On the Difference Equation

\[ x_n = \frac{a_n x_{n-k}}{b_n + c_n x_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \]

where \( k \in \mathbb{N} \) is fixed, the sequences \( b_n \) and \( c_n \) are real, \( (b_n, c_n) \neq (0,0) \), \( n \in \mathbb{N}_0 \), and the initial values \( x_{-k}, \ldots, x_{-1} \) are real numbers, is described.

1. Introduction

Recently there has been a huge interest in studying nonlinear difference equations and systems (see, e.g., [1–33] and the references therein). Here we study the difference equation

\[ x_n = \frac{x_{n-k}}{b_n + c_n x_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \]

where \( k \in \mathbb{N} \) is fixed, the sequences \( b_n \) and \( c_n \) are real, \( (b_n, c_n) \neq (0,0) \), \( n \in \mathbb{N}_0 \), and the initial values \( x_{-k}, \ldots, x_{-1} \) are real numbers. Equation (1.1) is a particular case of the equation

\[ x_n = \frac{\tilde{a}_n x_{n-k}}{\tilde{b}_n + \tilde{c}_n x_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \]

where \( \tilde{a}_n, \tilde{b}_n, \tilde{c}_n \) are real sequences.
with real sequences \( \tilde{a}_n, \tilde{b}_n \) and \( \tilde{c}_n \). For \( \tilde{a}_n = 0, n \in \mathbb{N}_0 \), the equation is trivial, and, for \( \tilde{a}_n \neq 0, n \in \mathbb{N}_0 \), it is reduced to equation (1.1) with \( b_n = \tilde{b}_n / \tilde{a}_n \) and \( c_n = \tilde{c}_n / \tilde{a}_n \).

Equation

\[
x_n = \frac{x_{n-k}}{b + cx_{n-1} \cdots x_{n-k}}, \quad n \in \mathbb{N}_0,
\]

(1.3)

where \( b, c \in \mathbb{R} \), which was treated in [32], is a particular case of equation (1.1).

As in [32], here, we employ our idea of using a change of variables in equation (1.1) which extends the one in our paper [21] and is later also used, for example, in [4]. For similar methods see also [22, 25]. Equation (1.3) in the case \( k = 2 \) was also studied in [1, 2], in a different way. The case when the sequences \( b_n \) and \( c_n \) are two-periodic was studied in [31] (some related results are also announced in talk [3]). For related symmetric systems of difference equations, see [27, 29]. For some other recent results on difference equations and systems which can be solved, see, for example, [6, 7, 20–22, 30, 31, 33]. Some classical results can be found, for example, in [11].

Equation (1.1) is a particular case of the equation

\[
y_n = f(y_{n-1}, \ldots, y_{n-k}, n) y_{n-k}, \quad n \in \mathbb{N}_0,
\]

(1.4)

where \( f : \mathbb{R}^{k+1} \rightarrow \mathbb{R} \) is a continuous function. Numerous particular cases of (1.4) have been investigated, for example, in [9, 21, 23]. In this paper we adopt the customary notation \( \prod_{i=k+1}^{k} g_i = 1 \) and \( \sum_{i=k+1}^{k} g_i = 0 \).

2. Case \( c_n = 0, n \in \mathbb{N}_0 \)

Here we consider the case \( c_n = 0, n \in \mathbb{N}_0 \). In this case equation (1.1) becomes

\[
x_n = \frac{x_{n-k}}{b_n}, \quad n \in \mathbb{N}_0,
\]

(2.1)

\( b_n \neq 0, n \in \mathbb{N}_0 \), from which it follows that for each \( i \in \{1, \ldots, k\} \)

\[
x_{km-i} = \frac{x_i}{\prod_{j=i}^{k} b_j}, \quad m \in \mathbb{N}_0.
\]

(2.2)

Using formula (2.2) the following theorem can be easily proved.
Theorem 2.1. Consider equation (1.1) with \( c_n = 0, b_n \neq 0, n \in \mathbb{N}_0 \). Then the following statements are true:

(a) if

\[
\lim \inf_{m \to \infty} |b_{km-i}| = p_i > 1,
\]

for some \( i \in \{1, \ldots, k\} \), then \( x_{km-i} \to 0 \) as \( m \to \infty \);

(b) if, for each \( i \in \{1, \ldots, k\} \), the limits \( p_i \) in (2.3) are greater than 1, then \( x_n \to 0 \) as \( n \to \infty \);

(c) if \( b_{km-i} = 1 \) for every \( m \in \mathbb{N} \) and for some \( i \in \{1, \ldots, k\} \), then \( x_{km-i} = x_i \), \( m \in \mathbb{N}_0 \);

(d) if \( b_{km-i} = -1 \), for every \( m \in \mathbb{N} \) and for some \( i \in \{1, \ldots, k\} \), then \( x_{km-i} = (-1)^m x_i \), \( m \in \mathbb{N}_0 \);

(e) if

\[
\lim \sup_{m \to \infty} |b_{km-i}| = q_i \in [0,1),
\]

and \( x_i \neq 0 \), for some \( i \in \{1, \ldots, k\} \), then \( |x_{km-i}| \to \infty \), as \( m \to \infty \);

(f) if, for each \( i \in \{1, \ldots, k\} \), the limits \( q_i \) in (2.4) belong to the interval \([0,1)\) and \( x_i \neq 0 \), then \( |x_n| \to \infty \) as \( n \to \infty \).

3. Case \( b_n = 0, n \in \mathbb{N}_0 \)

In this section we consider the case \( b_n = 0, n \in \mathbb{N}_0 \). Note that in this case equation (1.1) becomes

\[
x_n = \frac{x_{n-k}}{c_n x_{n-1} \cdots x_{n-k+1} x_{n-k}}, \quad n \in \mathbb{N}_0,
\]

where \( c_n \neq 0, n \in \mathbb{N}_0 \). If \( x_n \) is a well-defined solution of equation (3.1) (i.e., a solution with initial values \( x_i \neq 0, i = 1, \ldots, k \), which implies \( x_i \neq 0, n \in \mathbb{N}_0 \)), then

\[
x_n = \frac{1}{c_n x_{n-1} (x_{n-2} \cdots x_{n-k+1})} \cdot \frac{c_{n-1} x_{n-2} \cdots x_{n-k}}{c_n x_{n-2} \cdots x_{n-k+1}} = \frac{c_{n-1}}{c_n} x_{n-k}, \quad n \in \mathbb{N}.
\]

Hence for each \( i \in \{0,1, \ldots, k-1\} \)

\[
x_{km-i} = x_i \prod_{j=1}^{m} \frac{c_{k-j-1}}{c_{k-j}}, \quad m \in \mathbb{N}_0.
\]

Using formula (3.3) we easily prove the next theorem.
Theorem 3.1. Consider equation (1.1) with $b_n = 0$, $c_n \neq 0$, $n \in \mathbb{N}_0$. Then the following statements are true:

(a) if

$$
\lim \inf_{m \to \infty} \left| \frac{c_{km-i}}{c_{km-i-1}} \right| = \hat{p}_i > 1,
$$

for some $i \in \{0, 1, \ldots, k-1\}$, then $x_{km-i} \to 0$ as $m \to \infty$;

(b) if, for each $i \in \{0, 1, \ldots, k-1\}$, the limits $\hat{p}_i$ in (3.4) are greater than 1, then $x_n \to 0$ as $n \to \infty$;

(c) if $c_{km-i-1} = c_{km-i}$ for every $m \in \mathbb{N}$ and for some $i \in \{0, 1, \ldots, k-1\}$, then $x_{km-i} = x_i$, $m \in \mathbb{N}_0$;

(d) if $c_{km-i-1} = -c_{km-i}$ for every $m \in \mathbb{N}$ and for some $i \in \{0, 1, \ldots, k-1\}$, then $x_{km-i} = (-1)^m x_i$, $m \in \mathbb{N}_0$;

(e) if

$$
\lim \sup_{m \to \infty} \left| \frac{c_{km-i}}{c_{km-i-1}} \right| = \hat{q}_i \in [0, 1),
$$

and $x_i \neq 0$ for some $i \in \{0, 1, \ldots, k-1\}$, then $|x_{km-i}| \to \infty$ as $m \to \infty$;

(f) if, for each $i \in \{0, 1, \ldots, k-1\}$, $x_i \neq 0$ and the limits $\hat{q}_i$ in (3.5) belong to the interval $[0, 1)$, then $|x_n| \to \infty$ as $n \to \infty$.

4. Case $b_n \neq 0$ and $c_n \neq 0$

The case when $b_n \neq 0$ and $c_n \neq 0$ for every $n \in \mathbb{N}_0$ is considered in this section.

If $x_{i_0} = 0$ for some $i_0 \in \{1, \ldots, k\}$, then from (1.1) we have that

$$x_{km-i_0} = 0, \quad m \in \mathbb{N}_0. \tag{4.1}$$

From (4.1) and (1.1) we have that for each $i \in \{1, \ldots, k\} \setminus \{i_0\}$

$$x_{km-i} = \frac{x_{k(m-1)-i}}{b_{km-i}} = \frac{x_{-i}}{\prod_{j=1}^{m} b_{k-j}}, \quad m \in \mathbb{N}_0. \tag{4.2}$$

From (4.1) we see that, for $i = i_0$, (4.2) also holds. Hence Theorem 2.1 can be applied in this case. Note that if $x_n = 0$ for some $n \in \mathbb{N}_0$, then (1.1) implies that there is an $i_0 \in \{1, \ldots, k\}$ such that $x_{i_0} = 0$, and by the previous consideration we have that (4.2) also holds.

If $x_i \neq 0$, for each $i \in \{1, \ldots, k\}$, then for every well-defined solution we have $x_n \neq 0$ for $n \geq -k$ (note that there are solutions which are not well defined, that is, those for which $x_{n-1} \cdots x_{n-k} = -b_n/c_n$, for some $n \in \mathbb{N}_0$).
Multiplying equation (1.1) by $x_{n-1} \cdots x_{n-k+1}$ and using the transformation

$$y_n = \frac{1}{x_n x_{n-1} \cdots x_{n-k+1}}, \quad n \geq -1,$$  \hspace{1cm} (4.3)

we obtain equation

$$y_n = b_n y_{n-1} + c_n, \quad n \in \mathbb{N}_0.$$  \hspace{1cm} (4.4)

Note that from (4.3), for every well-defined solution $(x_n)_{n \geq k}$ of equation (1.1) such that $x_{-i} \neq 0$, for each $i \in \{1, \ldots, k\}$, it follows that $y_n \neq 0$, $n \geq -1$.

Since $b_n \neq 0$, $n \in \mathbb{N}_0$, we have that

$$y_n = \left( \prod_{i=0}^{n} b_i \right) \left( y_{-1} + \sum_{j=0}^{n} \frac{c_j}{\prod_{i=0}^{j} b_i} \right), \quad n \in \mathbb{N}_0.$$  \hspace{1cm} (4.5)

From (4.3) and (4.5) we have that

$$x_n = \frac{1}{y_n x_{n-1} \cdots x_{n-k+1}} = \frac{y_{n-1} x_{n-k}}{y_n} = \frac{y_{-1} + \sum_{j=0}^{n-1} \left( c_j / \prod_{i=0}^{j} b_i \right) x_{n-k}}{b_n \left( y_{-1} + \sum_{j=0}^{n} \left( c_j / \prod_{i=0}^{j} b_i \right) \right)} x_{n-k},$$  \hspace{1cm} (4.6)

for every $n \in \mathbb{N}_0$.

Hence, from (4.6), we obtain that

$$x_{m-k-i} = x_{-i} \prod_{l=1}^{m} \frac{1}{b_{kl-i} \left( 1/\alpha + \sum_{j=0}^{kl-i} \left( c_j / \prod_{i=0}^{j} b_i \right) \right)},$$  \hspace{1cm} (4.7)

for every $m \in \mathbb{N}_0$ and each $i = 1, 2, \ldots, k$, where

$$\alpha = \prod_{l=1}^{k} x_{-i}.$$  \hspace{1cm} (4.8)

5. Case $b_n = 1$, $n \in \mathbb{N}_0$

Here we consider the case $b_n = 1$, $n \in \mathbb{N}_0$. In this case, from (4.7) we have that for each $i \in \{1, \ldots, k\}$

$$x_{m-k-i} = x_{-i} \prod_{l=1}^{m} \frac{1 + \alpha \sum_{j=0}^{kl-i-1} c_j}{1 + \alpha \sum_{j=0}^{kl-i} c_j}, \quad m \in \mathbb{N}_0.$$  \hspace{1cm} (5.1)

Note that this formula includes also the case when $x_{-i_0} = 0$ for some $i_0 \in \{1, \ldots, k\}$.  

Now we formulate and prove a result in this case by using formula (5.1).

**Theorem 5.1.** Consider equation (1.1) with \( b_n = 1, \ n \in \mathbb{N}_0, \) sign \( c_n = \text{sign} \ c_0, \ n \in \mathbb{N}, \alpha \neq 0, \) and

\[
\alpha \sum_{j=0}^{n} c_j \neq -1, \quad n \in \mathbb{N}_0. \quad (5.2)
\]

Then the following statements hold:

(a) if for some \( i \in \{1, \ldots, k\} \)

\[
\sum_{l=1}^{\infty} \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} c_j} = +\infty, \quad (5.3)
\]

\[
\lim_{l \to \infty} \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} c_j} = 0, \quad (5.4)
\]

then \( x_{mk-i} \to 0 \) as \( m \to \infty; \)

(b) if (5.3) and (5.4) hold for every \( i \in \{1, \ldots, k\}, \) then \( x_n \to 0 \) as \( n \to \infty; \)

(c) if for some \( i \in \{1, \ldots, k\} \) the sum

\[
\sum_{l=1}^{\infty} \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} c_j} \quad (5.5)
\]

converges, then the sequence \( x_{mk-i} \) is also convergent;

(d) if the sum in (5.5) is finite for every \( i \in \{1, \ldots, k\}, \) then the sequences \( x_{km-i} \) are convergent.

**Proof.** Let \( (x_n)_{n \geq k} \) be a solution of equation (1.1). Using condition sign \( c_n = \text{sign} \ c_0, \ n \in \mathbb{N}, \) it is easy to see that if (5.4) holds for some \( i \in \{1, \ldots, k\}, \) there is an \( m_0 \in \mathbb{N} \) such that for \( j \geq m_0 + 1 \) the terms in the product in (5.1) are positive and that the following asymptotic formula

\[
\ln(1 + x) = x + O(x^2) \quad (5.6)
\]
can be used with \( x \) being the fraction in the limit (5.4). From (5.1) and (5.6) we have that

\[
|x_{km-i}| = |x_{-i}| \prod_{l=1}^{m} \frac{1 + \alpha \sum_{j=0}^{kl-1} c_j}{1 + \alpha \sum_{j=0}^{kl-1} c_j}
\]

\[
= |x_{-i}|c(m_0) \exp\left( \sum_{l=m_0+1}^{m} \ln \frac{1 + \alpha \sum_{j=0}^{kl-1} c_j}{1 + \alpha \sum_{j=0}^{kl-1} c_j} \right)
\]

\[
= |x_{-i}|c(m_0) \exp\left( \sum_{l=m_0+1}^{m} \ln \left( 1 - \frac{\alpha c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-1} c_j} \right) \right)
\]

\[
= |x_{-i}|c(m_0) \exp\left( - \sum_{l=m_0+1}^{m} \frac{\alpha c_{kl-i}(1 + o(1))}{1 + \alpha \sum_{j=0}^{kl-1} c_j} \right),
\]

where

\[
c(m_0) = \prod_{l=1}^{m_0} \frac{1 + \alpha \sum_{j=0}^{kl-1} c_j}{1 + \alpha \sum_{j=0}^{kl-1} c_j}.
\]

Using formula (5.7), the assumptions regarding the sum \( \sum_{j=m_0+1}^{\infty} (ac_{kl-i}/(1 + \alpha \sum_{j=0}^{kl-1} c_j)) \) and the comparison test for the series whose terms are of eventually the same sign, the results in the theorem easily follow.

6. Case \( b_n = -1, \ n \in \mathbb{N}_0 \)

Here we consider the case \( b_n = -1, \ n \in \mathbb{N}_0 \). In this case from (4.7) we have

\[
x_{mk-i} = (-1)^m x_{-i} \prod_{l=1}^{m} \frac{1 + \alpha \sum_{j=0}^{kl-1} (-1)^{j+1} c_j}{1 + \alpha \sum_{j=0}^{kl-1} (-1)^{j+1} c_j},
\]

for every \( m \in \mathbb{N}_0 \) and each \( i = 1, 2, \ldots, k \), where \( \alpha \) is defined by (4.8).
\textbf{Theorem 6.1.} Consider equation (1.1) with $\alpha \neq 0$, $b_n = -1$, $n \in \mathbb{N}_0$, and
\begin{equation}
\alpha \sum_{j=0}^{n} (-1)^{j+1} c_j \neq -1, \quad n \in \mathbb{N}_0.
\end{equation}

Then the following statements hold:

(a) if for some $i \in \{1, \ldots, k\}$
\begin{equation}
\sum_{i=1}^{\infty} \frac{\alpha(-1)^{kl-i} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} = +\infty,
\end{equation}
\begin{equation}
\lim_{i \to \infty} \frac{\alpha(-1)^{kl-i} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} = 0,
\end{equation}
\begin{equation}
\sum_{i=1}^{\infty} \frac{c_{kl-i}^2}{\left(1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j\right)^2} < +\infty,
\end{equation}
then $x_{mk-i} \to 0$ as $m \to \infty$;

(b) if for every $i \in \{1, \ldots, k\}$, (6.3), (6.4), and (6.5) hold, then $x_n \to 0$ as $n \to \infty$;

(c) if for some $i \in \{1, \ldots, k\}$
\begin{equation}
\sum_{i=1}^{\infty} \frac{\alpha(-1)^{kl-i} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} = -\infty,
\end{equation}
conditions (6.4) and (6.5) hold, and $x_{-i} \neq 0$, then $|x_{mk-i}| \to \infty$ as $m \to \infty$;

(d) if for every $i \in \{1, \ldots, k\}$, conditions (6.4), (6.5), and (6.6) hold, and $x_{-i} \neq 0$, $i \in \{1, \ldots, k\}$, then $|x_n| \to \infty$ as $n \to \infty$;

(e) if for some $i \in \{1, \ldots, k\}$ the sum
\begin{equation}
\sum_{i=1}^{\infty} \frac{\alpha(-1)^{kl-i} c_{kl-i}}{1 + \alpha \sum_{j=0}^{kl-i} (-1)^{j+1} c_j}
\end{equation}
converges and condition (6.5) holds, then the sequences $x_{2mk-i}$ and $x_{(2m+1)k-i}$ are also convergent;

(f) if for every $i \in \{1, \ldots, k\}$ the sum in (6.7) converges and condition (6.5) holds, then the sequences $x_{2km-j}$, $j = 1, \ldots, 2k$ are convergent.
Proof. Let \( (x_n)_{n \geq k} \) be a solution of equation (1.1). By (6.4) we see that irrespectively on \( i \in \{1, \ldots, k\} \), there is an \( m_1 \in \mathbb{N} \) such that for \( j \geq m_1 + 1 \) the terms in the product in (6.1) belong to the interval \((1/2, 3/2)\) and that asymptotic formulae

\[
\ln(1 + x) = x - \frac{x^2}{2} + O(x^3)
\]  

(6.8)
can be used with \(-x\) being the fraction in (6.4). From this and (6.1) we have that

\[
|x_{km-1}| = |x_{m+1}| \prod_{l=1}^{m} \left| \frac{1 + a \sum_{j=0}^{kl-i} (-1)^{j+1} c_j}{1 + a \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} \right|
\]

(6.10)

\[
= |x_{m+1}| c_1(m_1) \exp \left( \sum_{l=m+1}^{m} \ln \left( \frac{1 + a \sum_{j=0}^{kl-i} (-1)^{j+1} c_j}{1 + a \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} \right) \right)
\]

(6.9)

where

\[
c_1(m_1) = \prod_{l=1}^{m_1} \left| \frac{1 + a \sum_{j=0}^{kl-i} (-1)^{j+1} c_j}{1 + a \sum_{j=0}^{kl-i} (-1)^{j+1} c_j} \right|.
\]  

Using formula (6.9), the assumptions of the theorem and some well-known convergence tests for series, the results in (a)–(f) easily follow. \( \square \)

7. Case \( b_n = b_{n+k}, c_n = c_{n+k}, n \in \mathbb{N}_0 \)

In this section we consider equation (1.1) for the case \( b_n = b_{n+k}, c_n = c_{n+k}, n \in \mathbb{N}_0 \), that is, when the sequences \( b_n \) and \( c_n \) are \( k \)-periodic.

First we show the existence of \( k \)-periodic solutions of equation (4.4). If

\[
(\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{k-1})
\]  

(7.1)
is such a solution, then we have that

\[
\bar{y}_1 = b_1 \bar{y}_0 + c_1, \quad \bar{y}_2 = b_2 \bar{y}_1 + c_2, \ldots, \quad \bar{y}_0 = b_0 \bar{y}_{k-1} + c_k.
\]  

(7.2)
By successive elimination, or by Kronecker theorem (note that system (7.2) is linear), we get

\[
\overline{y}_i = \frac{\sum_{j=0}^{k-1} c_{\sigma(j)} \prod_{s=0}^{j-1} b_{\sigma(s)} j}{1 - \prod_{j=1}^{k} b_j}, \quad i = 1, k,
\]

(7.3)

if \(\prod_{j=1}^{k} b_j \neq 1\), where \(\sigma\) is the permutation defined by

\[
\sigma(i) = i - 1, \quad i = 2, k, \quad \sigma(1) = k,
\]

(7.4)

and \(\sigma^{[i]} = \sigma \circ \sigma^{[i-1]}, \sigma^{[0]} = \text{Id}\), where \(\text{Id}\) denotes the identity.

It is easy to see that (4.4) along with \(k\) periodicity of sequences \(b_n\) and \(c_n\) implies

\[
y_{km+i} = \left( \prod_{j=1}^{k} b_j \right)^m y_i + \sum_{j=0}^{k-1} c_{\sigma(j)} \prod_{s=0}^{j-1} b_{\sigma(s)} j
\]

(7.5)

for every \(m \in \mathbb{N}_0\) and \(i \in \{1, 2, \ldots, k\}\), such that \(k(m-1) + i \geq -1\).

Since (7.5) is a linear first-order difference equation, we have that when \(\prod_{j=1}^{k} b_j \neq 1\), its general solution is

\[
y_{km+i} = \left( \prod_{j=1}^{k} b_j \right)^m y_i + \frac{1 - \left( \prod_{j=1}^{k} b_j \right)^m}{1 - \prod_{j=1}^{k} b_j} \sum_{j=0}^{k-1} c_{\sigma(j)} \prod_{s=0}^{j-1} b_{\sigma(s)} j.
\]

(7.6)

By letting \(m \to \infty\) in (7.6) we obtain the following corollary.

**Corollary 7.1.** Consider equation (4.4) with \(b_n = b_{n+k}, c_n = c_{n+k}, n \in \mathbb{N}_0\). Assume that

\[
\left| \prod_{j=1}^{k} b_j \right| < 1.
\]

(7.7)

Then for every solution \(y_n\) of the equation we have that

\[
\lim_{m \to \infty} y_{km+i} = \overline{y}_i,
\]

(7.8)

for every \(i \in \{1, 2, \ldots, k\}\), that is, \(y_n\) converges to the \(k\)-periodic solution in formula (7.3).

Let

\[
L_i := \sum_{j=0}^{k-1} c_{\sigma(j)} \prod_{s=0}^{j-1} b_{\sigma(s)} j, \quad i = 1, k, \quad q := \prod_{j=1}^{k} b_j.
\]

(7.9)
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From now on we will use the following convention: if \( i, j \in \mathbb{N}_0 \), then we regard that \( L_j = L_i \), if \( i \equiv j \) (mod \( k \)). Also if a sequence \( (m_i)_{i \in \mathbb{N}_0} \) is defined by the relation \( m_j = f(L_i) \), where \( f \) is a real function, then we will assume that \( m_j = m_i \), if \( i \equiv j \) (mod \( k \)).

Using (7.6) and notation (7.9) in the relation \( x_n = (y_{n-1}/y_n)x_{n-k} \) (see (4.6)), for the case \( q \neq 1 \), we have that

\[
x_{km+1} = x_{i-k} \prod_{j=0}^{m} \left( \frac{(y_{i-1} - L_{i-1}/(1-q))q^j + L_{i-1}/(1-q)}{(y_i - L_i/(1-q))q^j + L_i/(1-q)} \right)
\]

for every \( m \in \mathbb{N}_0 \) and each \( i \in \{2, \ldots, k\} \), and

\[
x_{km+1} = x_{1-k} \prod_{j=0}^{m} \left( \frac{(y_{k-1} - L_{k-1}/(1-q))q^{i-1} + L_{k-1}/(1-q)}{(y_{k-1} - L_{k-1}/(1-q))q^{i-1} + L_{k-1}/(1-q)} \right)
\]

Now we present some results, which are applications of formulae (7.10) and (7.11).

**7.1. Case \( q = -1 \)**

If \( q = -1 \), then by (7.5) we get

\[
y_{km+i} = -y_{k(m-1)+i} + L_i = L_i - (L_i - y_{k(m-2)+i}) = y_{k(m-2)+i}, \quad m \in \mathbb{N},
\]

for \( k(m-2) + i \geq -1 \); that is, \( y_{km+i} \) is two-periodic for each \( i \in \{1, \ldots, k\} \). Hence \( y_n \) is a 2\( k \)-periodic solution of equation (4.4), in this case.

Hence from the relation \( x_n = (y_{n-1}/y_n)x_{n-k} \) (see (4.6)), for each \( i \in \{1,2,\ldots,k\} \), we have

\[
x_{km-i} = \frac{y_{km-i-1}}{y_{km-i}} x_{km-i-k} = \frac{y_{km-i-1} y_{km-i-1}}{y_{km-i}} x_{km-i-2k},
\]

for \( k(m-1) \geq i \).

From (7.13) and by 2\( k \) periodicity of \( y_n \), we get

\[
x_{2kl+j} = \left( \frac{y_{j-1} y_{j+k-1}}{y_j y_{j+k}} \right)^l x_j, \quad l \in \mathbb{N}_0,
\]

for each \( j \in \{-k+1, \ldots, -1, 0, 1, \ldots, k\} \).
From (7.14), the behavior of solutions of equation (1.1), in this case, easily follows. For example, if

$$p_j := \frac{y_{j-1}y_{j+k-1}}{y_jy_{j+k}} = 1,$$  \hspace{1cm} (7.15)$$

for each $j \in \{-k+1, \ldots, -1, 0, 1, \ldots, k\}$, then the solution $(x_n)_{n \geq k}$ of (1.1) is $2k$-periodic.

### 7.2. Case $q = 1$

If $q = 1$ and $\alpha \neq 0$, then from (7.5) we obtain

$$y_{km+i} = y_{k(m-1)+i} + L_i, \hspace{0.5cm} m \in \mathbb{N}_0, \hspace{0.5cm} i = 1, k,$$  \hspace{1cm} (7.16)$$

when $k(m-1) + i \geq -1$, from which along with (4.6), it follows that

$$x_{km+i} = x_1 \prod_{j=1}^{m} \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i}, \hspace{0.5cm} m \in \mathbb{N}, \hspace{0.5cm} i = 1, k,$$

$$x_{km+1} = x_1 \prod_{j=1}^{m} \frac{y_k + (j-1)L_k}{y_1 + jL_1}, \hspace{0.5cm} m \in \mathbb{N}.$$  \hspace{1cm} (7.17)$$

**Corollary 7.2.** Consider equation (1.1). Let $q = 1$, $\alpha \neq 0$, and $\hat{p}_i := L_{i-1}/L_i$, $i \in \{1, \ldots, k\}$. Then the following statements hold true.

(a) If $|\hat{p}_i| < 1$, for some $i \in \{1, \ldots, k\}$, then $x_{km+i} \to 0$ as $m \to \infty$.

(b) If $|\hat{p}_i| > 1$, or $L_i = 0$ and $L_{i-1} \neq 0$, for some $i \in \{1, \ldots, k\}$, then $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_1 \neq 0$.

(c) If $\hat{p}_i = 1$, for some $i \in \{2, \ldots, k\}$, and $(y_{i-1} - y_i)/L_i > 0$, then $|x_{km+i}| \to \infty$ as $m \to \infty$, if $x_1 \neq 0$.

(d) If $\hat{p}_i = 1$, for some $i \in \{2, \ldots, k\}$, and $(y_{i-1} - y_i)/L_i < 0$, then $x_{km+i} \to 0$ as $m \to \infty$.

(e) If $\hat{p}_i = 1$, for some $i \in \{2, \ldots, k\}$, and $y_{i-1} = y_i$, then the sequence $(x_{km+i})_{m \in \mathbb{N}_0}$ is convergent.

(f) If $\hat{p}_1 = 1$, and $(y_k - L_1 - y_1)/L_1 > 0$, then $|x_{km+1}| \to \infty$ as $m \to \infty$, if $x_1 \neq 0$.

(g) If $\hat{p}_1 = 1$, and $(y_k - L_1 - y_1)/L_1 < 0$, then $x_{km+1} \to 0$ as $m \to \infty$.

(h) If $\hat{p}_1 = 1$, and $y_k = L_1 + y_1$, then the sequence $(x_{km+1})_{m \in \mathbb{N}_0}$ is convergent.

**Proof.** The statements in (a) and (b) follow from the facts that

$$\lim_{j \to \infty} \left| \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i} \right| = |\hat{p}_i|, \hspace{0.5cm} i \in \{2, \ldots, k\}.$$  \hspace{1cm} (7.18)$$
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if $L_i \neq 0$,

$$\lim_{j \to \infty} \left| \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i} \right| = +\infty, \quad i \in \{2, \ldots, k\} \tag{7.19}$$

if $L_i = 0$ and $L_{i-1} \neq 0$,

$$\lim_{j \to \infty} \left| \frac{y_{k} + (j-1)L_k}{y_1 + jL_1} \right| = |\tilde{p}_1|, \tag{7.20}$$

if $L_1 \neq 0$, and

$$\lim_{j \to \infty} \left| \frac{y_{k} + (j-1)L_k}{y_1 + jL_1} \right| = +\infty, \tag{7.21}$$

if $L_1 = 0$ and $L_k \neq 0$.

Now assume that $\tilde{p}_1 = 1$ and let $(x_{e})_{e=2-k}$ be a solution of equation (1.1). It is easy to see that there is an $m_2 \in \mathbb{N}$ such that for $j \geq m_2 + 1$ the terms in the products in (7.17) are positive and that the following asymptotic formulae

$$(1 + x)^{-1} = 1 - x + O(x^2), \quad \ln(1 + x) = x + O(x^2) \tag{7.22}$$

can be applied with $x = (y_{i-1} - y_i) / (jL_i)$, when $i \in \{2, \ldots, k\}$ or with $x = (y_k - L_1 - y_1) / (jL_1)$. Using these formulae, for the case $i \in \{2, \ldots, k\}$, we have that

$$x_{km+i} = x_i \prod_{j=1}^{m} \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i} \tag{7.23}$$

$$= x_i c(m_2) \exp \left( \sum_{j=m_2+1}^{m} \ln \left( \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i} \right) \right)$$

$$= x_i c(m_2) \exp \left( \sum_{j=m_2+1}^{m} \ln \left( 1 + \frac{y_{i-1} - y_i}{jL_i} + O \left( \frac{1}{j^2} \right) \right) \right)$$

$$= x_i c(m_2) \exp \left( \sum_{j=m_2+1}^{m} \left( \frac{y_{i-1} - y_i}{jL_i} + O \left( \frac{1}{j^2} \right) \right) \right)$$

where

$$c(m_2) = \prod_{j=1}^{m} \frac{y_{i-1} + jL_{i-1}}{y_i + jL_i}. \tag{7.24}$$
Letting \( m \to \infty \) in (7.23), using the facts that
\[
\sum_{j=m_2+1}^{m} \frac{1}{j} \to +\infty \quad \text{as} \quad m \to \infty \tag{7.25}
\]
and that the series \( \sum_{j=m_2+1}^{\infty} O(1/j^2) \) converges, we get statements (c)–(e).

If \( \hat{p}_1 = 1 \), that is \( L_1 = L_k \neq 0 \), then by using (7.22) we get
\[
x_{km+1} = x_1 \prod_{j=1}^{m} \frac{y_k + (j-1)L_k}{y_1 + jL_1}
= x_1 d(m_2) \exp \left( \sum_{j=m_2+1}^{m} \ln \frac{y_k + (j-1)L_k}{y_1 + jL_1} \right)
= x_1 d(m_2) \exp \left( \sum_{j=m_2+1}^{m} \ln \left( 1 + \frac{y_k - L_1 - y_1}{jL_1} + O \left( \frac{1}{j^2} \right) \right) \right)
= x_1 d(m_2) \exp \left( \sum_{j=m_2+1}^{m} \left( \frac{y_k - L_1 - y_1}{jL_1} + O \left( \frac{1}{j^2} \right) \right) \right),
\]
where
\[
d(m_2) = \prod_{j=1}^{m} \frac{y_k + (j-1)L_k}{y_1 + jL_1}. \tag{7.27}
\]

Letting \( m \to \infty \) in (7.26), using (7.25) and the fact that the series \( \sum_{j=m_2+1}^{\infty} O(1/j^2) \) converges, we get statements (f)–(h), as desired.

\[ 7.3. \text{Case } q \neq \pm 1 \]

If \( q \neq \pm 1 \), then from (7.6) we get
\[
y_{km+i} = q^m s_i + t_i, \quad m \in \mathbb{N}_0, \tag{7.28}
\]
where
\[
s_i = y_i + \frac{L_i}{q-1}, \quad t_i = \frac{L_i}{1-q}, \quad i = 1, k, \tag{7.29}
\]
from (4.6) it follows that
\[
x_{km+i} = x_1 \prod_{j=1}^{m} \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i}, \quad m \in \mathbb{N}_0, \tag{7.30}
\]
for \( i \in \{2, \ldots, k\} \), and

\[
x_{km+1} = \frac{x_{1}y_{k}}{qs_{1} + t_{1}} \prod_{j=2}^{m} \frac{q^{-1}s_{k} + t_{k}}{q^{j}s_{1} + t_{1}}, \quad m \in \mathbb{N}_{0}.
\] (7.31)

Note that \( t_{i} = t_{j} \), if \( i \equiv j \pmod{k} \).

**Corollary 7.3.** If \( 0 < |q| < 1, \alpha \neq 0, \) and \( q^{i}s_{i} + t_{i} \neq 0 \), for every \( j \in \mathbb{N}_{0} \) and \( i \in \{1, \ldots, k\} \), then the following statements hold true.

(a) If \( |t_{i-1}| < |t_{i}| \) for some \( i \in \{1, \ldots, k\} \), we have that \( x_{km+1} \to 0 \) as \( m \to \infty \).

(b) If \( |t_{i-1}| > |t_{i}| \) and \( s_{i} \neq 0 \) if \( t_{i} = 0 \) for some \( i \in \{1, \ldots, k\} \), we have that \( |x_{km+1}| \to \infty \) as \( m \to \infty \), if \( x_{1} \neq 0 \).

(c) If \( t_{i-1} = t_{i} \neq 0 \) for some \( i \in \{1, \ldots, k\} \), then \( x_{km+1} \) is convergent.

(d) If \( t_{i-1} = t_{i} = 0 \) and \( |s_{i-1}| < |s_{i}| \) for some \( i \in \{2, \ldots, k\} \), then \( |x_{km+1}| \to 0 \) as \( m \to \infty \).

(e) If \( t_{1} = t_{k} = 0 \) and \( |s_{k}| < |qs_{1}| \), then \( x_{km+1} \to 0 \) as \( m \to \infty \).

(f) If \( t_{i-1} = t_{i} = 0 \) and \( |s_{i-1}| > |s_{i}| \) for some \( i \in \{2, \ldots, k\} \), then \( |x_{km+1}| \to \infty \) as \( m \to \infty \), if \( x_{1} \neq 0 \).

(g) If \( t_{1} = t_{k} = 0 \) and \( |s_{k}| > |qs_{1}| \), then \( |x_{km+1}| \to \infty \) as \( m \to \infty \), if \( x_{1} \neq 0 \).

(h) If \( t_{i-1} = t_{i} = 0 \) and \( s_{i-1} = s_{i} \neq 0 \) for some \( i \in \{2, \ldots, k\} \), then \( x_{km+1} \) is constant.

(i) If \( t_{1} = t_{k} = 0 \) and \( s_{k} = qs_{1} \neq 0 \), then \( x_{km+1} \) is constant.

(j) If \( t_{i-1} = t_{i} = 0 \) and \( s_{i-1} = -s_{i} \neq 0 \) for some \( i \in \{2, \ldots, k\} \), then \( x_{km+1} = (-1)^{m}x_{i} \).

(k) If \( t_{1} = t_{k} = 0 \) and \( s_{k} = -qs_{1} \neq 0 \), then \( x_{km+1} = x_{1}y_{k}(-1)^{m-1}/(qs_{1}) \).

(l) If \( t_{i-1} = -t_{i} \neq 0 \), for some \( i \in \{1, \ldots, k\} \), then the subsequences \( x_{2km+i} \) and \( x_{2km+k+i} \) are convergent.

**Proof.** Since we have that

\[
\lim_{j \to \infty} \frac{q^{i}s_{i-1} + t_{i-1}}{q^{j}s_{i} + t_{i}} = \frac{t_{i-1}}{t_{i}}, \quad i \in \{2, \ldots, k\}
\] (7.32)

when \( t_{i} \neq 0 \),

\[
\lim_{j \to \infty} \left| \frac{q^{i}s_{i-1} + t_{i-1}}{q^{j}s_{i} + t_{i}} \right| = +\infty, \quad i \in \{2, \ldots, k\}
\] (7.33)

when \( |t_{i-1}| > t_{i} = 0 \) and \( s_{i} \neq 0 \),

\[
\lim_{j \to \infty} \frac{q^{j-1}s_{k} + t_{k}}{q^{j}s_{1} + t_{1}} = \frac{t_{k}}{t_{1}},
\] (7.34)
when $t_1 \neq 0$, and

$$\lim_{j \to \infty} \left| \frac{q^{j-1}s_k + t_k}{q^js_1 + t_1} \right| = +\infty,$$  \hspace{1cm} (7.35)

when $|t_k| > t_1 = 0$ and $s_1 \neq 0$, the statements in (a) and (b) easily follow from (7.30)--(7.35).

(c) If $t_{i-1} = t_i \neq 0$, then

$$x_{km+i} = x_i \prod_{j=1}^{m} \left( 1 + q^j \left( \frac{s_{i-1} - s_i}{t_i} \right) + o\left( q^j \right) \right),$$  \hspace{1cm} (7.36)

for $i \in \{2, \ldots, k\}$, and if $t_1 = t_k \neq 0$, then

$$x_{km+1} = \frac{x_1 y_k}{qs_1 + t_1} \prod_{j=2}^{m} \left( 1 + q^{j-1} \left( \frac{s_k - q s_1}{t_1} \right) + o\left( q^j \right) \right)$$  \hspace{1cm} (7.37)

from which the statement in (c) easily follows.

(d)–(k) If $t_{i-1} = t_i = 0$, then

$$x_{km+i} = x_i \prod_{j=1}^{m} \left( \frac{s_{i-1}}{s_i} \right),$$  \hspace{1cm} (7.38)

for $i \in \{2, \ldots, k\}$, and if $t_1 = t_k = 0$, then

$$x_{km+1} = \frac{x_1 y_k}{qs_1} \prod_{j=2}^{m} \frac{s_k}{qs_1}$$  \hspace{1cm} (7.39)

from which the statements in (d)–(k) easily follow.

(l) If $t_{i-1} = -t_i \neq 0$, then we have that

$$x_{km+i} = x_i \prod_{j=1}^{m} \left[ - \left( 1 - q^j \left( \frac{s_{i-1} + s_i}{t_i} \right) + o\left( q^j \right) \right) \right]$$  \hspace{1cm} (7.40)

for $i \in \{2, \ldots, k\}$, and if $t_1 = -t_k \neq 0$, then

$$x_{km+1} = \frac{x_1 y_k}{qs_1 + t_1} \prod_{j=2}^{m} \left[ - \left( 1 - q^{j-1} \left( \frac{s_k + q s_1}{t_1} \right) + o\left( q^j \right) \right) \right]$$  \hspace{1cm} (7.41)

from which the statement in (l) easily follows.
Corollary 7.4. If \( |q| > 1 \) and \( \alpha \neq 0 \), and \( q^j s_i + t_i \neq 0 \), for every \( j \in \mathbb{N}_0 \) and \( i \in \{1, \ldots, k\} \), then the following statements hold true.

(a) If \( |s_{i-1}| < |s_i| \), for some \( i \in \{2, \ldots, k\} \), then \( x_{km+i} \to 0 \) as \( m \to \infty \).

(b) If \( |s_k| < |qs_1| \), then \( x_{km+1} \to 0 \) as \( m \to \infty \).

(c) If \( |s_{i-1}| > |s_i| \), or \( s_i = 0 \), \( s_{i-1} \neq 0 \) and \( t_i \neq 0 \), for some \( i \in \{2, \ldots, k\} \), then \( |x_{km+i}| \to \infty \) as \( m \to \infty \), if \( x_i \neq 0 \).

(d) If \( |s_k| > |qs_1| \), or if \( s_1 = 0 \), \( s_k \neq 0 \) and \( t_1 \neq 0 \), then \( |x_{km+1}| \to \infty \) as \( m \to \infty \), if \( x_1 \neq 0 \).

(e) If \( s_{i-1} = s_i \neq 0 \), for some \( i \in \{2, \ldots, k\} \), then the sequence \( (x_{km+i})_{m \in \mathbb{N}_0} \) is convergent.

(f) If \( s_{i-1} = s_i = 0 \) and \( |t_{i-1}| < |t_i| \) for some \( i \in \{2, \ldots, k\} \), then \( x_{km+i} \to 0 \) as \( m \to \infty \).

(g) If \( s_1 = s_k = 0 \) and \( |t_k| < |t_1| \), then \( x_{km+1} \to 0 \) as \( m \to \infty \).

(h) If \( s_{i-1} = s_i = 0 \) and \( |t_{i-1}| > |t_i| \) for some \( i \in \{2, \ldots, k\} \), then \( |x_{km+i}| \to +\infty \) as \( m \to \infty \), if \( x_i \neq 0 \).

(i) If \( s_1 = s_k = 0 \) and \( |t_k| > |t_1| \), then \( |x_{km+1}| \to +\infty \) as \( m \to \infty \), if \( x_1 \neq 0 \).

(j) If \( s_{i-1} = s_i = 0 \) and \( t_{i-1} = t_i \) for some \( i \in \{2, \ldots, k\} \), then the sequence \( x_{km+i} \) is constant.

(k) If \( s_1 = s_k = 0 \) and \( t_1 = t_k \), then the sequence \( x_{km+1} \) is constant.

(l) If \( s_{i-1} = s_i = 0 \) and \( t_{i-1} = -t_i \) for some \( i \in \{2, \ldots, k\} \), then the sequence \( x_{km+i} \) is two-periodic.

(m) If \( s_1 = s_k = 0 \) and \( t_1 = -t_k \), then the sequence \( x_{km+1} \) is two-periodic.

(n) If \( s_k = qs_1 \neq 0 \), then the sequence \( (x_{km+i})_{m \in \mathbb{N}_0} \) is convergent.

(o) If \( s_{i-1} = -s_i \neq 0 \), for some \( i \in \{2, \ldots, k\} \), then the sequences \( (x_{2km+i})_{m \in \mathbb{N}_0} \) and \( (x_{2km+k+i})_{m \in \mathbb{N}_0} \) are convergent.

(p) If \( s_k = -qs_1 \neq 0 \), then the sequences \( (x_{2km+i})_{m \in \mathbb{N}_0} \) and \( (x_{2km+k+i})_{m \in \mathbb{N}_0} \) are convergent.

Proof. (a)–(d) These statements follow correspondingly from the next relations (which are derived using formulae (7.30) and (7.31)):

\[
\lim_{j \to \infty} \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} = \frac{s_{i-1}}{s_i} \quad (7.42)
\]

for \( i \in \{2, \ldots, k\} \) if \( s_i \neq 0 \), and

\[
\lim_{j \to \infty} \left| \frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} \right| = +\infty \quad (7.43)
\]

for \( i \in \{2, \ldots, k\} \) if \( s_i = 0 \), \( s_{i-1} \neq 0 \) and \( t_i \neq 0 \);

\[
\lim_{j \to \infty} \frac{q^{-1} s_k + t_k}{q^j s_i + t_i} = \frac{s_k}{qs_1} \quad (7.44)
\]
if $s_1 \neq 0$, and
\[
\lim_{j \to \infty} \left| \frac{q^{j-1}s_k + t_k}{q^j s_1 + t_1} \right| = +\infty \tag{7.45}
\]

if $s_1 = 0$, $s_k \neq 0$ and $t_1 \neq 0$.

(e) If $s_{i-1} = s_i \neq 0$, then from (7.30) we get
\[
x_{km+i} = x_i \prod_{j=1}^m \left( 1 + q^{-j} \left( \frac{t_{i-1} - t_i}{s_{i-1}} \right) + o\left(q^{-j}\right) \right), \tag{7.46}
\]

for $i \in \{2, \ldots, k\}$, from which (e) follows.

(f)–(m) If $s_{i-1} = s_i = 0$ for some $i \in \{2, \ldots, k\}$, then for $i \in \{2, \ldots, k\}$ we have
\[
\frac{q^j s_{i-1} + t_{i-1}}{q^j s_i + t_i} = \frac{t_{i-1}}{t_i} \tag{7.47}
\]

while when $s_1 = s_k = 0$, we have
\[
\frac{q^{i-1}s_k + t_k}{q^i s_1 + t_1} = \frac{t_k}{t_1} \tag{7.48}
\]

from which the statements (f)–(i) easily follow.

(n) If $s_k = q s_1 \neq 0$, then we have
\[
x_{km+i} = \frac{x_i y_k}{q s_1 + t_1} \prod_{j=2}^m \left( 1 + q^{-j} \left( \frac{t_k - t_1}{s_1} \right) + o\left(q^{-j}\right) \right), \tag{7.49}
\]

from which along with the assumption $|q| > 1$ the statement follows.

(o) and (p) If $s_{i-1} = -s_i \neq 0$, then
\[
x_{km+i} = x_i \prod_{j=1}^m \left[ - \left( 1 - q^{-j} \left( \frac{t_{i-1} + t_i}{s_i} \right) + o\left(q^{-j}\right) \right) \right] \tag{7.50}
\]

for $i \in \{2, \ldots, k\}$, and
\[
x_{km+1} = \frac{x_i y_k}{q s_1 + t_1} \prod_{j=2}^m \left[ - \left( 1 - q^{-j} \left( \frac{t_k + t_1}{s_1} \right) + o\left(q^{-j}\right) \right) \right]. \tag{7.51}
\]

From (7.50) and (7.51) the statements in (o) and (p) correspondingly follow. \qed
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