlinear equations," *Computers & Mathematics with Applications*, vol. 21, no. 5, pp. 101–127, 1991.
[5] A.-M. Wazwaz, "Approximate solutions to boundary value problems of higher order by the modified decomposition method," *Computers & Mathematics with Applications*, vol. 40, no. 6-7, pp. 679–691, 2000.
[6] J.-H. He, "Variational approach to the sixth-order boundary value problems," *Applied Mathematics and Computation*, vol. 143, no. 2-3, pp. 537–538, 2003.
[7] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.
[8] J.-H. He, "Homotopy perturbation method for solving boundary value problems," *Physics Letters. A*, vol. 350, no. 1-2, pp. 87–88, 2006.
[9] H. E. Ji-Huan, "A Note on the homotopy perturbation method," *Thermal Science*, vol. 14, no. 2, pp. 565–568, 2010.
[10] S. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, vol. 2 of *CRC Series: Modern Mechanics and Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.
[11] S. Liao, "Notes on the homotopy analysis method: some definitions and theorems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 983–997, 2009.
[12] J. K. Zhou, *Deferential Transformation and its Application for Electrical Circuits*, Huazhong University Press, Wuhan, China, 1986.
[13] V. Marinca, N. Herişanu, and I. Nemeş, "Optimal homotopy asymptotic method with application to thin film flow," *Central European Journal of Physics*, vol. 6, no. 3, pp. 648–653, 2008.
[14] N. Herisanu, V. Marinca, T. Dordea, and G. Madescu, "A new analytical approach to nonlinear vibration of an electric machine," *Proceedings of the Romanian Academy*, vol. 9, no. 3, 2008.
[15] V. Marinca, N. Herişanu, C. Bota, and B. Marinca, "An optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 245–251, 2009.
[16] V. Marinca and N. Herişanu, "Application of Optimal Homotopy Asymptotic Method for solving nonlinear equations arising in heat transfer," *International Communications in Heat and Mass Transfer*, vol. 35, no. 6, pp. 710–715, 2008.
[17] V. Marinca and N. Herişanu, "Optimal homotopy perturbation method for strongly nonlinear differential equations," *Nonlinear Science Letters A*, vol. 1, pp. 273–280, 2010.
[18] A.-M. Wazwaz, "The numerical solution of fifth-order boundary value problems by the decomposition method," *Journal of Computational and Applied Mathematics*, vol. 136, no. 1-2, pp. 259–270, 2001.
[19] A.-M. Wazwaz, "The numerical solution of sixth-order boundary value problems by the modified decomposition method," *Applied Mathematics and Computation*, vol. 118, no. 2-3, pp. 311–325, 2001.
[20] M. Aslam Noor and S. T. Mohyud-Din, "An efficient algorithm for solving fifth-order boundary value problems," *Mathematical and Computer Modelling*, vol. 45, no. 7-8, pp. 954–964, 2007.
[21] M. A. Noor and S. T. Mohyud-Din, "Variational iteration method for fifth-order boundary value problems using He's polynomials," *Mathematical Problems in Engineering*, vol. 2008, Article ID 954794, 12 pages, 2008.
[22] M. Aslam Noor and S. T. Mohyud-Din, "Variational iteration technique for solving higher order boundary value problems," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1929–1942, 2007.
[23] M. A. Noor and S. T. Mohyud-Din, "A reliable approach for solving linear and nonlinear sixth-order boundary value problems," *International Journal of Computational and Applied Mathematics*, vol. 2, no. 2, pp. 163–172, 2007.
[24] M. A. Noor and S. T. Mohyud-Din, "Homotopy perturbation method for solving sixth-order boundary value problems," *Computers & Mathematics with Applications*, vol. 55, no. 12, pp. 2953–2972, 2008.
[25] M. A. Noor, S. T. Mohyud-Din, and M. Tahir, "Variational iteration decomposition method for solving eighth-order boundary value problems," *Differential Equations and Nonlinear Mechanics*, vol. 2007, Article ID 19529, 16 pages, 2007.
[26] S. T. Mohyud-Din, A. Yildirim, and M. M. Hosseini, "An iterative algorithm for fifth-order boundary value problems," *World Applied Sciences Journal*, vol. 8, no. 5, pp. 531–535, 2010.
[27] S. T. Mohyud-Din and A. Yildirim, "Solutions of tenth and ninth-order boundary value problems by modified variational iteration method," *Applications and Applied Mathematics*, vol. 5, no. 1, pp. 11–25, 2010.

[28] K. N. S. Kasi Viswanadham and P. Murali Krishna, "Quintic B-Spline Galerkin method for fifth order boundary value problems," *ARPN Journal of Engineering and Applied Sciences*, vol. 5, no. 2, 2010.

[29] Siraj-ul-Islam and M. Azam Khan, "A numerical method based on polynomial sextic spline functions for the solution of special fifth-order boundary-value problems," *Applied Mathematics and Computation*, vol. 181, no. 1, pp. 356–361, 2006.

[30] Siraj-ul-Islam, I. A. Tirmizi, Fazal-i-Haq, and M. A. Khan, "Non-polynomial splines approach to the solution of sixth-order boundary-value problems," *Applied Mathematics and Computation*, vol. 195, no. 1, pp. 270–284, 2008.

[31] J. Ali, Siraj-ul-Islam, S. Islam, and G. Zaman, "The solution of multipoint boundary value problems by the Optimal Homotopy Asymptotic Method," *Computers and Mathematics with Applications*, vol. 59, no. 6, pp. 2000–2006, 2010.

[32] J. Ali, S. Islam, M. Tariq Rahim, and G. Zaman, "The solution of special twelfth order boundary value problems by the optimal homotopy asymptotic method," *World Applied Sciences Journal*, vol. 11, no. 3, pp. 371–378, 2010.

[33] C. H. Che Hussin and A. Kiliçman, "On the solutions of nonlinear higher-order boundary value problems by using differential transformation method and Adomian decomposition method," *Mathematical Problems in Engineering*, vol. 2011, Article ID 724927, 19 pages, 2011.