Research Article

Coefficient Conditions for Harmonic Close-to-Convex Functions

Toshio Hayami

Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan

Correspondence should be addressed to Toshio Hayami, ha_ya_to112@hotmail.com

Received 25 January 2012; Accepted 13 April 2012

Academic Editor: Roman Simon Hilscher

Copyright © 2012 Toshio Hayami. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

New sufficient conditions, concerned with the coefficients of harmonic functions $f(z) = h(z) + \overline{g(z)}$ in the open unit disk $\mathbb{U}$ normalized by $f(0) = h(0) = h'(0) - 1 = 0$, for $f(z)$ to be harmonic close-to-convex functions are discussed. Furthermore, several illustrative examples and the image domains of harmonic close-to-convex functions satisfying the obtained conditions are enumerated.

1. Introduction

For a continuous complex-valued function $f(z) = u(x, y) + iv(x, y)$ ($z = x + iy$), we say that $f(z)$ is harmonic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ if both $u(x, y)$ and $v(x, y)$ are real harmonic in $\mathbb{U}$, that is, $u(x, y)$ and $v(x, y)$ satisfy the Laplace equations

$$\Delta u = u_{xx} + u_{yy} = 0, \quad \Delta v = v_{xx} + v_{yy} = 0. \quad (1.1)$$

A complex-valued harmonic function $f(z)$ in $\mathbb{U}$ is given by $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in $\mathbb{U}$. We call $h(z)$ and $g(z)$ the analytic part and the coanalytic part of $f(z)$, respectively. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense preserving in $\mathbb{U}$ is $|h'(z)| > |g'(z)|$ in $\mathbb{U}$ (see [1] or [2]). Let $\mathcal{H}$ denote the class of harmonic functions $f(z)$ in $\mathbb{U}$ with $f(0) = h(0) = 0$ and $h'(0) = 1$. Thus, every normalized harmonic function $f(z)$ can be written by

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in \mathcal{H}, \quad (1.2)$$

where $a_1 = 1$ and $b_0 = 0$, for convenience.

We next denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f(z) \in \mathcal{H}$ that are univalent and sense preserving in $\mathbb{U}$. Due to the sense-preserving property of $f(z)$, we see that $|b_1| = |g'(0)| < |h'(0)| = 1$. If $g(z) \equiv 0$, then $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ consisting of normalized analytic univalent functions. Furthermore, for every function $f(z) \in \mathcal{S}_{\mathcal{H}}$, the function

$$F(z) = \frac{f(z) - \overline{b_1 f(z)}}{1 - |b_1|^2} = z + \sum_{n=2}^{\infty} \frac{a_n - \overline{b_1} b_n}{1 - |b_1|^2} z^n + \sum_{n=2}^{\infty} \frac{\overline{b_n - b_1 a_n}}{1 - |b_1|^2} \overline{z}^n \quad (1.3)$$

is also a member of $\mathcal{S}_{\mathcal{H}}$. Therefore, we consider the subclass $\mathcal{S}_{\mathcal{H}}^0$ of $\mathcal{S}_{\mathcal{H}}$ defined as

$$\mathcal{S}_{\mathcal{H}}^0 = \{f(z) \in \mathcal{S}_{\mathcal{H}} : b_1 = g'(0) = 0\}. \quad (1.4)$$

Conversely, if $F(z) \in \mathcal{S}_{\mathcal{H}}^0$, then $f(z) = F(z) + \overline{b_1 F(z)} \in \mathcal{S}_{\mathcal{H}}$ for any b_1 ($|b_1| < 1$).

We say that a domain $\mathbb{D}$ is a close-to-convex domain if the complement of $\mathbb{D}$ can be written as a union of nonintersecting half-lines (except that the origin of one half-line may lie on one of the other half-lines). Let $\mathcal{C}$, $\mathcal{C}_{\mathcal{H}}$, and $\mathcal{C}_{\mathcal{H}}^0$ be the respective subclasses of $\mathcal{S}$, $\mathcal{S}_{\mathcal{H}}$, and $\mathcal{S}_{\mathcal{H}}^0$ consisting of all functions $f(z)$, which map $\mathbb{U}$ onto a certain close-to-convex domain.

Bshouty and Lyzzaik [3] have stated the following result.

Theorem 1.1. *If $f(z) = h(z) + \overline{g(z)} \in \mathcal{H}$ satisfies*

$$g'(z) = zh'(z), \quad \operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2} \quad (1.5)$$

for all $z \in \mathbb{U}$, then $f(z) \in \mathcal{C}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}^0$.

A simple and interesting example is below.

Example 1.2. The function

$$f(z) = \frac{1 - (1-z)^2}{2(1-z)^2} + \frac{\overline{z^2}}{2(1-z)^2} = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n + \sum_{n=2}^{\infty} \frac{n-1}{2} \overline{z}^n \quad (1.6)$$

satisfies the conditions of Theorem 1.1, and therefore $f(z)$ belongs to the class $\mathcal{C}_{\mathcal{H}}^0$. We now show that $f(\mathbb{U})$ is actually a close-to-convex domain. It follows that

$$\begin{aligned} f(z) &= \left(\frac{z}{2(1-z)^2} + \frac{z}{2(1-z)} \right) + \overline{\left(\frac{z}{2(1-z)^2} - \frac{z}{2(1-z)} \right)} \\ &= \operatorname{Re} \left(\frac{z}{(1-z)^2} \right) + i \operatorname{Im} \left(\frac{z}{1-z} \right). \end{aligned} \quad (1.7)$$

Setting

$$f(re^{i\theta}) = \frac{-2r^2 + r(1+r^2)\cos\theta}{(1+r^2-2r\cos\theta)^2} + \frac{r\sin\theta}{1+r^2-2r\cos\theta}i = u + iv \quad (1.8)$$

for any $z = re^{i\theta} \in \mathbb{U}$ ($0 \leq r < 1$, $0 \leq \theta < 2\pi$), we see that

$$-4(u + v^2) = \frac{4r(r - \cos\theta)(1 - r\cos\theta)}{(1 + r^2 - 2r\cos\theta)^2} = \frac{4r(r - t)(1 - rt)}{(1 + r^2 - 2rt)^2} \equiv \phi(t) \quad (-1 \leq t = \cos\theta \leq 1). \quad (1.9)$$

Since

$$\phi'(t) = \frac{-4r(1-r^2)^2}{(1+r^2-2rt)^3} \leq 0, \quad (1.10)$$

we obtain that

$$\phi(t) \leq \phi(-1) = \frac{4r}{(1+r)^2} \equiv \psi(r). \quad (1.11)$$

Also, noting that

$$\psi'(r) = \frac{4(1-r)}{(1+r)^3} > 0, \quad (1.12)$$

we know that

$$\psi(r) < \psi(1) = 1, \quad (1.13)$$

which implies that

$$u > -v^2 - \frac{1}{4}. \quad (1.14)$$

Thus, $f(z)$ maps $\mathbb{U}$ onto the following close-to-convex domain as shown in Figure 1.

Remark 1.3. Let $\mathcal{M}$ be the class of all functions satisfying the conditions of Theorem 1.1. Then, it was earlier conjectured by Mocanu [4, 5] that $\mathcal{M} \subset \mathcal{S}_{\mathcal{L}}^0$. Furthermore, we can immediately see that the function $f(z)$ in Example 1.2 is a member of the class $\mathcal{M}$ and it shows that $f(z) \in \mathcal{M}$ is not necessarily starlike with respect to the origin in $\mathbb{U}$ ($f(z)$ is starlike with respect to the origin in $\mathbb{U}$ if and only if $tw \in f(\mathbb{U})$ for all $w \in f(\mathbb{U})$ and t ($0 \leq t \leq 1$)).

Figure 1: The image of $f(z) = (1 - (1 - z)^2)/2(1 - z)^2 + \overline{z^2/2(1 - z)^2}$.

Remark 1.4. For the function $f(z) = h(z) + \overline{g(z)} \in \mathcal{H}$ given by

$$g'(z) = z^{n-1}h'(z) \quad (n = 2, 3, 4, \dots), \quad (1.15)$$

letting $w(t) = f(e^{it}) = h(e^{it}) + \overline{g(e^{it})}$ ($-\pi \leq t < \pi$), we know that

$$\operatorname{Im}\left(\frac{w''(t)}{w'(t)}\right) \leq 0 \quad (-\pi \leq t < \pi), \quad (1.16)$$

which means that $f(z)$ maps the unit circle $\partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ onto a union of several concave curves (see [6, Theorem 2.1]).

Jahangiri and Silverman [7] have given the following coefficient inequality for $f(z) \in \mathcal{H}$ to be in the class $\mathcal{C}_{\mathcal{H}}$.

Theorem 1.5. *If $f(z) \in \mathcal{H}$ satisfies*

$$\sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq 1, \quad (1.17)$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Example 1.6. The function

$$f(z) = z + \frac{1}{5}\overline{z}^5 \quad (1.18)$$

Figure 2: The image of $f(z) = z + (1/5)\bar{z}^5$.

belongs to the class $\mathcal{C}_{\mathcal{H}}^0 \subset \mathcal{C}_{\mathcal{H}}$ and satisfies the condition of Theorem 1.5. Indeed, $f(z)$ maps $\mathbb{U}$ onto the following hypocycloid of six cusps (cf. [8] or [6]) as shown in Figure 2.

The object of this paper is to find some sufficient conditions for functions $f(z) \in \mathcal{H}$ to be in the class $\mathcal{C}_{\mathcal{H}}$. In order to establish our results, we have to recall here the following lemmas due to Clunie and Sheil-Small [1].

Lemma 1.7. *If $h(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $|h'(0)| > |g'(0)|$ and $h(z) + \varepsilon g(z)$ is close-to-convex for each ε ($|\varepsilon| = 1$), then $f(z) = h(z) + \overline{g(z)}$ is harmonic close-to-convex.*

Lemma 1.8. *If $f(z) = h(z) + \overline{g(z)}$ is locally univalent in $\mathbb{U}$ and $h(z) + \varepsilon g(z)$ is convex for some ε ($|\varepsilon| \leq 1$), then $f(z)$ is univalent close-to-convex.*

We also need the following result due to Hayami et al. [9].

Lemma 1.9. *If a function $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ is analytic in $\mathbb{U}$ and satisfies*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} j(j+1) \binom{\alpha}{k-j} A_j \right\} \binom{\beta}{n-k} \right| + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} j(j-1) \binom{\alpha}{k-j} A_j \right\} \binom{\beta}{n-k} \right| \right] \leq 2 \quad (1.19)$$

for some real numbers α and β , then $F(z)$ is convex in $\mathbb{U}$.

2. Main Results

Our first result is contained in the following theorem.

Theorem 2.1. *If $f(z) \in \mathcal{H}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left| n a_n - e^{i\varphi} (n-1) a_{n-1} \right| + \sum_{n=1}^{\infty} \left| n b_n - e^{i\varphi} (n-1) b_{n-1} \right| \leq 1 \quad (2.1)$$

for some real number φ ($0 \leq \varphi < 2\pi$), then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Proof. Let $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ be analytic in $\mathbb{U}$. If $F(z)$ satisfies

$$\sum_{n=2}^{\infty} \left| n A_n - e^{i\varphi} (n-1) A_{n-1} \right| \leq 1, \quad (2.2)$$

then it follows that

$$\begin{aligned} \left| (1 - e^{i\varphi} z) F'(z) - 1 \right| &= \left| \sum_{n=2}^{\infty} (n A_n - e^{i\varphi} (n-1) A_{n-1}) z^{n-1} \right| \\ &\leq \sum_{n=2}^{\infty} \left| n A_n - e^{i\varphi} (n-1) A_{n-1} \right| \cdot |z|^{n-1} \\ &< \sum_{n=2}^{\infty} \left| n A_n - e^{i\varphi} (n-1) A_{n-1} \right| \leq 1 \quad (z \in \mathbb{U}). \end{aligned} \quad (2.3)$$

This gives us that

$$\operatorname{Re} \left((1 - e^{i\varphi} z) F'(z) \right) > 0 \quad (z \in \mathbb{U}), \quad (2.4)$$

that is, $F(z) \in \mathcal{C}$. Then, it is sufficient to prove that

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b_1} = z + \sum_{n=2}^{\infty} \frac{a_n + \varepsilon b_n}{1 + \varepsilon b_1} z^n \in \mathcal{C} \quad (2.5)$$

for each ε ($|\varepsilon| = 1$) by Lemma 1.7. From the assumption of the theorem, we obtain that

$$\begin{aligned} &\sum_{n=2}^{\infty} \left| n \frac{a_n + \varepsilon b_n}{1 + \varepsilon b_1} - e^{i\varphi} (n-1) \frac{a_{n-1} + \varepsilon b_{n-1}}{1 + \varepsilon b_1} \right| \\ &\leq \frac{1}{1 - |b_1|} \sum_{n=2}^{\infty} \left[\left| n a_n - e^{i\varphi} (n-1) a_{n-1} \right| + \left| n b_n - e^{i\varphi} (n-1) b_{n-1} \right| \right] \leq \frac{1 - |b_1|}{1 - |b_1|} = 1. \end{aligned} \quad (2.6)$$

This completes the proof of the theorem. $\square$

Figure 3: The image of $f(z) = -\bar{z} - 2 \log|1 - z|$.

Example 2.2. The function

$$f(z) = -\log(1 - z) + \overline{(-mz - \log(1 - z))} = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n + (1 - m)\bar{z} + \sum_{n=2}^{\infty} \frac{1}{n} \bar{z}^n \quad (0 < m \leq 1) \quad (2.7)$$

satisfies the condition of Theorem 2.1 with $\varphi = 0$ and belongs to the class $\mathcal{C}_{\mathcal{A}}$. In particular, putting $m = 1$, we obtain Figure 3.

By making use of Lemma 1.8 with $\varepsilon = 0$ and applying Lemma 1.9, we readily obtain the next theorem.

Theorem 2.3. *If $f(z) \in \mathcal{A}$ is locally univalent in $\mathbb{U}$ and satisfies*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} j(j+1) \binom{\alpha}{k-j} a_j \right\} \binom{\beta}{n-k} \right| + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} j(j-1) \binom{\alpha}{k-j} a_j \right\} \binom{\beta}{n-k} \right| \right] \leq 2 \quad (2.8)$$

for some real numbers α and β , then $f(z) \in \mathcal{C}_{\mathcal{A}}$.

Putting $\alpha = \beta = 0$ in the above theorem, we arrive at the following result due to Jahangiri and Silverman [7].

Theorem 2.4. *If $f(z) \in \mathcal{H}$ is locally univalent in $\mathbb{U}$ with*

$$\sum_{n=2}^{\infty} n^2 |a_n| \leq 1, \quad (2.9)$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Furthermore, taking $\alpha = 1$ and $\beta = 0$ in the theorem, we have the following corollary.

Corollary 2.5. *If $f(z) \in \mathcal{H}$ is locally univalent in $\mathbb{U}$ and satisfies*

$$\sum_{n=2}^{\infty} \{n|(n+1)a_n - (n-1)a_{n-1}| + (n-1)|na_n - (n-2)a_{n-1}|\} \leq 2, \quad (2.10)$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Example 2.6. The function

$$f(z) = -\int_0^z \frac{\log(1-t)}{t} dt + \overline{(z + (1-z)\log(1-z))} = z + \sum_{n=2}^{\infty} \frac{1}{n^2} z^n + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \bar{z}^n \quad (2.11)$$

satisfies the conditions of Corollary 2.5 and belongs to the class $\mathcal{C}_{\mathcal{H}}$ as shown in Figure 4.

3. Appendix

A sequence $\{c_n\}_{n=0}^{\infty}$ of nonnegative real numbers is called a convex null sequence if $c_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$c_n - c_{n+1} \geq c_{n+1} - c_{n+2} \geq 0 \quad (3.1)$$

for all n ($n = 0, 1, 2, \dots$).

The next lemma was obtained by Fejér [10].

Lemma 3.1. *Let $\{c_n\}_{k=0}^{\infty}$ be a convex null sequence. Then, the function*

$$p(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \quad (3.2)$$

is analytic and satisfies $\operatorname{Re}(p(z)) > 0$ in $\mathbb{U}$.

Applying the above lemma, we deduce the following theorem.

Figure 4: The image of $f(z) = -\int_0^z (\log(1-t)/t) dt + \overline{(z + (1-z)\log(1-z))}$.

Theorem 3.2. For some b ($|b| < 1$) and some convex null sequence $\{c_n\}_{n=0}^\infty$ with $c_0 = 2$, the function

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n + b \overline{\left(z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n \right)} \quad (3.3)$$

belongs to the class $\mathcal{C}_{\mathcal{L}}$.

Proof. Let us define $F(z)$ by

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b} = z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n \quad (3.4)$$

for each ε ($|\varepsilon| = 1$). Then, we know that

$$F'(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \quad (c_0 = 2). \quad (3.5)$$

By virtue of Lemmas 1.7 and 3.1, it follows that $\operatorname{Re}(F'(z)) > 0$ ($z \in \mathbb{U}$), that is, $F(z) \in \mathcal{C}$. Thus, we conclude that $f(z) = h(z) + \overline{g(z)} \in \mathcal{C}_{\mathcal{L}}$. $\square$

In the same manner, we also have the following theorem.

Theorem 3.3. For some b ($|b| < 1$) and some convex null sequence $\{c_n\}_{n=0}^\infty$ with $c_0 = 2$, the function

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} c_j \right) z^n + b \overline{\left(z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} c_j \right) z^n \right)} \quad (3.6)$$

belongs to the class $\mathcal{C}_{\mathcal{A}}$.

Proof. Let us define $F(z)$ by

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b} = z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} c_j \right) z^n \quad (3.7)$$

for each ε ($|\varepsilon| = 1$). Then, we know that

$$(1 - z)F'(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \quad (c_0 = 2). \quad (3.8)$$

Therefore, by the help of Lemmas 1.7 and 3.1, we obtain that $\operatorname{Re}((1 - z)F'(z)) > 0$ ($z \in \mathbb{U}$), that is, $F(z) \in \mathcal{C}$, which implies that $f(z) = h(z) + \overline{g(z)} \in \mathcal{C}_{\mathcal{A}}$. $\square$

Remark 3.4. The sequence

$$\{c_n\}_{n=0}^\infty = \left\{ 2, 1, \frac{2}{3}, \dots, \frac{2}{n+1}, \dots \right\} \quad (3.9)$$

is a convex null sequence because

$$\begin{aligned} \lim_{n \rightarrow \infty} c_n &= \lim_{n \rightarrow \infty} \left(\frac{2}{n+1} \right) = 0, & c_n - c_{n+1} &= \frac{2}{(n+1)(n+2)} \geq 0, \\ (c_n - c_{n+1}) - (c_{n+1} - c_{n+2}) &= \frac{4}{(n+1)(n+2)(n+3)} \geq 0 \quad (n = 0, 1, 2, \dots). \end{aligned} \quad (3.10)$$

Setting $b = 1/4$ in Theorem 3.2 with the above sequence $\{c_n\}_{n=0}^\infty$, we derive the following example.

Example 3.5. The function

$$f(z) = -z - 2 \int_0^z \frac{\log(1-t)}{t} dt - \frac{1}{4} \overline{\left(z + 2 \int_0^z \frac{\log(1-t)}{t} dt \right)} = z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n + \frac{1}{4} \overline{\left(z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n \right)} \quad (3.11)$$

is in the class $\mathcal{C}_{\mathcal{A}}$ as shown in Figure 5.

Figure 5: The image of $f(z)$ in Example 3.5.

Figure 6: The image of $f(z)$ in Example 3.7.

Moreover, we know the following remark.

Remark 3.6. The sequence

$$\{c_n\}_{n=0}^{\infty} = \left\{2, 1, \frac{1}{2}, \dots, 2^{1-n}, \dots\right\} \quad (3.12)$$

is a convex null sequence because

$$\begin{aligned} \lim_{n \rightarrow \infty} c_n &= \lim_{n \rightarrow \infty} 2^{1-n} = 0, & c_n - c_{n+1} &= 2^{-n} \geq 0, \\ (c_n - c_{n+1}) - (c_{n+1} - c_{n+2}) &= 2^{-(n+1)} \geq 0 \quad (n = 0, 1, 2, \dots). \end{aligned} \quad (3.13)$$

Hence, letting $b = 1/4$ in Theorem 3.3 with the sequence $\{c_n\}_{n=0}^{\infty} = \{2^{1-n}\}_{n=0}^{\infty}$, we have the following example.

Example 3.7. The function

$$\begin{aligned} f(z) &= -3 \log(1-z) + 4 \log\left(1 - \frac{z}{2}\right) + \overline{\left(-\frac{3}{4} \log(1-z) + \log\left(1 - \frac{z}{2}\right)\right)} \\ &= z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} 2^{1-j}\right) z^n + \frac{1}{4} \overline{\left(z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} 2^{1-j}\right) z^n\right)} \end{aligned} \quad (3.14)$$

is in the class $\mathcal{C}_{\mathcal{A}}$ as shown in Figure 6.

Dedication

This paper is dedicated to Professor Owa on the occasion of his retirement from Kinki University.

Acknowledgment

The author expresses his sincere thanks to the referees for their valuable suggestions and comments for improving this paper.

References

- [1] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiæ Scientiarum Fennicæ. Series A*, vol. 9, pp. 3–25, 1984.
- [2] H. Lewy, "On the non-vanishing of the Jacobian in certain one-to-one mappings," *Bulletin of the American Mathematical Society*, vol. 42, no. 10, pp. 689–692, 1936.
- [3] D. Bshouty and A. Lyzzaik, "Close-to-convexity criteria for planar harmonic mappings," *Complex Analysis and Operator Theory*, vol. 5, no. 3, pp. 767–774, 2011.
- [4] P. T. Mocanu, "Three-cornered hat harmonic functions," *Complex Variables and Elliptic Equations*, vol. 54, no. 12, pp. 1079–1084, 2009.
- [5] P. T. Mocanu, "Injectivity conditions in the complex plane," *Complex Analysis and Operator Theory*, vol. 5, no. 3, pp. 759–766, 2011.
- [6] T. Hayami and S. Owa, "Hypocycloid of $n + 1$ cusps harmonic function," *Bulletin of Mathematical Analysis and Applications*, vol. 3, pp. 239–246, 2011.
- [7] J. M. Jahangiri and H. Silverman, "Harmonic close-to-convex mappings," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 15, no. 1, pp. 23–28, 2002.
- [8] P. Duren, *Harmonic Mappings in the Plane*, vol. 156 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2004.
- [9] T. Hayami, S. Owa, and H. M. Srivastava, "Coefficient inequalities for certain classes of analytic and univalent functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 8, no. 4, article 95, pp. 1–10, 2007.
- [10] L. Fejér, "Über die positivität von summen, die nach trigonometrischen oder Legendreschen, funktionen fortschreiten. I," *Acta Szeged*, vol. 2, pp. 75–86, 1925.