Research Article

A Godunov-Mixed Finite Element Method on Changing Meshes for the Nonlinear Sobolev Equations

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A Godunov-mixed finite element method on changing meshes is presented to simulate the nonlinear Sobolev equations. The convection term of the nonlinear Sobolev equations is approximated by a Godunov-type procedure and the diffusion term by an expanded mixed finite element method. The method can simultaneously approximate the scalar unknown and the vector flux effectively, reducing the continuity of the finite element space. Almost optimal error estimates in $L^2$-norm under very general changes in the mesh can be obtained. Finally, a numerical experiment is given to illustrate the efficiency of the method.

1. Introduction

We consider the following nonlinear Sobolev equations:

$$
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot (a(x,t,u) \nabla u) + b(x,t,u) \cdot \nabla u &= f(x,t,u), & x \in \Omega, \ t \in (0,T], \\
\frac{\partial u}{\partial t} &= 0, & x \in \partial \Omega, \ t \in [0,T], \\
\frac{\partial u}{\partial t} &= u_0(x), & x \in \Omega,
\end{align*}
$$

where $\Omega$ is a bounded subset of $\mathbb{R}^n$ ($n \leq 3$) with smooth boundary $\partial \Omega$, $u_0(x)$ and $f(x,t,u)$ are known functions, and the coefficients $a(x,t,u), b(x,t,u), c(x,t,u) = (c_1(x,t,u), c_2(x,t,u))^T$ satisfy the following condition:

$$
0 < a_0 \leq a(x,t,u) \leq a_1, \quad \left| \frac{\partial a(x,t,u)}{\partial t} \right| \leq K_1, \\
0 < b_0 \leq b(x,t,u) \leq b_1,
$$
Section 3, we introduce three projections and a lemma. We derive almost optimal error the continuity of the finite element space. We describe this method in Section 2. In 1993, he expanded this method to multidimensions.

In advective flow problems in one space dimension by high-order Godunov-mixed conservation of mass with numerical stability and minimal numerical diffusion. The convection term of the nonlinear Sobolev equations is approximated by a Godunov-type procedure and the diffusion term by an expanded mixed finite element method. This method can simultaneously approximate the scalar unknown and the vector flux effectively, reducing the continuity of the finite element space. We describe this method in Section 2.

The object of this paper is to present a Godunov-mixed finite element method on changing meshes for the nonlinear Sobolev equations. The convection term \( \nabla \cdot u \) of the nonlinear Sobolev equations is approximated by a Godunov-type procedure and the diffusion term by an expanded mixed finite element method. This method can simultaneously approximate the scalar unknown and the vector flux effectively, reducing the continuity of the finite element space. We describe this method in Section 2.

\[ |c(x,t,u)| = \sqrt{c_1^2(x,t,u) + c_2^2(x,t,u)} \leq c_0, \quad \left| \frac{\partial c(x,t,u)}{\partial u} \right| \leq K_2, \]
\[ a(x,t,u), b(x,t,u), c(x,t,u), f(x,t,u), \frac{\partial c(x,t,u)}{\partial u} \text{ are Lipschitz continuous about } u, \]

(1.2)

where \( a_0, a_1, b_0, b_1, c_0, K_1, \) and \( K_2 \) are positive constants. We assume that \( u(x,t) \) satisfy the smooth condition in the following analysis.

For time-changing localized phenomena, such as sharp fronts and layers, the finite element method on changing meshes [1–3] is advantageous over fixed finite element method. The reason is that the former treats the problem with the finite element method on space domain by using different meshes and different basic functions at different time levels so that it has the capability of self-adaptive local grid modification (refinement or unrefinement) to efficiently capture propagating fronts or moving layers. The work [4] had combined this method with mixed finite element method to study parabolic problems. In [5], an upwind-mixed method on changing meshes was considered for two-phase miscible flow in porous media.

Sobolev equations have important applications in many mathematical and physical problems, such as the percolation theory when the fluid flows through the cracks [6], the transfer problem of the moisture in the soil [7], and the heat conduction problem in different materials [8]. So there exists great and actual significance to research Sobolev equations. Many works had researched on numerical treatments for Sobolev equations. More attentions were paid for treating a damping term \( \nabla \cdot [a \nabla u] \), which is a distinct character of Sobolev equations different from parabolic equation. For example, time stepping Galerkin method was presented for nonlinear Sobolev equations in [9, 10]. In [11, 12], nonlinear Sobolev equations with convection term were researched by using finite difference streamline-diffusion method and discontinuous Galerkin method, respectively. Two new least-squares mixed finite element procedures were formulated for solving convection-dominated Sobolev equations in [13].

Methods which combined Godunov-type schemes for advection with mixed finite elements for diffusion were introduced in [14] and had been applied to flow problems in reservoir engineering, contaminant transport, and computational fluid dynamics. Applications of these types of methods to single and two-phase flow in oil reservoirs were discussed in [15, 16]; application to the Navier-Stokes equations was given in [17]. These methods had proven useful for advective flow problems because they combined element-by-element conservation of mass with numerical stability and minimal numerical diffusion. Dawson [18] researched advective flow problems in one space dimension by high-order Godunov-mixed method. In 1993, he expanded this method to multidimensions [19] and presented three variations. In these methods, advection was approximated by a Godunov-type procedure, and diffusion was approximated by a low-order-mixed finite element method.

The object of this paper is to present a Godunov-mixed finite element method on changing meshes for the nonlinear Sobolev equations. The convection term \( \nabla \cdot u \) of the nonlinear Sobolev equations is approximated by a Godunov-type procedure and the diffusion term by an expanded mixed finite element method. This method can simultaneously approximate the scalar unknown and the vector flux effectively, reducing the continuity of the finite element space. We describe this method in Section 2. In Section 3, we introduce three projections and a lemma. We derive almost optimal error
estimates in $L^2$-norm under very general changes in the mesh in Section 4. In Section 5, we present results of numerical experiment, which confirm our theoretical results.

Throughout the analysis, the symbol $K$ will denote a generic constant, which is independent of mesh parameters $\Delta t$ and $h$ and not necessarily the same at different occurrences.

2. The Godunov-Mixed Method on Changing Meshes

At first we give some notation and basic assumptions. The usual Sobolev spaces and norms are adopted on $\Omega$. The inner product on $L^2(\Omega)$ is denoted by $(f, g) = \int_{\Omega} f g \, dx$. Define the following spaces and norms:

\[ H^m(\text{div}, \Omega) = \{ f = (f_x, f_y); f_x, f_y, \nabla \cdot f \in H^m(\Omega) \}, \quad m \geq 0, \]

\[ ||f||^2_{H^m(\text{div})} = ||f_x||^2_m + ||f_y||^2_m + ||\nabla \cdot f||^2_m, \]

\[ H(\text{div}, \Omega) = H^0(\text{div}, \Omega), \quad W = \frac{L^2(\Omega)}{\{ \varphi \equiv \text{constant on } \Omega \}}, \]

\[ V = \{ v \in H(\text{div}; \Omega) | \ v \cdot n = 0 \text{ on } \partial \Omega, n \text{ is the unit outward norm to } \partial \Omega \}. \]

Let $\Delta t^n > 0$ $(n = 1, 2, \ldots, N^*)$ denote different time steps, $t^n = \sum_{k=1}^{n} \Delta t^k$, $T = \sum_{m=1}^{N^*} \Delta t^n$, $\Delta t = \max_n \Delta t^n$. Assume that the time steps $\Delta t^n$ do not change too rapidly; that is, we assume there exist positive constants $t_*$ and $t^*$ which are independent of $n$ and $\Delta t$ such that

\[ t_* \leq \frac{\Delta t^n}{\Delta t^{n-1}} \leq t^*. \]

For a given function $g(x, t)$, let $g^n = g(x, t^n)$.

Assume $\Omega = (0, 1) \times (0, 1)$. At each time level $t^n$, we construct a quasiuniform rectangular partition $K^n_h = \{ e^n_i \}$ of $\Omega$:

\[ \delta^n_x : 0 = x^n_{1/2} < x^n_{3/2} < \cdots < x^n_{i+1/2} < \cdots < x^n_{n+1/2} = 1, \]

\[ \delta^n_y : 0 = y^n_{1/2} < y^n_{3/2} < \cdots < y^n_{j+1/2} < \cdots < y^n_{n+1/2} = 1. \]

Let $h^n_{i,x} = x^n_{i+1/2} - x^n_{i-1/2}$, $h^n_{i,y} = y^n_{j+1/2} - y^n_{j-1/2}$, $h^n = \max_{i,j} \{ h^n_{i,x}, h^n_{i,y} \}$, and $h = \max_n h^n$. And $x^n_i = (x^n_{i+1/2} + x^n_{i-1/2})/2$ is the midpoint of $[x^n_{i-1/2}, x^n_{i+1/2}]$, $h^n_{i+1/2,x} = x^n_{i+1} - x^n_i$, $\Delta t^n = O(h^n)$. Let $y^n_j, h^n_{j+1/2,y}$ be defined analogously.

Let

\[ M_{-1}^k(\delta^n_x) = \{ f | f|_{B^n_{i,x}} \in P^k(B^n_{i,x}), i = 1, \ldots, l^n \}, \quad M_{-1}^k(\delta^n_y) = M_{-1}^k(\delta^n_x) \cap C^0([0, 1]), \]

where $B^n_{i,x} = [x^n_{i-1/2}, x^n_{i+1/2}]$ and $P^k(B^n_{i,x})$ is the set of all polynomials of degree less than or equal to $k$ defined on $B^n_{i,x}$. Similar definitions are given to $M_{-1}^k(\delta^n_y)$ and $M_{-1}^k(\delta^n_y)$.\]
Then the lowest order Raviart-Thomas spaces $W^n_h$ and $V^n_h$ are given by

\[
W^n_h = M^0_{-1}(\delta^n_x) \otimes M^0_{-1}(\delta^n_y), \quad V^n_h = V^n_{h,x} \times V^n_{h,y},
\]
\[
V^n_{h,x} = M^1_0(\delta^n_x) \otimes M^0_{-1}(\delta^n_y), \quad V^n_{h,y} = M^0_{-1}(\delta^n_x) \otimes M^1_0(\delta^n_y).
\] (2.5)

That is, the space $W^n_h$ is the space of functions which are constant on each element $e^n_i \in K^n_h$, and $V^n_h$ is the space of vector valued functions whose components are continuous and linear on each element $e^n_i \in K^n_h$. The degrees of freedom of a function $v^n \in V^n_h$ correspond to the values of $v^n \cdot \gamma$ at the midpoints of the sides of $e^n_i$, where $\gamma$ is the unit outward normal to $\partial e^n_i$.

It is easy to see that $W^n_h \times V^n_h \subset W \times V$ and $\text{div} V^n_h = W^n_h$.

By introducing variables $\mathbf{z} = -\nabla u$, $\mathbf{z} = b\mathbf{Z} + a\mathbf{Z}$, $\mathbf{g} = cu = (c_1 u, c_2 u)^T = (g_1, g_2)^T$, and $\mathbf{c}(x, t, u) = (\partial c_1/\partial u, \partial c_2/\partial u)^T$, we modify the first equation in (1.1) as

\[
u_t + \nabla \cdot \mathbf{z} + (\nabla \cdot \mathbf{g} + u\mathbf{c}(u) \cdot \mathbf{z}) = f(u).
\] (2.6)

Here we are using the so-called “expanded” mixed finite element method, proposed by Arbogast et al. [20], which gives a gradient approximation $\mathbf{z}$ as well as an approximation to the diffusion term $z$.

The weak form of (2.6) is

\[
(u_t, w) + (\nabla \cdot \mathbf{z}, w) + (\nabla \cdot \mathbf{g}, w) + (u\mathbf{c}(u) \cdot \mathbf{z}, w) = (f(u), w),
\] (2.7)

so that

\[
(u_t, w) + (\nabla \cdot \mathbf{z}, w) + (\nabla \cdot \mathbf{g}, w) + (u\mathbf{c}(u) \cdot \mathbf{z}, w) = (f(u), w), \quad \forall w \in W,
\]
\[
(\mathbf{z}, v) = (u, \nabla \cdot v), \quad \forall v \in V,
\] (2.8)
\[
(\mathbf{z}, v) = (b(u)\mathbf{Z} + a(u)\mathbf{Z}, v), \quad \forall v \in V.
\]

The Godunov-mixed method on changing meshes is presented as follows: at each time level $n$, find $U^n \in W^n_h$, $Z^n \in V^n_h$ such that

\[
\frac{(U^n - R^n U^{n-1})}{\Delta t^n}, w^n) + (\nabla \cdot Z^n, w^n) + (U^n \mathbf{c}(U^n) \cdot \mathbf{Z}^n, w^n) = (f(U^n), w^n) - (\nabla \cdot G^{n-1}, w^n), \quad \forall w^n \in W^n_h,
\]
We construct a piecewise linear function $RU^{n-1}$ on each element $e_i^{n-1}$:

\[ RU^{n-1}|_{e_i} = U_i^{n-1} + (x - x_i)\delta_x U_{i,j}^{n-1} + (y - y_j)\delta_y U_{i,j}^{n-1}, \]

where $\delta_x U_{i,j}^{n-1}$ and $\delta_y U_{i,j}^{n-1}$ are $x$, $y$ slopes. The $x$ slope calculation can be performed as

\[ \delta_x U_{i,j}^{n-1} = \begin{cases} \frac{\Delta_x U_{i,j}^{n-1}}{\Delta i}, & \text{if } |\Delta_x U_{i,j}^{n-1}| \leq |\Delta_y U_{i,j}^{n-1}|, \\ \frac{\Delta_y U_{i,j}^{n-1}}{\Delta j}, & \text{otherwise}, \end{cases} \]

(2.12)
where $\Delta^+ x$ and $\Delta^- x$ are forward and backward difference operators in the $x$ direction, respectively:

\[
\Delta^+ x U_{i,j}^{n-1} = \frac{U_{i+1,j}^{n-1} - U_{i,j}^{n-1}}{h_{i+1/2,x}^{n-1}},
\]
\[
\Delta^- x U_{i,j}^{n-1} = \frac{U_{i,j}^{n-1} - U_{i-1,j}^{n-1}}{h_{i-1/2,x}^{n-1}},
\]

and $\delta_x$ satisfy

\[
\frac{h_{i,x}^{n-1}}{2} \left| \delta_x U_{i,j}^{n-1} \right| \leq C \sum_{j-1 \leq j+i} \sum_{i-1 \leq i+j} \left| U_{i,j}^{n-1} \right|,
\]
\[
\delta_x U_{i,j}^{n-1} = u_x(x_i, y_j, t^{n-1}) + O(h),
\]

where $L$ and $R$ are positive integers independent of $h$. The $y$ slope is defined analogously.

**Second Step**

By the Taylor expansion, for $u$ with smooth second derivatives, we have

\[
u(x_{i+1/2}, y_j, t^{n-1/2}) = u + \frac{h_{i,x}^{n}}{2} u_x + \frac{\Delta t^n}{2} u_t + O(h^2 + \Delta t^2).
\]

By (2.6), we see that

\[
u^{n-1/2} = u + \left( \frac{h_{i,x}^{n}}{2} - \frac{\Delta t^n}{2} g_{1u} \right) u_x - \frac{\Delta t^n}{2} g_{2y} + \sigma,
\]

where

\[
\sigma = \frac{\Delta t^n}{2} (f - u \zeta(u) \cdot \nabla - \nabla \cdot z) + O(h^2 + \Delta t^2) = O(h^2 + \Delta t).
\]

Based on (2.17) and a similar expansion about $(x_{i+1}, y_j, t^{n-1})$, we define $U^L, U^R$:

\[
U_{i+1/2,j}^{L,n-1} = U_{i,j}^{n-1} + \left( \frac{h_{i,x}^{n-1}}{2} - \frac{\Delta t^{n-1}}{2} g_{1u} \left( U_{i,j}^{n-1} \right) \right) \delta_x U_{i,j}^{n-1} - \frac{\Delta t^{n-1}}{2 h_{j,y}^{n-1}} \left[ \Gamma_{i,j+1/2}^{n-1} - \Gamma_{i,j-1/2}^{n-1} \right],
\]
\[
U_{i+1/2,j}^{R,n-1} = U_{i,j}^{n-1} + \left( \frac{h_{i,x}^{n-1}}{2} - \frac{\Delta t^{n-1}}{2} g_{1u} \left( U_{i,j}^{n-1} \right) \right) \delta_x U_{i,j}^{n-1} - \frac{\Delta t^{n-1}}{2 h_{j,y}^{n-1}} \left[ \Gamma_{i+1,j+1/2}^{n-1} - \Gamma_{i+1,j-1/2}^{n-1} \right],
\]

(2.19)
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where

\[
\Gamma_{i,j+1/2}^{n-1} = H_g \left( U_{i,j+1/2}^{L,n-1} U_{i,j+1/2}^{R,n-1} \right),
\]

\[
U_{i,j+1/2}^{L,n-1} = U_{i,j}^{n-1} + \frac{h_{j,y}^{n-1}}{2} \delta_y U_{i,j}^{n-1},
\]

\[
U_{i,j+1/2}^{R,n-1} = U_{i,j+1}^{n-1} + \frac{h_{j,y}^{n-1}}{2} \delta_y U_{i,j+1}^{n-1}.
\] (2.20)

Third Step

With the above definitions, \((\overline{S}^{n-1}_{i+1/2,j}, \overline{S}^{n-1}_{2,j+1/2})\) is calculated as follows.

If \(c_1(U_{i+1/2,j}^{n-1}) > 0\), define

\[
\overline{S}^{n-1}_{i+1/2,j} \triangleq H_{S_i} \left( U_{i+1/2,j}^{L,n-1} U_{i+1/2,j}^{L,n-1} \right) \triangleq c_1 \left( U_{i+1/2,j}^{L,n-1} \right) U_{i+1/2,j}^{L,n-1}.
\] (2.21)

If \(c_1(U_{i+1/2,j}^{n-1}) \leq 0\), define

\[
\overline{S}^{n-1}_{i+1/2,j} \triangleq H_{S_i} \left( U_{i+1/2,j}^{R,n-1} U_{i+1/2,j}^{R,n-1} \right) \triangleq c_1 \left( U_{i+1/2,j}^{R,n-1} \right) U_{i+1/2,j}^{R,n-1}.
\] (2.22)

For \(\overline{S}^{n-1}_{2,j+1/2}\), the definition is similar.

The equations (2.19)–(2.22) hold for elements at least one element away from the boundary. At the left and right boundaries, we can set

\[
U_{1/2,j}^{L,n-1} = U_{I^n+1/2,j}^{R,n-1} = 0, \quad j = 1, 2, \ldots, J^{n-1}, \quad n = 1, 2, \ldots, N,
\] (2.23)

and in the slope calculation procedure, we define \(\delta_x U_{i,j}^{n-1}\) and \(\delta_x U_{i-1,j}^{n-1}\) using (2.12) with

\[
\Delta_x U_{i,j}^{n-1} = \frac{2U_{i,j}^{n-1}}{h_{i,x}^{n-1}}, \quad \Delta_x U_{i-1,j}^{n-1} = \frac{-2U_{i-1,j}^{n-1}}{h_{i-1,x}^{n-1}}.
\] (2.24)

On the bottom and top boundaries, we set

\[
\Gamma_{i,1/2}^{n-1} = \Gamma_{i,J^{n-1}+1/2}^{n-1} = 0, \quad i = 1, 2, \ldots, I^{n-1},
\] (2.25)

in (2.19).
3. Projections and Lemma

We introduce three projections and a lemma to obtain error estimates. Let \( \{ \tilde{u}_n, \tilde{z}_n \} : J \to W_h^n \times V_h^n \) be the projection of \( \{ u, z \} \) in mixed finite element space such that

\[
(\tilde{z}_n, v^n) - (\tilde{u}_n, \nabla \cdot v^n) = 0, \quad \forall v^n \in V_h^n,
\]

and define \( \pi z^n \) as the \( \pi \)-projection of \( z^n \) such that

\[
(\nabla \cdot (z^n - \pi z^n), \omega^n) = 0, \quad \forall \omega^n \in W_h^n.
\]

These projections satisfy ([21, 22]):

\[
\| u^n - \tilde{u}_n\| + \| z^n - \tilde{z}_n\|_{H(\text{div})} \leq K h^n,
\]

\[
\left\| \frac{\partial}{\partial t}(u^n - \tilde{u}_n)\right\| + \left\| \frac{\partial}{\partial t}(z^n - \tilde{z}_n)\right\|_{H(\text{div})} \leq K h^n,
\]

and it is easy to see that \( \|\tilde{u}_n\|_{L_\infty} \leq K \).

Let \( \tilde{g} \in V_h^n \) satisfies that the value of \( \tilde{g} \cdot \gamma \) at the midpoint of the boundaries is equal to the average value of \( g \cdot \gamma \) on boundaries, that is,

\[
(\nabla \cdot (\tilde{g} - g^n), \omega^n) = 0, \quad \forall \omega^n \in W_h^n.
\]

Let \( \tilde{g} = (\tilde{g}_1, \tilde{g}_2) \). From (3.4), we can define

\[
\tilde{g}_1(x_{i+1/2}, y_j, t^n) \triangleq \frac{1}{h_{j,y}} \int_{y_{l/2}}^{y_{l+1/2}} c_1(x_{i+1/2}, y_j, t^n, u(x_{i+1/2}, y, t^n)) dy
\]

\[
\triangleq \frac{1}{h_{j,y}} \int_{y_{l/2}}^{y_{l+1/2}} g_1(u(x_{i+1/2}, y, t^n)) dy.
\]

Then, \( \tilde{g}_2(x_i, y_{j+1/2}, t^n) \) can be defined similarly.

**Lemma 3.1.** Let \( \xi = U - \tilde{u} \) and assume that \( u \) is sufficiently smooth. Then

\[
\left\| G^{n-1}(\cdot, U^{n-1}) - \tilde{g}^{n-1}\right\| \leq K \left( \| g^{n-1}\| + h + \Delta t \right).
\]

**Proof.** Firstly, construct \( \tilde{u}_i^{L,n-1}, \tilde{u}_i^{R,n-1} \) from \( \tilde{u} \) given in (3.1) and (3.2) by

\[
\tilde{u}_i^{L,n-1} = u_i^{n-1} + \left( \frac{h_{i,y}}{2} - \frac{\Delta t^n}{2} \frac{\delta u}{\delta x} u_i^{n-1} \right) \delta x u_i^{n-1}
\]

\[
- \frac{\Delta t^n}{2h_{j,y}} \left[ \Gamma_{i+1/2,j}^{n-1}(\tilde{u}) - \Gamma_{i,j-1/2}^{n-1}(\tilde{u}) \right].
\]
at interior edges, where

\[
\delta_x \tilde{u}_{i,j}^{n-1} = \begin{cases} 
\Delta_x \tilde{u}_{i,j}^{n-1}, & \text{if } \delta_x U_{ij}^{n-1} = \Delta_x U_{ij}^{n-1}, \\
\Delta_x \tilde{u}_{i,j}^{n-1}, & \text{otherwise},
\end{cases}
\]

\[
\Gamma_{i,j+1/2}^{n-1}(\tilde{u}) = H_{g_i} \left( \tilde{u}_{i,j+1/2}^{L,n-1}, \tilde{u}_{i,j+1/2}^{R,n-1} \right),
\]

(3.8)

\[
\tilde{u}_{i,j+1/2}^{L,n-1} = \tilde{u}_{i,j}^{n-1} + \frac{h_{ij}^{n-1}}{2} \delta_y \tilde{u}_{i,j}^{n-1},
\]

\[
\tilde{u}_{i,j+1/2}^{R,n-1} = \tilde{u}_{i,j}^{n-1} + \frac{h_{ij}^{n-1}}{2} \delta_y \tilde{u}_{i,j+1}^{n-1}.
\]

And define \( \tilde{u}_{i+1/2,j}^{R,n-1} \) similarly.

Then, assuming \( g_1 \) is twice differentiable and Lipschitz continuous, and using the Lipschitz continuity of the Godunov flux and (2.14), it can be shown that

\[
\left| U_{i+1/2}^{L,n-1} - \tilde{u}_{i+1/2}^{L,n-1} \right| \leq K \sum_{i=1}^{i+2} \sum_{m=1}^{i+2} \left| U_{i,m}^{n-1} - \tilde{u}_{i,m}^{n-1} \right|.
\]

(3.9)

A similar bound holds for \( |U_{i+1/2,j}^{R,n-1} - \tilde{u}_{i+1/2,j}^{R,n-1}| \).

At the boundaries, we follow (2.23) and (2.25) and define

\[
\tilde{u}_{1,2j}^{L,n-1} = \tilde{u}_{1,n+1/2,j}^{R,n-1} = 0, \quad j = 1, 2, \ldots, J^{n-1},
\]

(3.10)

\[
\Gamma_{i,1/2}^{n-1}(\tilde{u}) = \Gamma_{i,n^{1/2}+1/2}^{n-1}(\tilde{u}) = 0, \quad i = 1, 2, \ldots, I^{n-1}.
\]

Define \( u_{i+1/2,j}^{L,n-1}, u_{i+1/2,j}^{R,n-1} \) analogous to \( \tilde{u}_{i+1/2,j}^{L,n-1}, \tilde{u}_{i+1/2,j}^{R,n-1} \).

If \( c_1 (U_{i+1/2,j}^{n-1}) > 0 \), we consider

\[
\delta_{i+1/2,j}^{n-1} - \tilde{g}_i \left( x_{i+1/2,j}, y_{i+1/2,j}^{n-1} \right)
\]

\[
= H_{g_i} \left( U_{i+1/2,j}^{L,n-1}, U_{i+1/2,j}^{R,n-1} \right) - \frac{1}{h_{i,j}^{n-1}} \int_{y_i^{1/2}}^{y_i^{n-1/2}} g_i \left( u \left( x_{i+1/2,j}, y, t^{n-1} \right) \right) dy.
\]
\[
\begin{align*}
&= \left[ H_{g_1}(U_{i+1/2,j}^{L,n-1}, U_{i+1/2,j}^{L,n-1}) - H_{g_1}(\tilde{u}_{i+1/2,j}^{L,n-1}, \tilde{u}_{i+1/2,j}^{L,n-1}) \right] \\
&\quad + \left[ H_{g_1}(\tilde{u}_{i+1/2,j}^{L,n-1}, \tilde{u}_{i+1/2,j}^{L,n-1}) - H_{g_1}(u_{i+1/2,j}^{L,n-1}, u_{i+1/2,j}^{L,n-1}) \right] \\
&\quad + \left[ H_{g_1}(u_{i+1/2,j}^{L,n-1}, u_{i+1/2,j}^{L,n-1}) - g_1(u_{i+1/2,j}^{n-1/2}) \right] \\
&\quad + \left[ g_1(u_{i+1/2,j}^{n-1/2}) - \frac{1}{h_{j,y}^{n-1}} \int_{y_{j-1/2}}^{y_{j+1/2}} g_1(u(x_{i+1/2}, y, t^{n-1})) \, dy \right] \\
&= R_1 + R_2 + R_3 + R_4. 
\end{align*}
\] (3.11)

By the Lipschitz continuity of \( H_{g_1} \) and (3.9), we have
\[
|R_1| \leq K \left\{ \left| U_{i+1/2,j}^{L,n-1} - \tilde{u}_{i+1/2,j}^{L,n-1} \right| + \left| U_{i+1/2,j}^{R,n-1} - \tilde{u}_{i+1/2,j}^{R,n-1} \right| \right\} 
\leq K \sum_{i=1}^{j+2} \sum_{m=1}^{j+2} \left| U_{i,m}^{n-1} - \tilde{u}_{i,m}^{n-1} \right|. 
\] (3.12)

The similar arguments are applied to \( R_2 \) to get
\[
|R_2| \leq K \sum_{i=1}^{j+2} \sum_{m=1}^{j+2} \left| \tilde{u}_{i,m}^{n-1} - u_{i+1/2,j}^{n-1} \right|. 
\] (3.13)

By the consistency of \( H_{g_1} \), we see
\[
|R_3| \leq K \left| u_{i+1/2,j}^{L,n-1} - u_{i+1/2,j}^{n-1/2} \right| = O(\Delta t + h^2). 
\] (3.14)

For \( R_4 \), it is easy to have
\[
|R_4| \leq K(\Delta t + h^2). 
\] (3.15)

Using (3.3), (3.12)–(3.15), and equivalence of norms, we have
\[
\left\| \tilde{s}_1^{n-1} - \tilde{s}_1^{n-1} \right\| \leq K \sum_j \sum_i \left\| \tilde{s}_{1,j+1/2,j}^{n-1} - \tilde{s}_{1,j+1/2,j}^{n-1} \right\| \right\| \\
\leq K \left( \left\| s_1^{n-1} \right\| + \Delta t + h \right). 
\] (3.16)
The similar approach can be used when $c_1(U_{i_{i+1/2}}^{n-1}) \leq 0$. Defining $\overline{g}_2$ and $\tilde{g}_2$ similarly and following analogous arguments yields

$$\left\| \overline{g}_2^{n-1} - \tilde{g}_2^{n-1} \right\| \leq C \left( \left\| g^{n-1} \right\| + \Delta t + \bar{h} \right).$$

(3.17)

Thus, Lemma 3.1 holds.

\[\square\]

4. Error Estimates

At time level $t^n$, for all $w^n \in W_h^n$, $v^n \in V_h^n$, the exact solutions satisfy

$$\left( \frac{u^n - u^{n-1}}{\Delta t^n}, w^n \right) + (\nabla \cdot z^n, w^n) + (u^n \bar{c}(u^n) \cdot \bar{z}^n, w^n) = (f(u^n), w^n) - (\nabla \cdot g^n, w^n) - (\rho^n, w^n),$$

$$\left( \bar{z}, v^n \right) = (u^n, \nabla \cdot v^n),$$

$$\left( \bar{z}^n, v^n \right) = \left( b(u^n) \bar{z}^n + a(u^n) \frac{\bar{z}^n - \bar{z}^{n-1}}{\Delta t^n}, v^n \right) + (\tau^n, v^n),$$

(4.1)

where $\rho^n = u^n_t - (u^n - u^{n-1})/\Delta t^n$, $\tau^n = a(u^n)(\bar{z}^n - (\bar{z} - \bar{z}^{n-1})/\Delta t^n)$.

Let

$$\xi_u = U - \bar{u}, \quad \beta_u = u - \bar{u},$$

$$\xi_z = Z - \bar{z}, \quad \beta_z = z - \bar{z},$$

(4.2)

Using projections (3.1) and (3.2), we subtract (2.9) from (4.1) to get

$$\left( \frac{\xi_{u}^{n} - \xi_{u}^{n-1}}{\Delta t^n}, w^n \right) + (\nabla \cdot \xi_{z}^{n}, w^n) + \left( U^n \bar{c}(U^n) \cdot \bar{z}^n - u^n \bar{c}(u^n) \cdot \bar{z}^n, w^n \right) = (f(U^n) - f(u^n), w^n) + (\rho^n, w^n) + \left( \nabla \cdot \left( \bar{g}^{n} - G^{n-1} \right), w^n \right) + \left( \frac{\beta_u^n - \beta_u^{n-1}}{\Delta t^n}, w^n \right),$$

$$\left( \xi_{z}, v^n \right) = (\xi_{u}, \nabla \cdot v^n),$$

$$\left( \xi_{z}^{n}, v^n \right) = \left( b(U^n) \bar{z}^n - b(u^n) \bar{z}^n, v^n \right) - (\tau^n, v^n) - (\beta_z^n, v^n)$$

$$+ \left( a(U^n) \frac{\bar{z}^n - \bar{z}^{n-1}}{\Delta t^n} - a(u^n) \frac{\bar{z}^n - \bar{z}^{n-1}}{\Delta t^n}, v^n \right),$$

(4.3)
where the last term of the first equation in (4.3) is related to the changing meshes. If the meshes do not change, this term will be zero.

Taking \( w = \zeta^n_u \), \( v = \zeta^n_z \) in the second equation and \( v = \bar{\zeta}^n_z \) in the third, then adding them together, we have

\[
\left( \frac{\zeta^n - \zeta^{n-1}}{\Delta t^n}, \zeta^n_u \right) + \left( b(U^n) \tilde{Z}^n - b(u^n) \tilde{Z}^n, \tilde{z}^n_z \right) + \left( a(U^n) \frac{\tilde{Z}^n - \tilde{Z}^{n-1}}{\Delta t^n}, \zeta^n_u \right) - \left( a(u^n) \frac{\tilde{Z}^n - \tilde{Z}^{n-1}}{\Delta t^n}, \tilde{z}^n_z \right)
\]

\[
= \left( f(U^n) - f(u^n), \tilde{z}^n_u \right) + \left( \rho^n, \tilde{z}^n_u \right) + \left( \nabla \cdot \left( \tilde{g}^n - G^{n-1} \right), \tilde{z}^n_u \right)
\]

\[
- \left( U^n \tilde{c}(U^n) \cdot \tilde{Z}^n - u^n \tilde{c}(u^n) \cdot \tilde{Z}^n, \tilde{z}^n_u \right) + \left( \tilde{v}^n, \tilde{z}^n_z \right) - \left( \beta^n_z, \tilde{z}^n_z \right) + \left( \frac{\beta^n_u - \beta^{n-1}_u}{\Delta t^n}, \tilde{z}^n_u \right)
\]

(4.4)

For (4.4), we see

(I)

\[
\left( b(U^n) \tilde{Z}^n - b(u^n) \tilde{Z}^n, \tilde{z}^n_z \right) = \left( b(U^n) \tilde{z}^n_z, \tilde{z}^n_z \right) - \left( b(U^n) \tilde{z}^n_z, \tilde{z}^n_z \right)
\]

\[
+ \left( [b(U^n) - b(u^n)] \tilde{Z}^n, \tilde{z}^n_z \right)
\]

(4.5)

(II)

\[
\left( a(U^n) \frac{\tilde{Z}^n - \tilde{Z}^{n-1}}{\Delta t^n}, \tilde{z}^n_z \right) - \left( a(u^n) \frac{\tilde{Z}^n - \tilde{Z}^{n-1}}{\Delta t^n}, \tilde{z}^n_z \right)
\]

\[
= \left( a(U^n) \frac{\tilde{Z}^n - \tilde{Z}^{n-1}}{\Delta t^n}, \tilde{z}^n_z \right) - \left( a(U^n) \frac{\tilde{Z}^n - \tilde{Z}^{n-1}}{\Delta t^n}, \tilde{z}^n_z \right) + \left( a(U^n) - a(u^n) \right) \frac{\tilde{Z}^n - \tilde{Z}^{n-1}}{\Delta t^n}
\]

(4.6)

(III)

\[
\left( U^n \tilde{c}(U^n) \cdot \tilde{Z}^n - u^n \tilde{c}(u^n) \cdot \tilde{Z}^n, \tilde{z}^n_u \right)
\]

\[
= \left( [\tilde{z}^n_u - \tilde{z}^{n-1}_u] \tilde{c}(U^n) \cdot \tilde{Z}^n, \tilde{z}^n_u \right) + \left( u^n \tilde{c}(U^n) \cdot \tilde{z}^n_u - \tilde{z}^{n-1}_u \right) + \left( u^n \tilde{c}(U^n) - \tilde{c}(u^n) \right) \cdot \tilde{Z}^n, \tilde{z}^n_u
\]

(4.7)

The last second term in (4.6) is related to the changing meshes. If the meshes do not change, this term will be zero.
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Substitute (4.5)–(4.7) into (4.4) to get

\[
\left( \frac{\dot{u}_n - \frac{u_{n-1}}{\Delta_t}}{\frac{u_n}{\Delta_t}} \right) + \left( b(U^n) \frac{\bar{z}_n}{\bar{z}_x} \right) + \left( a(U^n) \frac{\bar{z}_n - \frac{u_{n-1}}{\Delta_t}}{\bar{z}_x} \right) \\
= (f(U^n) - f(u^n), \dot{\xi}_n) + (\dot{\rho}^n, \dot{\xi}_n) + (\nabla \cdot (\dot{g}^n - \dot{C}^{n-1}), \dot{\xi}_n) + \left( \frac{\ddot{r}^n}{\bar{z}_x} \right) \\
- \left( \frac{\ddot{\beta}^n}{\bar{z}_x} \right) - \left( \frac{\ddot{\xi}^n}{\bar{z}_x} \right) - \left( a^n [\bar{e}(U^n) - \bar{e}(u^n)] : \bar{Z}^n, \frac{\dot{\xi}^n}{\bar{z}_x} \right) - \left( a^n \frac{\ddot{\xi}^n}{\bar{z}_x} - \ddot{\beta}^n \right) \\
- \left( \frac{\ddot{u}^n}{\bar{z}_x} \right) - \left( \frac{\ddot{b}^n}{\bar{z}_x} \right) - \left( \frac{\ddot{\xi}^n}{\bar{z}_x} \right) - \left( \frac{\ddot{\beta}^n}{\bar{z}_x} \right) \\
+ \left( \frac{\ddot{u}^n}{\bar{z}_x} \right) - \left( \frac{\ddot{b}^n}{\bar{z}_x} \right) - \left( \frac{\ddot{\xi}^n}{\bar{z}_x} \right) - \left( \frac{\ddot{\beta}^n}{\bar{z}_x} \right) \\
+ \left( a^n \frac{\ddot{\xi}^n}{\bar{z}_x} - \ddot{\beta}^n \right) = T_1 + T_2 + \cdots + T_{13}.
\]

Furthermore, for the first and third terms on the left-hand side of (4.8), we have

\[
\left( \frac{\dot{u}_n - \frac{u_{n-1}}{\Delta_t}}{\frac{u_n}{\Delta_t}} \right) = \frac{1}{2\Delta_t} \left[ (\dot{u}_n, \frac{u_n}{\Delta_t}) - (\dot{u}_{n-1}, \frac{u_{n-1}}{\Delta_t}) \right] + \frac{1}{2\Delta_t} \left\| \frac{u_n - u_{n-1}}{\Delta_t} \right\|^2,
\]

where

\[
\left( a(U^n) \frac{\bar{z}_n}{\bar{z}_x} - \frac{\bar{z}_{n-1}}{\frac{\Delta_t}{\bar{z}_x}} \right) \geq \frac{1}{2\Delta_t} \left[ (a(U^n) \frac{\bar{z}_n}{\bar{z}_x}) - (a(U^{n-1}) \frac{\bar{z}_{n-1}}{\bar{z}_x}) \right] \]

\[
\cdot \frac{\alpha_2}{2} \left\| \frac{\bar{z}_{n-1}}{\bar{z}_x} \right\|^2 + \frac{\alpha_0}{2\Delta_t} \left\| \frac{\bar{z}_n - \frac{u_{n-1}}{\Delta_t}}{\bar{z}_x} \right\|^2.
\]
By the Lipschitz continuity of \( f(u), a(u), b(u), \partial c/\partial u \), and (3.3), the following terms on the right-hand side of (4.10) can be obtained:

\[
T_1 \leq K\|U^n - u^n\|^2 + K\|\xi_{u}^{n}\|^2 \leq K\|\xi_{u}^{n}\|^2 + Kh^2,
\]

\[
T_2 \leq K\Delta t^n \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2((m^{-1}, h; H^1)}^2 + K\|\xi_{u}^{n}\|^2,
\]

\[
T_3 \leq K\Delta t^n \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2((m^{-1}, h; H^1)}^2 + K\|\xi_{u}^{n}\|^2,
\]

\[
T_4 \leq K\Delta t^n \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2((m^{-1}, h; H^1)}^2 + K\|\xi_{u}^{n}\|^2,
\]

\[
T_5 \leq K\|\xi_{u}^{n}\|^2 + K\|\xi_{z}^{n}\|^2 + Kh^2,
\]

\[
T_6 \leq K\|\xi_{u}^{n}\|^2 + K\|\xi_{z}^{n}\|^2 + Kh^2,
\]

\[
T_7 \leq K\|\xi_{u}^{n}\|^2 + K\|\xi_{z}^{n}\|^2 + Kh^2,
\]

\[
T_8 \leq K\|U^n - u^n\|^2 + K\|\xi_{u}^{n}\|^2 \leq K\|\xi_{u}^{n}\|^2 + Kh^2,
\]

\[
T_{10} \leq (b_1)h^2 + K\left\| \xi_{z}^{n} \right\|^2,
\]

\[
T_{11} \leq K\|\xi_{u}^{n}\|^2 + K\|\xi_{z}^{n}\|^2 + Kh^2,
\]

\[
T_{12} \leq K\left\| \xi_{z}^{n} \right\|^2 + K\|\xi_{u}^{n}\|^2 + K(\Delta t^n)^{-1} \left\| \frac{\partial u}{\partial t} \right\|_{L^2((m^{-1}, h; H^1)}^2 + Kh^2.
\]

\[
(4.11)
\]

We turn to consider \( T_3 \). Letting \( v = \tilde{g}^n - G^{n-1} \) in the second equation of (4.10), then we have

\[
(\tilde{\xi}_{z}^{n}, \tilde{g}^n - G^{n-1}) = (\xi_{u}^{n}, \nabla \cdot (\tilde{g}^n - G^{n-1})).
\]

(4.12)

By Lemma 3.1, we have

\[
T_3 \leq \frac{1}{2} \left( b(U^n)\tilde{\xi}_{z}^{n}, \tilde{\xi}_{z}^{n} \right) + K\left( b_0^{-1} \right) \left\| \tilde{g}^n - G^{n-1} \right\|^2
\]

\[
\leq \frac{1}{2} \left( b(U^n)\tilde{\xi}_{z}^{n}, \tilde{\xi}_{z}^{n} \right) + K\left( b_0^{-1} \right) \left\| \tilde{g}^n - G^{n-1} \right\|^2 + \left\| \tilde{g}^{n-1} - G^{n-1} \right\|^2
\]

\[
\leq \frac{1}{2} \left( b(U^n)\tilde{\xi}_{z}^{n}, \tilde{\xi}_{z}^{n} \right) + K\left( b_0^{-1} \right) \left( \left\| \xi_{u}^{n-1} \right\|^2 + h^2 + \Delta t^2 \right).
\]

(4.13)
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Substituting (4.11) and (4.13) into (4.10), we have

\[
\begin{align*}
&\frac{1}{2\Delta t_n} \left[ (\xi^n_{u}, \xi^n_{u}) - (\xi^{n-1}_{u}, \xi^{n-1}_{u}) \right] + \frac{1}{2\Delta t_n} \left\| b(L^n) \xi^n_{x}, \xi^n_{x} \right\|^2 + \frac{1}{2} \left( b(L^n) \xi^n_{x}, \xi^n_{x} \right) \\
&+ a_0 \frac{h}{\Delta t_n} \left\| \xi^n_{x} - \xi^{n-1}_{x} \right\|^2 + \frac{1}{2\Delta t_n} \left[ \left( a^n \xi^n_{x}, \xi^n_{x} \right) - \left( a^{n-1} \xi^{n-1}_{x}, \xi^{n-1}_{x} \right) \right] \\
&\leq K \left( \left\| \xi^n_{u} \right\|^2 + \left\| \xi^n_{u} \right\|^2 \right) + K \left( \left\| \xi^n_{u} \right\|^2 + \left\| \xi^n_{u} \right\|^2 + \Delta t^2 + h^2 \right) \\
&+ K(\Delta t^n) \left( \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2((t^n, t^{n+1}; H^1))} + \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L^2((t^n, t^{n+1}; H^1))} \right) + T_0 + T_{13}.
\end{align*}
\]

(4.14)

Let \( N \) be the time step at which \( \left\| \xi^n_{u} \right\| \) is maximum, that is,

\[
\left\| \xi^n_{u} \right\|^2 = \max_{1 \leq n \leq N^*} \left\| \xi^n_{u} \right\|^2.
\]

(4.15)

Multiplying (4.14) by \( 2\Delta t_n \) and summing on \( n \) from 1 to \( N \), we obtain

\[
\begin{align*}
\left\| \xi^n_{u} \right\|^2 + \sum_{n=1}^{N} \left\| \xi^n_{u} - \xi^{n-1}_{u} \right\|^2 + \sum_{n=1}^{N} \left( b(L^n) \xi^n_{x}, \xi^n_{x} \right) \Delta t^n + a_0 \sum_{n=1}^{N} \left\| \xi^n_{x} - \xi^{n-1}_{x} \right\|^2 + a_0 \left\| \xi^n_{x} \right\|^2 \\
&\leq K \left( \Delta t^2 + h^2 \right) + K \sum_{n=1}^{N} \left\| \xi^n_{u} \right\|^2 \Delta t^n + \left\| \xi^0_{u} \right\|^2 + a_1 \left\| \xi^n_{x} \right\|^2 + K \sum_{n=1}^{N} \left\| \xi^n_{x} \right\|^2 \Delta t^n \\
&+ \sum_{n=1}^{N} \left( \frac{\cdot^n_{u} - \cdot^{n-1}_{u}}{\Delta t_n}, \cdot^n_{u} \right) \Delta t^n + \sum_{n=1}^{N} \left( a(L^n) \frac{\cdot^n_{x} - \cdot^{n-1}_{x}}{\Delta t_n}, \cdot^n_{x} \right) \Delta t^n.
\end{align*}
\]

(4.16)

Assuming that the mesh is modified at most \( M \) times, and \( M \leq M^* \), where \( M^* \) is independent of \( h \) and \( \Delta t \), we get [3]:

\[
\begin{align*}
\sum_{n=1}^{N} \left( \frac{\beta^n_{u} - \beta^{n-1}_{u}}{\Delta t_n}, \beta^n_{u} \right) \Delta t^n = \sum_{n=1}^{N} \left( (\beta^n_{u} - \beta^{n-1}_{u}), \beta^n_{u} \right) \leq K(M^* h)^2 + \frac{1}{4} \left\| \xi^n_{u} \right\|^2, \\
\sum_{n=1}^{N} \left( a(L^n) \frac{\beta^n_{x} - \beta^{n-1}_{x}}{\Delta t_n}, \beta^n_{x} \right) \Delta t^n \leq K h^2 + \epsilon \sum_{n=1}^{N} \left\| \xi^n_{x} \right\|^2 \Delta t^n,
\end{align*}
\]

(4.17)

where \( \epsilon \) is a sufficiently small positive constant.
Substituting (4.17) into (4.16), we find
\[
\| {\xi}^n_u \|^2 + \sum_{n=1}^{N} \| \xi^n_u - \xi^{n-1}_u \|^2 + \sum_{n=1}^{N} \left( b(U^n) \xi^n_z / \xi^n_x \right) \Delta t^n + a_0 \sum_{n=1}^{N} \| \xi^n_z - \xi^{n-1}_z \|^2 + a_0 \| \xi^n_z \|^2 \\
\leq K \left( \Delta t^2 + h^2 + (M^*)^2 \right) + \| \xi^0_u \|^2 + a_1 \| \xi^0_z \|^2 + K \sum_{n=1}^{N} \| \xi^n_{bu} \|^2 \Delta t^n \\
+ K \sum_{n=1}^{N} \| \xi^n_z \|^2 \Delta t^n + \epsilon \sum_{n=1}^{N} \| \xi^n_z \|^2 \Delta t^n + \frac{1}{4} \| \xi^N_u \|^2.
\] (4.18)

Using the discrete Gronwall’s lemma, we have
\[
\max_n \| \xi^n_{bu} \|^2 \leq K \left( h^2 + \Delta t^2 + (M^*)^2 \right).
\] (4.19)

Similarly as (4.14), we can derive
\[
\max_n \| \xi^n_z \|^2 \leq K \left( h^2 + \Delta t^2 + (M^*)^2 \right).
\] (4.20)

By the triangle inequality and (3.3), we can obtain the following result.

**Theorem 4.1.** Assuming that the coefficients satisfy condition (1.2), the mesh is modified at most \( M \) times, \( M \leq M^* \), and \( u \in L^2(H^1) \), \( u_t \in L^2(H^1) \), \( u_{tt} \in L^2(H^1) \), then we have
\[
\max_n \| u^n - U^n \| \leq K(h + \Delta t) + K(M^*)h, \\
\max_n \| \xi^n - \xi^n \| \leq K(h + \Delta t) + K(M^*)h,
\] (4.21)

where \( K \) and \( M^* \) are positive constants independent of \( h \) and \( \Delta t \).

## 5. Numerical Example

In this section, we give numerical experiment of the following nonlinear equations to illustrate the efficiency of the Godunov–mixed finite element method on changing meshes:

\[
u_t = (a(x,t)u_{x_1} + b(x,t,u)u_{x_2}) - c(x,t)u_x + f(x,t,u), \quad (x,t) \in (0,1) \times (0,1),
\]
\[
u(0,t) = \nu(1,t) = 0, \quad t \in [0,1],
\]
\[
u(\cdot,0) = \sin \pi x, \quad x \in [0,1],
\] (5.1)

where \( (x,t) \in [0,1] \times [0,1] \), \( a(x,t) = t^2(x + 0.25) + 0.25, \) \( b(x,t,u) = 0.005ut^2(x + 0.05), \) \( c(x,t) = 0.05t(x + 1) + 1, \) and \( f(x,t,u) \) is chosen properly so that the exact solution is \( u = e^{-t} \sin \pi x \).
Table 1: $L^2$-norm estimates of $U^n - u^n$.

<table>
<thead>
<tr>
<th>Time</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0.4$</td>
<td>0.0454</td>
<td>0.0232</td>
<td>0.0117</td>
<td>0.0057</td>
</tr>
<tr>
<td>$t = 0.6$</td>
<td>0.0253</td>
<td>0.0135</td>
<td>0.0064</td>
<td>0.0032</td>
</tr>
<tr>
<td>$t = 1.0$</td>
<td>0.0288</td>
<td>0.0160</td>
<td>0.0080</td>
<td>0.0039</td>
</tr>
</tbody>
</table>

Table 2: $L^2$-norm estimates of $\tilde{Z}^n - \bar{z}^n$.

<table>
<thead>
<tr>
<th>Time</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0.4$</td>
<td>0.0537</td>
<td>0.0307</td>
<td>0.0161</td>
<td>0.0069</td>
</tr>
<tr>
<td>$t = 0.6$</td>
<td>0.0636</td>
<td>0.0353</td>
<td>0.0174</td>
<td>0.0038</td>
</tr>
<tr>
<td>$t = 1.0$</td>
<td>0.0643</td>
<td>0.0395</td>
<td>0.0198</td>
<td>0.0099</td>
</tr>
</tbody>
</table>

Table 3: Convergence rate of $L^2$-norm.

<table>
<thead>
<tr>
<th>Time</th>
<th>Convergence rates of $|u^n - U^n|$</th>
<th>Convergence rates of $|\tilde{Z}^n - \bar{z}^n|$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Case 1/Case 2</td>
<td>Case 2/Case 3</td>
</tr>
<tr>
<td>$t = 0.4$</td>
<td>0.9670</td>
<td>0.9852</td>
</tr>
<tr>
<td>$t = 0.6$</td>
<td>0.9023</td>
<td>1.0724</td>
</tr>
<tr>
<td>$t = 1.0$</td>
<td>0.8464</td>
<td>1.0075</td>
</tr>
</tbody>
</table>

In the numerical example, $h$ and $\Delta t$ change as the following four cases.

Case 1. $t \in [0, 0.4]$, set $h = \Delta t = 0.1$; $t \in (0.4, 0.6]$, set $h = \Delta t = 0.05$; $t \in (0.6, 1.0]$, set $h = \Delta t = 0.1$ and calculate 40 steps at every time interval.

Case 2. $t \in [0, 0.4]$, set $h = \Delta t = 0.05$; $t \in (0.4, 0.6]$, set $h = \Delta t = 0.025$; $t \in (0.6, 1.0]$, set $h = \Delta t = 0.05$ and calculate 80 steps at every time interval.

Case 3. $t \in [0, 0.4]$, set $h = \Delta t = 0.025$; $t \in (0.4, 0.6]$, set $h = \Delta t = 0.0125$; $t \in (0.6, 1.0]$, set $h = \Delta t = 0.025$ and calculate 160 steps at every time interval.

Case 4. $t \in [0, 0.4]$, set $h = \Delta t = 0.0125$; $t \in (0.4, 0.6]$, set $h = \Delta t = 0.00625$; $t \in (0.6, 1.0]$, set $h = \Delta t = 0.0125$ and calculate 320 steps at every time interval.

The numerical solutions $U^n, \tilde{Z}^n$ are calculated and the $L^2$ error estimates of $U^n - u^n, \tilde{Z}^n - \bar{z}^n$ are obtained; see Tables 1 and 2.

In Table 3, convergence rates of $U^n - u^n, \tilde{Z}^n - \bar{z}^n$ are given. The figures of convergence rate are shown by Figure 1. The convergence rate of $U^n - u^n$ in Figure 1(a) is one order, and in Figure 1(b) the convergence rate of $\tilde{Z}^n - \bar{z}^n$ is a little smaller than one order at the beginning. But the convergence rate $\tilde{Z}^n - \bar{z}^n$ will be close to one order when $h$ diminishes, which is consistent with the analysis in this paper.

For Case 4, we compare the exact solution $\{u, \bar{z}\}$ with the approximate solution $\{U, \tilde{Z}\}$ at time $t = 0.25, 0.5, and 0.75$, respectively; see Figure 2. From Figures 2(a) and 2(b), we can see that the approximate solutions are very close to the exact solutions.
Figure 1: Convergence rate figures.

Figure 2: Compare figures for Case 4 at time $t = 0.25$, 0.5, and 0.75.

Acknowledgments

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