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Research Article

On the Difference Equation $x_n = a_n x_{n-k}/(b_n + c_n x_{n-1} \cdots x_{n-k})$

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The behavior of well-defined solutions of the difference equation $x_n = a_n x_{n-k}/(b_n + c_n x_{n-1} \cdots x_{n-k})$, $n \in \mathbb{N}_0$, where $k \in \mathbb{N}$ is fixed, the sequences a_n, b_n and c_n are real, $(b_n, c_n) \neq (0,0)$, $n \in \mathbb{N}_0$, and the initial values $x_{-k}, \ldots, x_{-1}$ are real numbers, is described.

1. Introduction

Recently there has been a huge interest in studying nonlinear difference equations and systems (see, e.g., [1–33] and the references therein). Here we study the difference equation

$$x_n = \frac{x_{n-k}}{b_n + c_n x_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $k \in \mathbb{N}$ is fixed, the sequences b_n and c_n are real, $(b_n, c_n) \neq (0,0)$, $n \in \mathbb{N}_0$, and the initial values $x_{-k}, \ldots, x_{-1}$ are real numbers. Equation (1.1) is a particular case of the equation

$$x_n = \frac{\widehat{a}_n x_{n-k}}{\widehat{b}_n + \widehat{c}_n x_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

with real sequences $\widehat{a}_n, \widehat{b}_n$ and $\widehat{c}_n$. For $\widehat{a}_n = 0, n \in \mathbb{N}_0$, the equation is trivial, and, for $\widehat{a}_n \neq 0,\ n \in \mathbb{N}_0$, it is reduced to equation (1.1) with $b_n = \widehat{b}_n/\widehat{a}_n$ and $c_n = \widehat{c}_n/\widehat{a}_n$.

Equation

$$x_n = \frac{x_{n-k}}{b + cx_{n-1}\cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $b, c \in \mathbb{R}$, which was treated in [32], is a particular case of equation (1.1).

As in [32], here, we employ our idea of using a change of variables in equation (1.1) which extends the one in our paper [21] and is later also used, for example, in [4]. For similar methods see also [22, 25]. Equation (1.3) in the case $k = 2$ was also studied in [1, 2], in a different way. The case when the sequences b_n and c_n are two-periodic was studied in [31] (some related results are also announced in talk [3]). For related symmetric systems of difference equations, see [27, 29]. For some other recent results on difference equations and systems which can be solved, see, for example, [6, 7, 20–22, 30, 31, 33]. Some classical results can be found, for example, in [11].

Equation (1.1) is a particular case of the equation

$$y_n = f(y_{n-1}, \ldots, y_{n-k}, n)y_{n-k}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where $f : \mathbb{R}^{k+1} \to \mathbb{R}$ is a continuous function. Numerous particular cases of (1.4) have been investigated, for example, in [9, 21, 23]. In this paper we adopt the customary notation $\prod_{i=k+1}^{k} g_i = 1$ and $\sum_{i=k+1}^{k} g_i = 0$.

2. Case $c_n = 0,\ n \in \mathbb{N}_0$

Here we consider the case $c_n = 0,\ n \in \mathbb{N}_0$. In this case equation (1.1) becomes

$$x_n = \frac{x_{n-k}}{b_n}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

$b_n \neq 0, n \in \mathbb{N}_0$, from which it follows that for each $i \in \{1, \ldots, k\}$

$$x_{km-i} = \frac{x_{-i}}{\prod_{j=1}^{m} b_{kj-i}}, \quad m \in \mathbb{N}_0. \tag{2.2}$$

Using formula (2.2) the following theorem can be easily proved.

Theorem 2.1. *Consider equation* (1.1) *with* $c_n = 0$, $b_n \neq 0, n \in \mathbb{N}_0$. *Then the following statements are true:*

(a) *if*

$$\liminf_{m\to\infty} |b_{km-i}| = p_i > 1, \tag{2.3}$$

for some $i \in \{1,\dots,k\}$, *then* $x_{km-i} \to 0$ *as* $m \to \infty$;

(b) *if, for each* $i \in \{1,\dots,k\}$, *the limits* p_i *in* (2.3) *are greater than* 1, *then* $x_n \to 0$ *as* $n \to \infty$;

(c) *if* $b_{km-i} = 1$, *for every* $m \in \mathbb{N}$ *and for some* $i \in \{1,\dots,k\}$, *then* $x_{km-i} = x_{-i}$, $m \in \mathbb{N}_0$;

(d) *if* $b_{km-i} = -1$, *for every* $m \in \mathbb{N}$ *and for some* $i \in \{1,\dots,k\}$, *then* $x_{km-i} = (-1)^m x_{-i}$, $m \in \mathbb{N}_0$;

(e) *if*

$$\limsup_{m\to\infty} |b_{km-i}| = q_i \in [0,1), \tag{2.4}$$

and $x_{-i} \neq 0$, *for some* $i \in \{1,\dots,k\}$, *then* $|x_{km-i}| \to \infty$, *as* $m \to \infty$;

(f) *if, for each* $i \in \{1,\dots,k\}$, *the limits* q_i *in* (2.4) *belong to the interval* $[0,1)$ *and* $x_{-i} \neq 0$, *then* $|x_n| \to \infty$ *as* $n \to \infty$.

3. Case $b_n = 0$, $n \in \mathbb{N}_0$

In this section we consider the case $b_n = 0$, $n \in \mathbb{N}_0$. Note that in this case equation (1.1) becomes

$$x_n = \frac{x_{n-k}}{c_n x_{n-1} \cdots x_{n-k+1} x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

where $c_n \neq 0$, $n \in \mathbb{N}_0$. If x_n is a well-defined solution of equation (3.1) (i.e., a solution with initial values $x_{-i} \neq 0$, $i = 1,\dots,k$, which implies $x_n \neq 0$, $n \in \mathbb{N}_0$), then

$$x_n = \frac{1}{c_n x_{n-1}(x_{n-2} \cdots x_{n-k+1})} = \frac{c_{n-1} x_{n-2} \cdots x_{n-k}}{c_n x_{n-2} \cdots x_{n-k+1}} = \frac{c_{n-1}}{c_n} x_{n-k}, \quad n \in \mathbb{N}. \tag{3.2}$$

Hence for each $i \in \{0,1,\dots,k-1\}$

$$x_{km-i} = x_{-i} \prod_{j=1}^{m} \frac{c_{kj-i-1}}{c_{kj-i}}, \quad m \in \mathbb{N}_0. \tag{3.3}$$

Using formula (3.3) we easily prove the next theorem.

Theorem 3.1. *Consider equation* (1.1) *with* $b_n = 0$, $c_n \neq 0$, $n \in \mathbb{N}_0$. *Then the following statements are true:*

(a) *if*

$$\liminf_{m\to\infty} \left| \frac{c_{km-i}}{c_{km-i-1}} \right| = \widehat{p}_i > 1, \tag{3.4}$$

for some $i \in \{0, 1, \ldots, k-1\}$, *then* $x_{km-i} \to 0$ *as* $m \to \infty$;

(b) *if, for each* $i \in \{0, 1, \ldots, k-1\}$, *the limits* $\widehat{p}_i$ *in* (3.4) *are greater than* 1, *then* $x_n \to 0$ *as* $n \to \infty$;

(c) *if* $c_{km-i-1} = c_{km-i}$, *for every* $m \in \mathbb{N}$ *and for some* $i \in \{0, 1, \ldots, k-1\}$, *then* $x_{km-i} = x_{-i}$, $m \in \mathbb{N}_0$;

(d) *if* $c_{km-i-1} = -c_{km-i}$, *for every* $m \in \mathbb{N}$ *and for some* $i \in \{0, 1, \ldots, k-1\}$, *then* $x_{km-i} = (-1)^m x_{-i}$, $m \in \mathbb{N}_0$;

(e) *if*

$$\limsup_{m\to\infty} \left| \frac{c_{km-i}}{c_{km-i-1}} \right| = \widehat{q}_i \in [0, 1), \tag{3.5}$$

and $x_{-i} \neq 0$ *for some* $i \in \{0, 1, \ldots, k-1\}$, *then* $|x_{km-i}| \to \infty$, *as* $m \to \infty$;

(f) *if, for each* $i \in \{0, 1, \ldots, k-1\}$, $x_{-i} \neq 0$ *and the limits* $\widehat{q}_i$ *in* (3.5) *belong to the interval* $[0, 1)$, *then* $|x_n| \to \infty$ *as* $n \to \infty$.

4. Case $b_n \neq 0$ and $c_n \neq 0$

The case when $b_n \neq 0$ and $c_n \neq 0$ for every $n \in \mathbb{N}_0$ is considered in this section.

If $x_{-i_0} = 0$ for some $i_0 \in \{1, \ldots, k\}$, then from (1.1) we have that

$$x_{km-i_0} = 0, \quad \text{for } m \in \mathbb{N}_0. \tag{4.1}$$

From (4.1) and (1.1) we have that for each $i \in \{1, \ldots, k\} \setminus \{i_0\}$

$$x_{km-i} = \frac{x_{k(m-1)-i}}{b_{km-i}} = \frac{x_{-i}}{\prod_{j=1}^{m} b_{kj-i}}, \quad m \in \mathbb{N}_0. \tag{4.2}$$

From (4.1) we see that, for $i = i_0$, (4.2) also holds. Hence Theorem 2.1 can be applied in this case. Note that if $x_n = 0$ for some $n \in \mathbb{N}_0$, then (1.1) implies that there is an $i_0 \in \{1, \ldots, k\}$ such that $x_{-i_0} = 0$, and by the previous consideration we have that (4.2) also holds.

If $x_{-i} \neq 0$, for each $i \in \{1, \ldots, k\}$, then for every well-defined solution we have $x_n \neq 0$ for $n \geq -k$ (note that there are solutions which are not well defined, that is, those for which $x_{n-1} \cdots x_{n-k} = -b_n/c_n$, for some $n \in \mathbb{N}_0$).

Multiplying equation (1.1) by $x_{n-1}\cdots x_{n-k+1}$ and using the transformation

$$y_n = \frac{1}{x_n x_{n-1}\cdots x_{n-k+1}}, \quad n \geq -1, \tag{4.3}$$

we obtain equation

$$y_n = b_n y_{n-1} + c_n, \quad n \in \mathbb{N}_0. \tag{4.4}$$

Note that from (4.3), for every well-defined solution $(x_n)_{n\geq -k}$ of equation (1.1) such that $x_{-i} \neq 0$, for each $i \in \{1,\ldots,k\}$, it follows that $y_n \neq 0$, $n \geq -1$.

Since $b_n \neq 0$, $n \in \mathbb{N}_0$, we have that

$$y_n = \left(\prod_{i=0}^{n} b_i\right)\left(y_{-1} + \sum_{j=0}^{n} \frac{c_j}{\prod_{i=0}^{j} b_i}\right), \quad n \in \mathbb{N}_0. \tag{4.5}$$

From (4.3) and (4.5) we have that

$$x_n = \frac{1}{y_n x_{n-1}\cdots x_{n-k+1}} = \frac{y_{n-1}}{y_n} x_{n-k} = \frac{y_{-1} + \sum_{j=0}^{n-1}\left(c_j / \prod_{i=0}^{j} b_i\right)}{b_n\left(y_{-1} + \sum_{j=0}^{n}\left(c_j / \prod_{i=0}^{j} b_i\right)\right)} x_{n-k}, \tag{4.6}$$

for every $n \in \mathbb{N}_0$.

Hence, from (4.6), we obtain that

$$x_{mk-i} = x_{-i} \prod_{l=1}^{m} \frac{1/\alpha + \sum_{j=0}^{kl-i-1}\left(c_j / \prod_{i=0}^{j} b_i\right)}{b_{kl-i}\left(1/\alpha + \sum_{j=0}^{kl-i}\left(c_j / \prod_{i=0}^{j} b_i\right)\right)}, \tag{4.7}$$

for every $m \in \mathbb{N}_0$ and each $i = 1, 2, \ldots, k$, where

$$\alpha = \prod_{l=1}^{k} x_{-l}. \tag{4.8}$$

5. Case $b_n = 1$, $n \in \mathbb{N}_0$

Here we consider the case $b_n = 1$, $n \in \mathbb{N}_0$. In this case, from (4.7) we have that for each $i \in \{1,\ldots,k\}$

$$x_{mk-i} = x_{-i} \prod_{l=1}^{m} \frac{1 + \alpha \sum_{j=0}^{kl-i-1} c_j}{1 + \alpha \sum_{j=0}^{kl-i} c_j}, \quad m \in \mathbb{N}_0. \tag{5.1}$$

Note that this formula includes also the case when $x_{-i_0} = 0$ for some $i_0 \in \{1,\ldots,k\}$.

Now we formulate and prove a result in this case by using formula (5.1).

Theorem 5.1. *Consider equation* (1.1) *with* $b_n = 1$, $n \in \mathbb{N}_0$, $\operatorname{sign} c_n = \operatorname{sign} c_0$, $n \in \mathbb{N}$, $\alpha \neq 0$, *and*

$$\alpha \sum_{j=0}^{n} c_j \neq -1, \quad n \in \mathbb{N}_0. \tag{5.2}$$

Then the following statements hold:

(a) *if for some* $i \in \{1, \ldots, k\}$

$$\sum_{l=1}^{\infty} \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} c_j} = +\infty, \tag{5.3}$$

$$\lim_{l \to \infty} \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} c_j} = 0, \tag{5.4}$$

then $x_{mk-i} \to 0$ *as* $m \to \infty$;

(b) *if* (5.3) *and* (5.4) *hold for every* $i \in \{1, \ldots, k\}$, *then* $x_n \to 0$ *as* $n \to \infty$;

(c) *if for some* $i \in \{1, \ldots, k\}$ *the sum*

$$\sum_{l=1}^{\infty} \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} c_j} \tag{5.5}$$

converges, then the sequence x_{mk-i} *is also convergent;*

(d) *if the sum in* (5.5) *is finite for every* $i \in \{1, \ldots, k\}$, *then the sequences* x_{km-i} *are convergent.*

Proof. Let $(x_n)_{n \geq -k}$ be a solution of equation (1.1). Using condition $\operatorname{sign} c_n = \operatorname{sign} c_0$, $n \in \mathbb{N}$, it is easy to see that if (5.4) holds for some $i \in \{1, \ldots, k\}$, there is an $m_0 \in \mathbb{N}$ such that for $j \geq m_0 + 1$ the terms in the product in (5.1) are positive and that the following asymptotic formula

$$\ln(1 + x) = x + O\left(x^2\right) \tag{5.6}$$

can be used with x being the fraction in the limit (5.4). From (5.1) and (5.6) we have that

$$\begin{aligned}
|x_{km-i}| &= |x_{-i}| \prod_{l=1}^{m} \left| \frac{1+\alpha \sum_{j=0}^{kl-i-1} c_j}{1+\alpha \sum_{j=0}^{kl-i} c_j} \right| \\
&= |x_{-i}| c(m_0) \exp\left(\sum_{l=m_0+1}^{m} \ln \frac{1+\alpha \sum_{j=0}^{kl-i-1} c_j}{1+\alpha \sum_{j=0}^{kl-i} c_j} \right) \\
&= |x_{-i}| c(m_0) \exp\left(\sum_{l=m_0+1}^{m} \ln \left(1 - \frac{\alpha c_{kl-i}}{1+\alpha \sum_{j=0}^{kl-i} c_j} \right) \right) \\
&= |x_{-i}| c(m_0) \exp\left(- \sum_{l=m_0+1}^{m} \frac{\alpha c_{kl-i}(1+o(1))}{1+\alpha \sum_{j=0}^{kl-i} c_j} \right),
\end{aligned} \tag{5.7}$$

where

$$c(m_0) = \prod_{l=1}^{m_0} \left| \frac{1+\alpha \sum_{j=0}^{kl-i-1} c_j}{1+\alpha \sum_{j=0}^{kl-i} c_j} \right|. \tag{5.8}$$

Using formula (5.7), the assumptions regarding the sum $\sum_{j=m_0+1}^{\infty} (\alpha c_{kl-i} / (1+\alpha \sum_{j=0}^{kl-i} c_j))$ and the comparison test for the series whose terms are of eventually the same sign, the results in the theorem easily follow. □

6. Case $b_n = -1$, $n \in \mathbb{N}_0$

Here we consider the case $b_n = -1$, $n \in \mathbb{N}_0$. In this case from (4.7) we have

$$x_{mk-i} = (-1)^m x_{-i} \prod_{l=1}^{m} \frac{1+\alpha \sum_{j=0}^{kl-i-1} (-1)^{j+1} c_j}{1+\alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j}, \tag{6.1}$$

for every $m \in \mathbb{N}_0$ and each $i = 1, 2, \ldots, k$, where α is defined by (4.8).

Theorem 6.1. *Consider equation* (1.1) *with* $\alpha \neq 0$, $b_n = -1$, $n \in \mathbb{N}_0$, *and*

$$\alpha \sum_{j=0}^{n} (-1)^{j+1} c_j \neq -1, \quad n \in \mathbb{N}_0. \tag{6.2}$$

Then the following statements hold:

(a) *if for some* $i \in \{1, \ldots, k\}$

$$\sum_{l=1}^{\infty} \frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} = +\infty, \tag{6.3}$$

$$\lim_{l \to \infty} \frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} = 0, \tag{6.4}$$

$$\sum_{l=1}^{\infty} \frac{c_{kl-i}^2}{\left(1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j\right)^2} < +\infty, \tag{6.5}$$

then $x_{mk-i} \to 0$ *as* $m \to \infty$;

(b) *if for every* $i \in \{1, \ldots, k\}$, (6.3), (6.4), *and* (6.5) *hold, then* $x_n \to 0$ *as* $n \to \infty$;

(c) *if for some* $i \in \{1, \ldots, k\}$

$$\sum_{l=1}^{\infty} \frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} = -\infty, \tag{6.6}$$

conditions (6.4) *and* (6.5) *hold, and* $x_{-i} \neq 0$, *then* $|x_{mk-i}| \to \infty$ *as* $m \to \infty$;

(d) *if for every* $i \in \{1, \ldots, k\}$, *conditions* (6.4), (6.5), *and* (6.6) *hold, and* $x_{-i} \neq 0$, $i \in \{1, \ldots, k\}$, *then* $|x_n| \to \infty$ *as* $n \to \infty$;

(e) *if for some* $i \in \{1, \ldots, k\}$ *the sum*

$$\sum_{l=1}^{\infty} \frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} \tag{6.7}$$

converges and condition (6.5) *holds, then the sequences* x_{2mk-i} *and* $x_{(2m+1)k-i}$ *are also convergent;*

(f) *if for every* $i \in \{1, \ldots, k\}$ *the sum in* (6.7) *converges and condition* (6.5) *holds, then the sequences* x_{2km-j}, $j = 1, \ldots, 2k$ *are convergent.*

Proof. Let $(x_n)_{n\geq -k}$ be a solution of equation (1.1). By (6.4) we see that irrespectively on $i \in \{1,\dots,k\}$, there is an $m_1 \in \mathbb{N}$ such that for $j \geq m_1+1$ the terms in the product in (6.1) belong to the interval $(1/2, 3/2)$ and that asymptotic formulae

$$\ln(1+x) = x - \frac{x^2}{2} + O\left(x^3\right) \tag{6.8}$$

can be used with $-x$ being the fraction in (6.4). From this and (6.1) we have that

$$\begin{aligned}
|x_{km-i}| &= |x_{-i}| \prod_{l=1}^{m} \left| \frac{1+\alpha \sum_{j=0}^{kl-i-1} (-1)^{j+1} c_j}{1+\alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} \right| \\
&= |x_{-i}| c_1(m_1) \exp\left(\sum_{l=m_1+1}^{m} \ln \frac{1+\alpha \sum_{j=0}^{kl-i-1} (-1)^{j+1} c_j}{1+\alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} \right) \\
&= |x_{-i}| c_1(m_1) \exp\left(\sum_{l=m_1+1}^{m} \ln\left(1 - \frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1+\alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} \right) \right) \\
&= |x_{-i}| c_1(m_1) \exp\left(- \sum_{l=m_1+1}^{m} \left(\frac{\alpha(-1)^{kl-i+1} c_{kl-i}}{1+\alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} + \frac{\alpha^2 c_{kl-i}^2 (1+o(1))}{2\left(1+\alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j\right)^2} \right) \right),
\end{aligned} \tag{6.9}$$

where

$$c_1(m_1) = \prod_{l=1}^{m_1} \left| \frac{1+\alpha \sum_{j=0}^{kl-i-1} (-1)^{j+1} c_j}{1+\alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} \right|. \tag{6.10}$$

Using formula (6.9), the assumptions of the theorem and some well-known convergence tests for series, the results in (a)–(f) easily follow. □

7. Case $b_n = b_{n+k}$, $c_n = c_{n+k}$, $n \in \mathbb{N}_0$

In this section we consider equation (1.1) for the case $b_n = b_{n+k}$, $c_n = c_{n+k}$, $n \in \mathbb{N}_0$, that is, when the sequences b_n and c_n are k-periodic.

First we show the existence of k-periodic solutions of equation (4.4). If

$$\left(\overline{y}_0, \overline{y}_1, \dots, \overline{y}_{k-1}\right) \tag{7.1}$$

is such a solution, then we have that

$$\overline{y}_1 = b_1 \overline{y}_0 + c_1,\ \overline{y}_2 = b_2 \overline{y}_1 + c_2, \dots,\ \overline{y}_0 = b_k \overline{y}_{k-1} + c_k. \tag{7.2}$$

By successive elimination, or by Kronecker theorem (note that system (7.2) is linear), we get

$$\overline{y}_i = \frac{\sum_{j=0}^{k-1} c_{\sigma^{[j]}(i)} \prod_{s=0}^{j-1} b_{\sigma^{[s]}(i)}}{1 - \prod_{j=1}^{k} b_j}, \quad i = \overline{1,k}, \tag{7.3}$$

if $\prod_{j=1}^{k} b_j \neq 1$, where σ is the permutation defined by

$$\sigma(i) = i - 1, \quad i = \overline{2,k}, \ \sigma(1) = k, \tag{7.4}$$

and $\sigma^{[i]} = \sigma \circ \sigma^{[i-1]}$, $\sigma^{[0]} = \mathrm{Id}$, where Id denotes the identity.

It is easy to see that (4.4) along with k periodicity of sequences b_n and c_n implies

$$y_{km+i} = \left(\prod_{j=1}^{k} b_j\right) y_{k(m-1)+i} + \sum_{j=0}^{k-1} c_{\sigma^{[j]}(i)} \prod_{s=0}^{j-1} b_{\sigma^{[s]}(i)}, \tag{7.5}$$

for every $m \in \mathbb{N}_0$ and $i \in \{1, 2, \ldots, k\}$, such that $k(m-1) + i \geq -1$.

Since (7.5) is a linear first-order difference equation, we have that when $\prod_{j=1}^{k} b_j \neq 1$, its general solution is

$$y_{km+i} = \left(\prod_{j=1}^{k} b_j\right)^m y_i + \frac{1 - \left(\prod_{j=1}^{k} b_j\right)^m}{1 - \prod_{j=1}^{k} b_j} \sum_{j=0}^{k-1} c_{\sigma^{[j]}(i)} \prod_{s=0}^{j-1} b_{\sigma^{[s]}(i)}. \tag{7.6}$$

By letting $m \to \infty$ in (7.6) we obtain the following corollary.

Corollary 7.1. *Consider equation* (4.4) *with* $b_n = b_{n+k}$, $c_n = c_{n+k}$, $n \in \mathbb{N}_0$. *Assume that*

$$\left|\prod_{j=1}^{k} b_j\right| < 1. \tag{7.7}$$

Then for every solution y_n *of the equation we have that*

$$\lim_{m\to\infty} y_{km+i} = \overline{y}_i, \tag{7.8}$$

for every $i \in \{1, 2, \ldots, k\}$, *that is,* y_n *converges to the* k*-periodic solution in formula* (7.3).

Let

$$L_i := \sum_{j=0}^{k-1} c_{\sigma^{[j]}(i)} \prod_{s=0}^{j-1} b_{\sigma^{[s]}(i)}, \quad i = \overline{1,k}, \qquad q := \prod_{j=1}^{k} b_j. \tag{7.9}$$

From now on we will use the following convention: if $i, j \in \mathbb{N}_0$, then we regard that $L_j = L_i$, if $i \equiv j \pmod{k}$. Also if a sequence $(m_j)_{j\in\mathbb{N}_0}$ is defined by the relation $m_j = f(L_j)$, where f is a real function, then we will assume that $m_j = m_i$, if $i \equiv j \pmod{k}$.

Using (7.6) and notation (7.9) in the relation $x_n = (y_{n-1}/y_n)x_{n-k}$ (see (4.6)), for the case $q \neq 1$, we have that

$$\begin{aligned} x_{km+i} &= x_{i-k}\prod_{j=0}^{m}\frac{(y_{i-1}-L_{i-1}/(1-q))q^j + L_{i-1}/(1-q)}{(y_i - L_i/(1-q))q^j + L_i/(1-q)} \\ &= x_{i-k}\prod_{j=0}^{m}\frac{L_{i-1}}{L_i}\cdot\frac{1+((1-q)y_{i-1}/L_{i-1}-1)q^j}{1+((1-q)y_i/L_i-1)q^j}, \end{aligned} \tag{7.10}$$

for every $m \in \mathbb{N}_0$ and each $i \in \{2,\ldots,k\}$, and

$$\begin{aligned} x_{km+1} &= x_{1-k}\prod_{j=0}^{m}\frac{(y_k-L_k/(1-q))q^{j-1} + L_k/(1-q)}{(y_1 - L_1/(1-q))q^j + L_1/(1-q)} \\ &= x_{1-k}\prod_{j=0}^{m}\frac{L_k}{L_1}\cdot\frac{1+((1-q)y_k/L_k-1)q^{j-1}}{1+((1-q)y_1/L_1-1)q^j}. \end{aligned} \tag{7.11}$$

Now we present some results, which are applications of formulae (7.10) and (7.11).

7.1. Case $q = -1$

If $q = -1$, then by (7.5) we get

$$y_{km+i} = -y_{k(m-1)+i} + L_i = L_i - \left(L_i - y_{k(m-2)+i}\right) = y_{k(m-2)+i}, \quad m \in \mathbb{N}, \tag{7.12}$$

for $k(m-2)+i \geq -1$; that is, y_{km+i} is two-periodic for each $i \in \{1,\ldots,k\}$. Hence y_n is a $2k$-periodic solution of equation (4.4), in this case.

Hence from the relation $x_n = (y_{n-1}/y_n)x_{n-k}$ (see (4.6)), for each $i \in \{1,2,\ldots,k\}$, we have

$$x_{km-i} = \frac{y_{km-i-1}}{y_{km-i}}x_{km-i-k} = \frac{y_{km-i-1}}{y_{km-i}}\frac{y_{k(m-1)-i-1}}{y_{k(m-1)-i}}x_{km-i-2k}, \tag{7.13}$$

for $k(m-1) \geq i$.

From (7.13) and by $2k$ periodicity of y_n, we get

$$x_{2kl+j} = \left(\frac{y_{j-1}y_{j+k-1}}{y_j y_{j+k}}\right)^l x_j, \quad l \in \mathbb{N}_0, \tag{7.14}$$

for each $j \in \{-k+1,\ldots,-1,0,1,\ldots,k\}$.

From (7.14), the behavior of solutions of equation (1.1), in this case, easily follows. For example, if

$$p_j := \frac{y_{j-1}y_{j+k-1}}{y_j y_{j+k}} = 1, \tag{7.15}$$

for each $j \in \{-k+1, \ldots, -1, 0, 1, \ldots, k\}$, then the solution $(x_n)_{n\geq k}$ of (1.1) is $2k$-periodic.

7.2. Case $q = 1$

If $q = 1$ and $\alpha \neq 0$, then from (7.5) we obtain

$$y_{km+i} = y_{k(m-1)+i} + L_i, \quad m \in \mathbb{N}_0,\ i = \overline{1,k}, \tag{7.16}$$

when $k(m-1)+i \geq -1$, from which along with (4.6), it follows that

$$\begin{aligned} x_{km+i} &= x_i \prod_{j=1}^{m} \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i}, \quad m \in \mathbb{N},\ i = \overline{2,k}, \\ x_{km+1} &= x_1 \prod_{j=1}^{m} \frac{y_k + (j-1)L_k}{y_1 + jL_1}, \quad m \in \mathbb{N}. \end{aligned} \tag{7.17}$$

Corollary 7.2. *Consider equation* (1.1). *Let $q = 1$, $\alpha \neq 0$, and $\widehat{p}_i := L_{i-1}/L_i$, $i \in \{1, \ldots, k\}$. Then the following statements hold true.*

(a) *If $|\widehat{p}_i| < 1$, for some $i \in \{1, \ldots, k\}$, then $x_{km+i} \to 0$ as $m \to \infty$.*

(b) *If $|\widehat{p}_i| > 1$, or $L_i = 0$ and $L_{i-1} \neq 0$, for some $i \in \{1, \ldots, k\}$, then $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_i \neq 0$.*

(c) *If $\widehat{p}_i = 1$, for some $i \in \{2, \ldots, k\}$, and $(y_{i-1} - y_i)/L_i > 0$, then $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_i \neq 0$.*

(d) *If $\widehat{p}_i = 1$, for some $i \in \{2, \ldots, k\}$, and $(y_{i-1} - y_i)/L_i < 0$, then $x_{km+i} \to 0$ as $m \to \infty$.*

(e) *If $\widehat{p}_i = 1$, for some $i \in \{2, \ldots, k\}$, and $y_{i-1} = y_i$, then the sequence $(x_{km+i})_{m\in\mathbb{N}_0}$ is convergent.*

(f) *If $\widehat{p}_1 = 1$, and $(y_k - L_1 - y_1)/\ L_1 > 0$, then $|x_{km+1}| \to \infty$ as $m \to \infty$, if $x_1 \neq 0$.*

(g) *If $\widehat{p}_1 = 1$, and $(y_k - L_1 - y_1)/L_1 < 0$, then $x_{km+1} \to 0$ as $m \to \infty$.*

(h) *If $\widehat{p}_1 = 1$, and $y_k = L_1 + y_1$, then the sequence $(x_{km+1})_{m\in\mathbb{N}_0}$ is convergent.*

Proof. The statements in (a) and (b) follow from the facts that

$$\lim_{j\to\infty} \left| \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i} \right| = |\widehat{p}_i|, \quad i \in \{2, \ldots, k\} \tag{7.18}$$

if $L_i \neq 0$,

$$\lim_{j\to\infty}\left|\frac{y_{i-1}+jL_{i-1}}{y_i+jL_i}\right| = +\infty, \quad i \in \{2,\dots,k\} \tag{7.19}$$

if $L_i = 0$ and $L_{i-1} \neq 0$,

$$\lim_{j\to\infty}\left|\frac{y_k+(j-1)L_k}{y_1+jL_1}\right| = \left|\widehat{p}_1\right|, \tag{7.20}$$

if $L_1 \neq 0$, and

$$\lim_{j\to\infty}\left|\frac{y_k+(j-1)L_k}{y_1+jL_1}\right| = +\infty, \tag{7.21}$$

if $L_1 = 0$ and $L_k \neq 0$.

Now assume that $\widehat{p}_i = 1$ and let $(x_n)_{n\geq -k}$ be a solution of equation (1.1). It is easy to see that there is an $m_2 \in \mathbb{N}$ such that for $j \geq m_2 + 1$ the terms in the products in (7.17) are positive and that the following asymptotic formulae

$$(1+x)^{-1} = 1 - x + O\left(x^2\right), \qquad \ln(1+x) = x + O\left(x^2\right) \tag{7.22}$$

can be applied with $x = (y_{i-1} - y_i)/(jL_i)$, when $i \in \{2,\dots,k\}$ or with $x = (y_k - L_1 - y_1)/(jL_1)$. Using these formulae, for the case $i \in \{2,\dots,k\}$, we have that

$$\begin{aligned}
x_{km+i} &= x_i \prod_{j=1}^{m} \frac{y_{i-1}+jL_{i-1}}{y_i+jL_i} \\
&= x_i c(m_2) \exp\left(\sum_{j=m_2+1}^{m} \ln \frac{y_{i-1}+jL_{i-1}}{y_i+jL_i}\right) \\
&= x_i c(m_2) \exp\left(\sum_{j=m_2+1}^{m} \ln\left(1 + \frac{y_{i-1}-y_i}{jL_i} + O\left(\frac{1}{j^2}\right)\right)\right) \\
&= x_i c(m_2) \exp\left(\sum_{j=m_2+1}^{m} \left(\frac{y_{i-1}-y_i}{jL_i} + O\left(\frac{1}{j^2}\right)\right)\right),
\end{aligned} \tag{7.23}$$

where

$$c(m_2) = \prod_{j=1}^{m_2} \frac{y_{i-1}+jL_{i-1}}{y_i+jL_i}. \tag{7.24}$$

Letting $m \to \infty$ in (7.23), using the facts that

$$\sum_{j=m_2+1}^{m} \frac{1}{j} \longrightarrow +\infty \quad \text{as } m \longrightarrow \infty \tag{7.25}$$

and that the series $\sum_{j=m_2+1}^{\infty} O(1/j^2)$ converges, we get statements (c)–(e).

If $\widehat{p}_1 = 1$, that is $L_1 = L_k \neq 0$, then by using (7.22) we get

$$\begin{aligned} x_{km+1} &= x_1 \prod_{j=1}^{m} \frac{y_k + (j-1)L_k}{y_1 + jL_1} \\ &= x_1 d(m_2) \exp\left(\sum_{j=m_2+1}^{m} \ln \frac{y_k + (j-1)L_k}{y_1 + jL_1} \right) \\ &= x_1 d(m_2) \exp\left(\sum_{j=m_2+1}^{m} \ln\left(1 + \frac{y_k - L_1 - y_1}{jL_1} + O\left(\frac{1}{j^2}\right) \right) \right) \\ &= x_1 d(m_2) \exp\left(\sum_{j=m_2+1}^{m} \left(\frac{y_k - L_1 - y_1}{jL_1} + O\left(\frac{1}{j^2}\right) \right) \right), \end{aligned} \tag{7.26}$$

where

$$d(m_2) = \prod_{j=1}^{m_2} \frac{y_k + (j-1)L_k}{y_1 + jL_1}. \tag{7.27}$$

Letting $m \to \infty$ in (7.26), using (7.25) and the fact that the series $\sum_{j=m_2+1}^{\infty} O(1/j^2)$ converges, we get statements (f)–(h), as desired. □

7.3. Case $q \neq \pm 1$

If $q \neq \pm 1$, then from (7.6) we get

$$y_{km+i} = q^m s_i + t_i, \quad m \in \mathbb{N}_0, \tag{7.28}$$

where

$$s_i = y_i + \frac{L_i}{q-1}, \quad t_i = \frac{L_i}{1-q}, \quad i = \overline{1,k}, \tag{7.29}$$

from (4.6) it follows that

$$x_{km+i} = x_i \prod_{j=1}^{m} \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i}, \quad m \in \mathbb{N}_0, \tag{7.30}$$

for $i \in \{2, \ldots, k\}$, and

$$x_{km+1} = \frac{x_1 y_k}{q s_1 + t_1} \prod_{j=2}^{m} \frac{q^{j-1} s_k + t_k}{q^j s_1 + t_1}, \quad m \in \mathbb{N}_0. \tag{7.31}$$

Note that $t_i = t_j$, if $i \equiv j \pmod{k}$.

Corollary 7.3. *If $0 < |q| < 1$, $\alpha \neq 0$, and $q^j s_i + t_i \neq 0$, for every $j \in \mathbb{N}_0$ and $i \in \{1, \ldots, k\}$, then the following statements hold true.*

(a) *If $|t_{i-1}| < |t_i|$, for some $i \in \{1, \ldots, k\}$, we have that $x_{km+i} \to 0$ as $m \to \infty$.*

(b) *If $|t_{i-1}| > |t_i|$, and $s_i \neq 0$ if $t_i = 0$ for some $i \in \{1, \ldots, k\}$, we have that $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_i \neq 0$.*

(c) *If $t_{i-1} = t_i \neq 0$, for some $i \in \{1, \ldots, k\}$, then x_{km+i} is convergent.*

(d) *If $t_{i-1} = t_i = 0$, and $|s_{i-1}| < |s_i|$ for some $i \in \{2, \ldots, k\}$, then $|x_{km+i}| \to 0$ as $m \to \infty$.*

(e) *If $t_1 = t_k = 0$, and $|s_k| < |q s_1|$, then $x_{km+1} \to 0$ as $m \to \infty$.*

(f) *If $t_{i-1} = t_i = 0$, and $|s_{i-1}| > |s_i|$ for some $i \in \{2, \ldots, k\}$, then $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_i \neq 0$.*

(g) *If $t_1 = t_k = 0$, and $|s_k| > |q s_1|$, then $|x_{km+1}| \to \infty$ as $m \to \infty$, if $x_1 \neq 0$.*

(h) *If $t_{i-1} = t_i = 0$, and $s_{i-1} = s_i \neq 0$ for some $i \in \{2, \ldots, k\}$, then x_{km+i} is constant.*

(i) *If $t_1 = t_k = 0$, and $s_k = q s_1 \neq 0$, then x_{km+1} is constant.*

(j) *If $t_{i-1} = t_i = 0$, and $s_{i-1} = -s_i \neq 0$ for some $i \in \{2, \ldots, k\}$, then $x_{km+i} = (-1)^m x_i$.*

(k) *If $t_1 = t_k = 0$, and $s_k = -q s_1 \neq 0$, then $x_{km+1} = x_1 y_k (-1)^{m-1} / (q s_1)$.*

(l) *If $t_{i-1} = -t_i \neq 0$, for some $i \in \{1, \ldots, k\}$, then the subsequences x_{2km+i} and $x_{2km+k+i}$ are convergent.*

Proof. Since we have that

$$\lim_{j \to \infty} \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} = \frac{t_{i-1}}{t_i}, \quad i \in \{2, \ldots, k\} \tag{7.32}$$

when $t_i \neq 0$,

$$\lim_{j \to \infty} \left| \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} \right| = +\infty, \quad i \in \{2, \ldots, k\} \tag{7.33}$$

when $|t_{i-1}| > t_i = 0$ and $s_i \neq 0$,

$$\lim_{j \to \infty} \frac{q^{j-1} s_k + t_k}{q^j s_1 + t_1} = \frac{t_k}{t_1}, \tag{7.34}$$

when $t_1 \neq 0$, and

$$\lim_{j\to\infty}\left|\frac{q^{j-1}s_k + t_k}{q^j s_1 + t_1}\right| = +\infty, \tag{7.35}$$

when $|t_k| > t_1 = 0$ and $s_1 \neq 0$, the statements in (a) and (b) easily follow from (7.30)–(7.35).

(c) If $t_{i-1} = t_i \neq 0$, then

$$x_{km+i} = x_i \prod_{j=1}^{m}\left(1 + q^j\left(\frac{s_{i-1} - s_i}{t_i}\right) + o\left(q^j\right)\right), \tag{7.36}$$

for $i \in \{2, \ldots, k\}$, and if $t_1 = t_k \neq 0$, then

$$x_{km+1} = \frac{x_1 y_k}{q s_1 + t_1}\prod_{j=2}^{m}\left(1 + q^{j-1}\left(\frac{s_k - q s_1}{t_1}\right) + o\left(q^j\right)\right) \tag{7.37}$$

from which the statement in (c) easily follows.

(d)–(k) If $t_{i-1} = t_i = 0$, then

$$x_{km+i} = x_i \prod_{j=1}^{m}\frac{s_{i-1}}{s_i}, \tag{7.38}$$

for $i \in \{2, \ldots, k\}$, and if $t_1 = t_k = 0$, then

$$x_{km+1} = \frac{x_1 y_k}{q s_1}\prod_{j=2}^{m}\frac{s_k}{q s_1} \tag{7.39}$$

from which the statements in (d)–(k) easily follow.

(l) If $t_{i-1} = -t_i \neq 0$, then we have that

$$x_{km+i} = x_i \prod_{j=1}^{m}\left[-\left(1 - q^j\left(\frac{s_{i-1} + s_i}{t_i}\right) + o\left(q^j\right)\right)\right] \tag{7.40}$$

for $i \in \{2, \ldots, k\}$, and if $t_1 = -t_k \neq 0$, then

$$x_{km+1} = \frac{x_1 y_k}{q s_1 + t_1}\prod_{j=2}^{m}\left[-\left(1 - q^{j-1}\left(\frac{s_k + q s_1}{t_1}\right) + o\left(q^j\right)\right)\right] \tag{7.41}$$

from which the statement in (l) easily follows. □

Corollary 7.4. *If $|q| > 1$ and $\alpha \neq 0$, and $q^j s_i + t_i \neq 0$, for every $j \in \mathbb{N}_0$ and $i \in \{1, \ldots, k\}$, then the following statements hold true.*

(a) *If $|s_{i-1}| < |s_i|$, for some $i \in \{2, \ldots, k\}$, then $x_{km+i} \to 0$ as $m \to \infty$.*

(b) *If $|s_k| < |qs_1|$, then $x_{km+1} \to 0$ as $m \to \infty$.*

(c) *If $|s_{i-1}| > |s_i|$, or $s_i = 0$, $s_{i-1} \neq 0$ and $t_i \neq 0$, for some $i \in \{2, \ldots, k\}$, then $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_i \neq 0$.*

(d) *If $|s_k| > |qs_1|$, or if $s_1 = 0$, $s_k \neq 0$ and $t_1 \neq 0$, then $|x_{km+1}| \to \infty$ as $m \to \infty$, if $x_1 \neq 0$.*

(e) *If $s_{i-1} = s_i \neq 0$, for some $i \in \{2, \ldots, k\}$, then the sequence $(x_{km+i})_{m\in\mathbb{N}_0}$ is convergent.*

(f) *If $s_{i-1} = s_i = 0$ and $|t_{i-1}| < |t_i|$ for some $i \in \{2, \ldots, k\}$, then $x_{km+i} \to 0$ as $m \to \infty$.*

(g) *If $s_1 = s_k = 0$ and $|t_k| < |t_1|$, then $x_{km+1} \to 0$ as $m \to \infty$.*

(h) *If $s_{i-1} = s_i = 0$ and $|t_{i-1}| > |t_i|$ for some $i \in \{2, \ldots, k\}$, then $|x_{km+i}| \to +\infty$ as $m \to \infty$, if $x_i \neq 0$.*

(i) *If $s_1 = s_k = 0$ and $|t_k| > |t_1|$, then $|x_{km+1}| \to +\infty$ as $m \to \infty$, if $x_1 \neq 0$.*

(j) *If $s_{i-1} = s_i = 0$ and $t_{i-1} = t_i$ for some $i \in \{2, \ldots, k\}$, then the sequence x_{km+i} is constant.*

(k) *If $s_1 = s_k = 0$ and $t_1 = t_k$, then the sequence x_{km+1} is constant.*

(l) *If $s_{i-1} = s_i = 0$ and $t_{i-1} = -t_i$ for some $i \in \{2, \ldots, k\}$, then the sequence x_{km+i} is two-periodic.*

(m) *If $s_1 = s_k = 0$ and $t_1 = -t_k$, then the sequence x_{km+1} is two periodic.*

(n) *If $s_k = qs_1 \neq 0$, then the sequence $(x_{km+1})_{m\in\mathbb{N}_0}$ is convergent.*

(o) *If $s_{i-1} = -s_i \neq 0$, for some $i \in \{2, \ldots, k\}$, then the sequences $(x_{2km+i})_{m\in\mathbb{N}_0}$ and $(x_{2km+k+i})_{m\in\mathbb{N}_0}$, are convergent.*

(p) *If $s_k = -qs_1 \neq 0$, then the sequences $(x_{2km+1})_{m\in\mathbb{N}_0}$ and $(x_{2km+k+1})_{m\in\mathbb{N}_0}$, are convergent.*

Proof. (a)–(d) These statements follow correspondingly from the next relations (which are derived using formulae (7.30) and (7.31)):

$$\lim_{j\to\infty} \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} = \frac{s_{i-1}}{s_i} \tag{7.42}$$

for $i \in \{2, \ldots, k\}$ if $s_i \neq 0$, and

$$\lim_{j\to\infty} \left| \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} \right| = +\infty \tag{7.43}$$

for $i \in \{2, \ldots, k\}$ if $s_i = 0$, $s_{i-1} \neq 0$ and $t_i \neq 0$;

$$\lim_{j\to\infty} \frac{q^{j-1} s_k + t_k}{q^j s_1 + t_1} = \frac{s_k}{qs_1}, \tag{7.44}$$

if $s_1 \neq 0$, and

$$\lim_{j\to\infty}\left|\frac{q^{j-1}s_k+t_k}{q^j s_1+t_1}\right| = +\infty \tag{7.45}$$

if $s_1 = 0$, $s_k \neq 0$ and $t_1 \neq 0$.

(e) If $s_{i-1} = s_i \neq 0$, then from (7.30) we get

$$x_{km+i} = x_i \prod_{j=1}^{m}\left(1+q^{-j}\left(\frac{t_{i-1}-t_i}{s_{i-1}}\right)+o\left(q^{-j}\right)\right), \tag{7.46}$$

for $i \in \{2,\ldots,k\}$, from which (e) follows.

(f)–(m) If $s_{i-1} = s_i = 0$ for some $i \in \{2,\ldots,k\}$, then for $i \in \{2,\ldots,k\}$ we have

$$\frac{q^j s_{i-1}+t_{i-1}}{q^j s_i+t_i} = \frac{t_{i-1}}{t_i} \tag{7.47}$$

while when $s_1 = s_k = 0$, we have

$$\frac{q^{j-1}s_k+t_k}{q^j s_1+t_1} = \frac{t_k}{t_1} \tag{7.48}$$

from which the statements (f)–(i) easily follow.

(n) If $s_k = qs_1 \neq 0$, then we have

$$x_{km+1} = \frac{x_1 y_k}{qs_1+t_1}\prod_{j=2}^{m}\left(1+q^{-j}\left(\frac{t_k-t_1}{s_1}\right)+o\left(q^{-j}\right)\right), \tag{7.49}$$

from which along with the assumption $|q| > 1$ the statement follows.

(o) and (p) If $s_{i-1} = -s_i \neq 0$, then

$$x_{km+i} = x_i \prod_{j=1}^{m}\left[-\left(1-q^{-j}\left(\frac{t_{i-1}+t_i}{s_i}\right)+o\left(q^{-j}\right)\right)\right] \tag{7.50}$$

for $i \in \{2,\ldots,k\}$, and

$$x_{km+1} = \frac{x_1 y_k}{qs_1+t_1}\prod_{j=2}^{m}\left[-\left(1-q^{-j}\left(\frac{t_k+t_1}{s_1}\right)+o\left(q^{-j}\right)\right)\right]. \tag{7.51}$$

From (7.50) and (7.51) the statements in (o) and (p) correspondingly follow. □

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