Research Article

The Expression of the Generalized Drazin Inverse of $A - CB$

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Received 5 August 2011; Accepted 5 December 2011

Academic Editor: Ondřej Došlý

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We investigate the generalized Drazin inverse of $A - CB$ over Banach spaces stemmed from the Drazin inverse of a modified matrix and present its expressions under some conditions.

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. We denote the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. In particular, we write $\mathcal{B}(\mathcal{X})$ instead of $\mathcal{B}(\mathcal{X}, \mathcal{X})$.

For any $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represent its range and null space, respectively. If $A \in \mathcal{B}(\mathcal{X})$, the symbols $\sigma(A)$ and $\text{acc}(\sigma(A))$ stand for its spectrum and the set of all accumulation points of $\sigma(A)$, respectively.

Recall the concept of the generalized Drazin inverse introduced by Koliha [1] that the element $T_d \in \mathcal{B}(\mathcal{X})$ is called the generalized Drazin inverse of $T \in \mathcal{B}(\mathcal{X})$ provided it satisfies

$$TT_d = T_dT, \quad T_dTT_d = T_d, \quad T - T^2T_d \text{ is quasinilpotent.} \quad (1.1)$$

If it exists then it is unique. The Drazin index $\text{Ind}(T)$ of T is the least positive integer k if $(T - T^2T_d)^k = 0$, and otherwise $\text{Ind}(T) = +\infty$.

From the definition of the generalized Drazin inverse, it is easy to see that if T is a quasinilpotent operator, then T_d exists and $T_d = 0$. It is well known that the generalized Drazin inverse of $T \in \mathcal{B}(\mathcal{X})$ exists if and only if $0 \notin \text{acc}(\sigma(T))$ (see [1, Theorem 4.2]).

If T is generalized Drazin invertible, then the spectral idempotent T^π of T corresponding to 0 is given by $T^\pi = I - TT^d$.

The generalized Drazin inverse is widely investigated because of its applications in singular differential difference equations, Markov chains, (semi-) iterative method numerical analysis (see, for example, [1–5, 7], and references therein).

In this paper, we aim to discuss the generalized Drazin inverse of $A - CB$ over Banach spaces. This question stems from the Drazin inverse of a modified matrix (see, e.g., [6]). In [3], Deng studied the generalized Drazin inverse of $A - CB$. Here we research the problem under more general conditions than those in [3]. Our results extend the relative results in [3, 4].

In this section, we will list some lemmas. In next section, we will present the expressions of the generalized Drazin inverse of $A - CB$. In final section, we illustrate a simple example.

Lemma 1.1 (see [4, Theorem 2.3]). *Let $A, B \in \mathcal{B}(\mathcal{X})$ be the generalized Drazin invertible. If $AB = 0$, then $A + B$ is generalized Drazin invertible and*

$$(A + B)_d = B^\pi \sum_{n=0}^{\infty} B^n A_d^{n+1} + \left(\sum_{n=0}^{\infty} B_d^{n+1} A^n \right) A^\pi. \quad (1.2)$$

Lemma 1.2 (see [7, Theorem 5.1]). *If $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ are generalized Drazin invertible and $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, then*

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad (1.3)$$

is also generalized Drazin invertible and

$$M^d = \begin{pmatrix} A^d & S \\ O & B^d \end{pmatrix}, \quad (1.4)$$

where

$$S = A_d^2 \left(\sum_{n=0}^{\infty} A_d^n C B^n \right) B^\pi + A^\pi \left(\sum_{n=0}^{\infty} A^n C B_d^n \right) B_d^2 - A_d C B_d. \quad (1.5)$$

2. Main Results

We start with our main result.

Theorem 2.1. Let $A \in \mathcal{B}(\mathcal{X})$ be the generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $BP = 0$. If $R = (I - P)(A - CB)$ and AP are generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and

$$\begin{aligned} (A - CB)_d &= \left[\sum_{n=0}^{\infty} (AP)_d^{n+1} (R^n + VR^{n-1} + V^2R^{n-2}) \right] R^\pi \\ &\quad - (AP)_d \left[VR_d + V^2R_d^2 + (AP)_d V^2R_d \right] \\ &\quad + (AP)^\pi \sum_{n=0}^{\infty} (AP)^n (R_d^{n+1} + VR_d^{n+2} + V^2R_d^{n+3}), \end{aligned} \quad (2.1)$$

where $V = PA - PCB - AP$ and the symbols $V^i R^j = 0, i = 1, 2$, if $j < 0$.

Proof. Let $S := AP$ and $T := (A - CB)(I - P)$. Then

$$TS = (A - CB)(I - P)AP = 0, \quad (2.2)$$

$$RP = (I - P)(A - CB)P = 0, \quad (2.3)$$

$$A - CB = AP + A(I - P) - CB(I - P) = S + T \quad (2.4)$$

since $AP = PAP$ and $BP = 0$. So, by Lemma 1.1,

$$(T + S)_d = S^\pi \sum_{n=0}^{\infty} S^n T_d^{n+1} + \sum_{n=0}^{\infty} S_d^{n+1} T^n T^\pi. \quad (2.5)$$

Next, we will give the representations of T_d , T^n , and T_d^n . In order to obtain the expression of T_d , rewrite T as

$$T = R + PA - PCB - PAP = R + V. \quad (2.6)$$

Since $VP = PAP - AP^2 = PAP(I - P)$,

$$V^2P = (PA - PCB - AP)PAP(I - P) = (PAPAP - APPAP)(I - P) = 0, \quad (2.7)$$

and then $V^n = 0$ for $n > 2$ since $V = PA - CB - AP$. So V_d exists and $V_d = 0$. By (2.3), $RV = RP(A - CB - AP) = 0$ and then $R_dV = R_dR_dRV = 0$. So, by Lemma 1.1,

$$T_d = (R + V)_d = R_d + VR_d^2 + V^2R_d^3, \quad (2.8)$$

and then

$$TT_d = RR_d + VR_d + V^2R_d^2. \quad (2.9)$$

Since $R(R + V)^k = R^{k+1}$ and $V^2(R + V)^k = V^2R^k$ for $k \geq 1$,

$$T^n = (R + V)^n = (R^2 + VR + V^2)(R + V)^{n-2} = R^n + VR^{n-1} + V^2R^{n-2}, \quad n \geq 2. \quad (2.10)$$

From $R_dV = 0$, it is easy to verify that

$$T_d^n = (R_d + VR_d^2 + V^2R_d^3)^n = R_d^n + VR_d^{n+1} + V^2R_d^{n+2}. \quad (2.11)$$

Hence,

$$\begin{aligned} \left(\sum_{n=0}^{\infty} S_d^{n+1} T^n \right) T^\pi &= (AP)_d \left[I + (AP)_d(R + V) + (AP)_d^2(R^2 + VR + V^2) \right] \\ &\quad \times (R^\pi - VR_d - V^2R_d^2) + \sum_{n=3}^{\infty} (AP)_d^{n+1} (R^n + VR^{n-1} + V^2R^{n-2}) R^\pi \\ &= (AP)_d \left[I + (AP)_d(R + V) + (AP)_d^2(R^2 + VR + V^2) \right] R^\pi \\ &\quad - (AP)_d (VR_d + V^2R_d^2 + (AP)_d V^2R_d) \\ &\quad + \sum_{n=3}^{\infty} (AP)_d^{n+1} (R^n + VR^{n-1} + V^2R^{n-2}) R^\pi, \\ S^\pi \sum_{n=0}^{\infty} S^n T_d^{n+1} &= (AP)^\pi \sum_{n=0}^{\infty} (AP)^n (R_d^{n+1} + VR_d^{n+2} + V^2R_d^{n+3}). \end{aligned} \quad (2.12)$$

Therefore, we reach (2.1). □

When $\text{Ind}(AP)$, $\text{Ind}(R) < +\infty$, we have the following corollary.

Corollary 2.2. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible. $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $BP = 0$. If $R = (I - P)(A - CB)$ and AP are generalized Drazin invertible and $\text{Ind}(R) = k < +\infty$ and $\text{Ind}(AP) = h < +\infty$, then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned} (A - CB)_d &= \left[\sum_{n=0}^{k-1} (AP)_d^{n+1} (R^n + VR^{n-1} + V^2R^{n-2}) \right] R^\pi \\ &\quad - (AP)_d [VR_d + V^2R_d^2 + (AP)_d V^2R_d] \\ &\quad + (AP)^\pi \sum_{n=0}^{h-1} (AP)^n (R_d^{n+1} + VR_d^{n+2} + V^2R_d^{n+3}), \end{aligned} \quad (2.13)$$

where $V = PA - PCB - AP$ and the symbols $V^i R^j = 0$, $i = 1, 2$, if $j < 0$.

If an operator T is quasinilpotent, $T_d = 0$ and $T^\pi = I$. So, the following corollary follows from Theorem 2.1.

Corollary 2.3. Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $BP = 0$. If $R = (I - P)(A - CB)$ is generalized Drazin invertible and AP is a quasinilpotent operator, then $A - CB$ is generalized Drazin invertible and

$$(A - CB)_d = \sum_{n=0}^{\infty} (AP)^n \left(R_d^{n+1} + V R_d^{n+2} + V^2 R_d^{n+3} \right), \quad (2.14)$$

where $V = PA - PCB - AP$.

Theorem 2.4. Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$ and $BP = B$. If $R = P(A - CB)$ is generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and

$$\begin{aligned} (A - CB)_d &= R_d + A_d(I - P) + \sum_{n=0}^{\infty} A_d^{n+2}(I - P)(A - CB)P(A - CB)^n R^\pi \\ &\quad + A^\pi \sum_{n=0}^{\infty} A^n(I - P)(A - CB)P R_d^{n+2} - A_d(I - P)(A - CB)R_d. \end{aligned} \quad (2.15)$$

Proof. Since $P^2 = P$, we have $\mathcal{X} = \mathcal{R}(P) \oplus \mathcal{N}(P)$ and can write P in the following matrix form:

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.16)$$

The condition $PA = PAP$, therefore, yields the matrix form of A as follows:

$$A = \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix}. \quad (2.17)$$

From $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$ and the hypothesis that A_d exists, $A_1 \in \mathcal{B}(\mathcal{R}(P))$ and $A_2 \in \mathcal{B}(\mathcal{N}(P))$ are generalized Drazin invertible since $0 \notin \text{acc}(\sigma(A))$ if and only if $0 \notin \text{acc}(\sigma(A_1))$ and $0 \notin \text{acc}(\sigma(A_2))$. And, by Lemma 1.2,

$$A_d = \begin{pmatrix} A_1^d & 0 \\ W & A_2^d \end{pmatrix}, \quad (2.18)$$

where W is some operator. Since

$$A(I - P) = \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}, \quad (2.19)$$

$(A(I - P))_d$ exists and

$$(A(I - P))_d = \begin{pmatrix} 0 & 0 \\ 0 & A_2^d \end{pmatrix} = A_d(I - P). \quad (2.20)$$

To use Theorem 2.1 to complete the proof, let $Q = (I - P)$. So $R = (I - Q)(A - CB)$ and AQ are generalized Drazin invertible. And from the conditions $PA = PAP$ and $BP = B$, we can obtain $AQ = QAQ$ and $BQ = 0$. Thus, by Theorem 2.1, we have

$$\begin{aligned} (A - CB)_d &= (AQ)_d R^\pi + (AQ)_d^2 (R + V) R^\pi + \left[\sum_{n=2}^{\infty} (AQ)_d^{n+1} (R^n + V R^{n-1} + V^2 R^{n-2}) \right] R^\pi \\ &\quad - (AQ)_d^d [V R_d + V^2 R_d^2 + (AQ)_d V^2 R_d] + (AQ)^\pi (R_d + V R_d^2 + V^2 R_d^3) \\ &\quad + (AQ)^\pi \sum_{n=1}^{\infty} (AP)^n (R_d^{n+1} + V R_d^{n+2} + V^2 R_d^{n+3}), \end{aligned} \quad (2.21)$$

where $V = QA - QCB - AQ$.

Since $P^2 = P$ and $Q^2 = Q$ and then $VQ = 0$ and $V = QV$. So $V^2 = 0$. Note that $QR = 0$ and then $QR_d = 0$ and $(AQ)_d R = 0$. Thus it follows from (2.21) that

$$\begin{aligned} (A - CB)_d &= (AQ)_d + (AQ)_d^2 V R^\pi + \left[\sum_{n=2}^{\infty} (AQ)_d^{n+1} V R^{n-1} \right] R^\pi - (AQ)_d V R_d \\ &\quad + R_d + (AQ)^\pi V R_d^2 + (AQ)^\pi \sum_{n=1}^{\infty} (AQ)^n V R_d^{n+2} \\ &= (AQ)_d + \left[\sum_{n=0}^{\infty} (AQ)_d^{n+2} V R^n \right] R^\pi - (AQ)_d V R_d + R_d \\ &\quad + (AQ)^\pi \sum_{n=0}^{\infty} (AQ)^n V (R_d)^{n+2}. \end{aligned} \quad (2.22)$$

Since $V = Q(A - CB) - (A - CB)Q = (A - CB)(I - Q) - (I - Q)(A - CB)$, $VR = Q(A - CB)R$ and $QV = Q(A - CB)(I - Q)$. Note that $R^n = P(A - CB)^n$ and $(AQ)^n = A^n Q$. Substituting V and $Q = I - P$ in (2.22) yields (2.15). $\square$

Adding the condition $PC = C$ in Theorem 2.4 yields a result below.

Corollary 2.5. *Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Z}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$, $BP = B$, and $PC = C$. If $R = P(A - CB)$ is generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and*

$$\begin{aligned} (A - CB)_d &= R_d + A_d(I - P) + \sum_{n=0}^{\infty} A_d^{n+2} (I - P) A P (A - CB)^n R^\pi \\ &\quad + A^\pi \sum_{n=0}^{\infty} A^n (I - P) A P R_d^{n+2} - A_d(I - P) A R_d. \end{aligned} \quad (2.23)$$

Adding the condition $PC = 0$ in Theorem 2.4 yields $R = PA$. So similar to the proof of $(A(I - P))^d = A^d(I - P)$ in Theorem 2.4, we can gain $(PA)^d = PA^d$.

Corollary 2.6. Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Z}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$, $BP = B$, and $PC = 0$; then $A - CB$ is generalized Drazin invertible and

$$(A - CB)_d = A_d + \sum_{n=0}^{\infty} A_d^{n+2} (I - P)(A - CB)PA^n A^\pi + A^\pi \sum_{n=0}^{\infty} A^n (I - P)(A - CB)PA_d^{n+2} - A_d(I - P)(A - CB)PA_d. \quad (2.24)$$

Analogously, we can deduce Theorem 2.7 and Corollary 2.9 below.

Theorem 2.7. Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $PC = C$. If $R = (A - CB)P$ is generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and

$$(A - CB)_d = R_d + (I - P)A_d + \sum_{n=0}^{\infty} R_d^{n+2} P(A - CB)(I - P)A^n A^\pi + R^\pi \sum_{n=0}^{\infty} (A - CB)^n P(A - CB)(I - P)A_d^{n+2} - R_d(A - CB)(I - P)A_d. \quad (2.25)$$

Remark 2.8 (see [4, Theorem 2.4]). It is a special case of Theorem 2.7.

Corollary 2.9. Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Z}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$, $PC = C$, and $BP = 0$; then $A - CB$ is generalized Drazin invertible and

$$(A - CB)_d = A_d + \sum_{n=0}^{\infty} A_d^{n+2} P(A - CB)(I - P)A^n A^\pi + A^\pi \sum_{n=0}^{\infty} A^n P(A - CB)(I - P)A_d^{n+2} - A_d P(A - CB)(I - P)A_d. \quad (2.26)$$

Similar to Theorem 2.1 and Corollary 2.2, we can show the following two results.

Theorem 2.10. Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$ and $PC = 0$. If $R = (A - CB)(I - P)$ and PA are generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and

$$(A - CB)_d = R^\pi \sum_{n=0}^{\infty} \left(R^n + R^{n-1}V + R^{n-2}V^2 \right) (PA)_d^{n+1} - \left[R_d V + R_d^2 V^2 + R_d V^2 (PA)_d \right] (PA)_d + \left[\sum_{n=0}^{\infty} \left(R_d^{n+1} + R_d^{n+2} V + R_d^{n+3} V^2 \right) (PA)^n \right] (PA)^\pi, \quad (2.27)$$

where $V = AP - CBP - PA$ and the symbols $R^i V^j = 0$, $j = 1, 2$, if $i < 0$.

Corollary 2.11. Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible. $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$ and $PC = 0$. If $R = (A - CB)(I - P)$ and PA are generalized Drazin invertible and $\text{Ind}(R) = k < +\infty$ and $\text{Ind}(PA) = h < +\infty$, then $A - CB$ is generalized Drazin invertible and

$$\begin{aligned} (A - CB)_d &= R^\pi \sum_{n=0}^{k-1} (R^n + R^{n-1}V + R^{n-2}V^2)(PA)_d^{n+1} \\ &\quad - [R_d V + R_d^2 V^2 + R_d V^2 (PA)_d] (PA)_d^d \\ &\quad + \left[\sum_{n=0}^{h-1} (R_d^{n+1} + R_d^{n+2}V + R_d^{n+3}V^2)(PA)^n \right] (PA)^\pi, \end{aligned} \quad (2.28)$$

where $V = AP - CBP - PA$ and the symbols $R^i V^j = 0$, $j = 1, 2$, if $i < 0$.

When $PA = AP$ and $P^2 = P$ in Theorem 2.10, we can obtain the following result since $R^n = (A - CB)^n(I - P)$.

Corollary 2.12 (see [3, Theorem 4.3]). Let $A \in \mathcal{B}(\mathcal{X})$ be the generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ commuting with A such that $PC = 0$. If $R = (A - CB)(I - P)$ is generalized Drazin invertible, then $A - CB$ is the generalized Drazin invertible and

$$(A - CB)_d = R_d + PA_d - R_d V A_d + R^\pi \sum_{n=0}^{\infty} (A - CB)^n V A_d^{n+2} + \sum_{n=0}^{\infty} R_d^{n+2} V A^n A^\pi, \quad (2.29)$$

where $V = -CBP$.

3. Example

Before ending this paper, we give an example as follows.

Example 3.1. Let

$$A = \begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = (0 \ 0 \ 0 \ 1), \quad C = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \quad (3.1)$$

Then

$$CB = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A - CB = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.2)$$

We will compute the Drazin inverse of $A - CB$. To do this, we choose the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.3)$$

Apparently, P is not idempotent and $PA \neq AP$. But $BP = 0$ and

$$AP = PAP = \begin{pmatrix} 1 & -2 & 8 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.4)$$

Obviously, $\text{Ind}(AP) = 2$. Computing

$$R = (I - P)(A - CB) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.5)$$

$$V = PA - PCB - AP = \begin{pmatrix} 0 & 4 & -4 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.6)$$

we have $\text{Ind}(R) = 2$. So, by Corollary 2.2,

$$(A - CB)_d = \begin{pmatrix} 1 & -4 & 10 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.7)$$

Acknowledgments

The authors would like to thank the referees for their helpful comments and suggestions. This work was supported by the National Natural Science Foundation of China (11061005), the Ministry of Education Science and Technology Key Project under Grant no. 210164, and Grants HCIC201103 of Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis Open Fund.

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