Research Article

Warped Product Pseudo-Slant Submanifolds of a Nearly Cosymplectic Manifold

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We study warped product pseudo-slant submanifolds of a nearly cosymplectic manifold. We obtain some characterization results on the existence or nonexistence of warped product pseudo-slant submanifolds of a nearly cosymplectic manifold in terms of the canonical structures $P$ and $F$.

1. Introduction

To study the manifolds with negative curvature, Bishop and O’Neill [1] introduced the notion of warped product manifolds by homothetically warping the product metric of a product manifold $N_1 \times N_2$ onto the fibers $p \times N_2$ for each $p \in N_1$. Later on, the geometrical aspect of these manifolds has been studied by many researchers (cf. [2–4]). Pseudo-slant submanifolds were introduced by Carriazo [5] as a special case of bislant submanifolds.

Almost contact manifolds with Killing structure tensors were defined in [6] as nearly cosymplectic manifolds, and it was shown that the normal nearly cosymplectic manifolds are cosymplectic (see also [7]). Later on, Blair and Showers [8] studied nearly cosymplectic structure $(\phi, \xi, \eta, g)$ on a Riemannian manifold $\overline{M}$ with $\eta$ closed from the topological viewpoint.

Recently, Sahin [9] studied the warped product hemislant (pseudo-slant) submanifolds of Kaehler manifolds. He proved that the warped product submanifolds of the type $M = N_1 \times f N_0$ of a Kaehler manifold $\overline{M}$ do not exist and obtained some characterization results on the existence of warped product submanifold $M = N_0 \times f N_1$, where $N_1$ and $N_0$ are totally real and proper slant submanifolds of a Kaehler manifold $\overline{M}$, respectively. After that, we have extended this study to the more general setting of nearly Kaehler manifolds [4]. The warped product semi-invariant submanifolds of a nearly cosymplectic manifold had been studied in [10].
In this paper, we study warped product pseudo-slant submanifolds of a nearly cosymplectic manifold. We obtain some characterization results of warped product submanifolds of the types \(N_{1} \times fN_{\theta}\) and \(N_{\theta} \times fN_{1}\) in terms of the canonical structures \(P\) and \(F\), where \(N_{1}\) and \(N_{\theta}\) are anti-invariant and proper slant submanifolds of a nearly cosymplectic manifold \(\overline{M}\), respectively.

### 2. Preliminaries

A \((2n + 1)\)-dimensional \(C^\infty\) manifold \(\overline{M}\) is said to have an almost contact structure if there exist on \(\overline{M}\) a tensor field \(\phi\) of type \((1, 1)\), a vector field \(\xi\), and a 1-form \(\eta\) satisfying [8]

\[
\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.
\]

There always exists a Riemannian metric \(g\) on an almost contact manifold \(\overline{M}\) satisfying the following compatibility condition:

\[
\eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

where \(X\) and \(Y\) are vector fields on \(\overline{M}\) [8].

An almost contact structure \((\phi, \xi, \eta)\) is said to be normal if the almost complex structure \(J\) on the product manifold \(\overline{M} \times \mathbb{R}\) given by

\[
J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),
\]

where \(f\) is a \(C^\infty\)-function on \(\overline{M} \times \mathbb{R}\) has no torsion, that is, \(J\) is integrable, the condition for normality in terms of \(\phi, \xi, \eta\) is \([\phi, \phi] + 2d\eta \otimes \xi = 0\) on \(\overline{M}\), where \([\phi, \phi]\) is the Nijenhuis tensor of \(\phi\). Finally the fundamental 2-form \(\Phi\) is defined by \(\Phi(X, Y) = g(X, \phi Y)\).

An almost contact metric structure \((\phi, \xi, \eta, g)\) is said to be cosymplectic, if it is normal and both \(\Phi\) and \(\eta\) are closed [8]. The structure is said to be nearly cosymplectic if \(\phi\) is Killing, that is, if

\[
(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X = 0,
\]

for any \(X, Y \in T\overline{M}\), where \(T\overline{M}\) is the tangent bundle of \(\overline{M}\) and \(\overline{\nabla}\) denotes the Riemannian connection of the metric \(g\). Equation (2.4) is equivalent to \((\overline{\nabla}_X\phi)X = 0\), for each \(X \in T\overline{M}\). The structure is said to be closely cosymplectic if \(\phi\) is Killing and \(\eta\) is closed. It is well known that an almost contact metric manifold is cosymplectic if and only if \(\overline{\nabla}\phi\) vanishes identically, that is, \((\overline{\nabla}_X\phi)Y = 0\) and \(\overline{\nabla}_X\xi = 0\).

**Proposition 2.1** (see [8]). On a nearly cosymplectic manifold the vector field \(\xi\) is Killing.

From the above proposition, one has \(\overline{\nabla}_X\xi = 0\), for any vector field \(X\) tangent to \(\overline{M}\), where \(\overline{M}\) is a nearly cosymplectic manifold.
Let $M$ be submanifold of an almost contact metric manifold $\overline{M}$ with induced metric $g$ and if $\nabla$ and $\nabla^\perp$ are the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively, then Gauss and Weingarten formulae are given by

\[
\nabla_X Y = \nabla_X Y + h(X, Y),
\]
\[
\nabla_X N = -A_N X + \nabla^\perp_X N,
\]\n
for each $X, Y \in TM$ and $N \in T^\perp M$, where $h$ and $A_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field $N$), respectively, for the immersion of $M$ into $\overline{M}$. They are related as

\[
g(h(X, Y), N) = g(A_N X, Y),
\]\n
where $g$ denotes the Riemannian metric on $\overline{M}$ as well as induced on $M$.

For any $X \in TM$, one writes

\[
\phi X = PX + FX,
\]

where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$.

Similarly for any $N \in T^\perp M$, one writes

\[
\phi N = BN + CN,
\]

where $BN$ is the tangential component and $CN$ is the normal component of $\phi N$.

Now, denote by $\rho_X Y$ and $Q_X Y$ the tangential and normal parts of $(\nabla_X \phi) Y$, that is,

\[
(\nabla_X \phi) Y = \rho_X Y + Q_X Y
\]

for all $X, Y \in TM$. Making use of (2.8), (2.10), and the Gauss and Weingarten formulae, the following equations may easily be obtained:

\[
\rho_X Y = (\nabla_X P) Y - A_{FY} X - Bh(X, Y),
\]
\[
Q_X Y = (\nabla_X F) Y + h(X, PY) - Ch(X, Y).
\]

Similarly, for any $N \in T^\perp M$, denoting tangential and normal parts of $(\nabla_X \phi) N$ by $\rho_X N$ and $Q_X N$, respectively, one obtains

\[
\rho_X N = (\nabla_X B) N + PA_N X - A_{CN} X,
\]
\[
Q_X N = (\nabla_X C) N + h(BN, X) + FA_N X,
\]
where the covariant derivatives of $P,F,B$, and $C$ are defined by

\[
\begin{align*}
\nabla_X P &= \nabla_X PY - P \nabla_X Y, \\
\nabla_X F &= \nabla_X FY - F \nabla_X Y, \\
\nabla_X B &= \nabla_X BN - B \nabla_X N, \\
\nabla_X C &= \nabla_X CN - C \nabla_X N,
\end{align*}
\]  

for all $X,Y \in TM$ and $N \in T^\perp M$.

It is straightforward to verify the following properties of $P$ and $Q$, which one enlists here for later use

\(p_1\) (i) $P_{X+Y} = P_X + P_Y$, (ii) $Q_{X+Y} = Q_X + Q_Y$,

\(p_2\) (i) $P_X(Y + W) = P_X Y + P_X W$, (ii) $Q_X(Y + W) = Q_X Y + Q_X W$,

\(p_3\) (i) $g(P_X Y, W) = -g(Y, P_X W)$, (ii) $g(Q_X Y, N) = -g(Y, Q_X N)$,

\(p_4\) $P_X \phi Y + Q_X \phi Y = -\phi(P_X Y + Q_X Y),$

for all $X,Y \in TM$ and $N \in T^\perp M$.

On a submanifold $M$ of a nearly cosymplectic manifold, by (2.4) and (2.10), one has

\[
\begin{align*}
&\text{(a) } P_X Y + P_Y X = 0, \\
&\text{(b) } Q_X Y + Q_Y X = 0,
\end{align*}
\]  

for any $X,Y \in TM$.

The submanifold $M$ is said to be invariant if $F$ is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand, $M$ is said to be anti-invariant if $P$ is identically zero, that is, $\phi X \in T^\perp M$, for any $X \in TM$.

One will always consider $\xi$ to be tangent to the submanifold $M$. There is another class of submanifolds that is called the slant submanifold. For each nonzero vector $X$ tangent to $M$ at any $x \in M$, such that $X$ is not proportional to $\xi_x$, one denotes by $0 \leq \theta(X) \leq \pi/2$, the angle between $\phi X$ and $T_x M$ is called the slant angle. If the slant angle $\theta(X)$ is constant for all $X \in T_x M - (\xi_x)$ and $x \in M$, then $M$ is said to be a slant submanifold [11]. Obviously, if $\theta = 0$, then $M$ is an invariant submanifold and if $\theta = \pi/2$, then $M$ is an anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant.

One recalls the following result for a slant submanifold.

**Theorem 2.2** (see [11]). Let $M$ be a submanifold of an almost contact metric manifold $\overline{M}$, such that $\xi \in TM$. Then $M$ is slant if and only if there exists a constant $\lambda \in [0,1]$ such that

\[
P^2 = \lambda (-I + \eta \otimes \xi).
\]  

Furthermore, if $\theta$ is slant angle, then $\lambda = \cos^2 \theta$. 

The following relations are straightforward consequence of (2.18):

\[
\begin{align*}
g(PX, PY) &= \cos^2 \theta (g(X, Y) - \eta(Y)\eta(X)), \\
g(FX, FY) &= \sin^2 \theta (g(X, Y) - \eta(Y)\eta(X)),
\end{align*}
\]

(2.19) (2.20)

for all \(X, Y \in TM\).

A submanifold \(M\) of an almost contact manifold \(\overline{M}\) is said to be a pseudo-slant submanifold if there exist two orthogonal complementary distributions \(D_1\) and \(D_2\) satisfying:

(i) \(TM = D_1 \oplus D_2 \oplus \langle \xi \rangle\),

(ii) \(D_1\) is a slant distribution with slant angle \(\theta \neq \pi/2\),

(iii) \(D_2\) is anti-invariant that is, \(\phi D_2 \subseteq T^\perp M\).

A pseudo-slant submanifold \(M\) of an almost contact manifold \(\overline{M}\) is mixed geodesic if

\[h(X, Z) = 0,\]

(2.21)

for any \(X \in D_1\) and \(Z \in D_2\).

If \(\mu\) is the invariant subspace of the normal bundle \(T^\perp M\), then in the case of pseudo-slant submanifold, the normal bundle \(T^\perp M\) can be decomposed as follows:

\[T^\perp M = FD_1 \oplus FD_2 \oplus \mu.\]

(2.22)

3. Warped Product Pseudo-Slant Submanifolds

Bishop and O’Neill [1] introduced the notion of warped product manifolds. These manifolds are the natural generalizations of Riemannian product manifolds. They defined these manifolds as follows. Let \((N_1, g_1)\) and \((N_2, g_2)\) be two Riemannian manifolds and \(f\), a positive differentiable function on \(N_1\). The warped product of \(N_1\) and \(N_2\) is the Riemannian manifold \(N_1 \times_f N_2 = (N_1 \times N_2, g)\), where

\[g = g_1 + f^2 g_2.\]

(3.1)

A warped product manifold \(N_1 \times_f N_2\) is said to be trivial if the warping function \(f\) is constant. We recall the following general formula on a warped product manifold [1]:

\[\nabla_X Z = \nabla_Z X = (X \ln f) Z,\]

(3.2)

where \(X\) is tangential to \(N_1\) and \(Z\) is tangential to \(N_2\).

Let \(M = N_1 \times_f N_2\) be a warped product manifold. This means that \(N_1\) is totally geodesic and \(N_2\) is a totally umbilical submanifold of \(M\), respectively [1].

Throughout this section, we consider the warped product pseudo-slant submanifolds which are either in the form \(N_\theta \times_f N_0\) or \(N_0 \times_f N_\perp\) in a nearly cosymplectic manifold \(\overline{M}\), where \(N_\theta\) and \(N_\perp\) are proper slant and anti-invariant submanifolds of a nearly cosymplectic
manifold $\overline{M}$, respectively. On a warped product submanifold $M = N_1 \times f N_2$ of a nearly cosymplectic manifold $\overline{M}$, we have the following result.

**Theorem 3.1** (see [10]). A warped product submanifold $M = N_1 \times f N_2$ of a nearly cosymplectic manifold $\overline{M}$ is an usual Riemannian product if the structure vector field $\xi$ is tangential to $M_2$, where $M_1$ and $M_2$ are the Riemannian submanifolds of $\overline{M}$.

Now, one considers the warped product pseudo-slant submanifolds in the form $M = N_\perp \times f N_\theta$ of a nearly cosymplectic manifold $\overline{M}$. If one considers the structure vector field $\xi \in TN_\theta$ then by Theorem 3.1, the warping function $f$ is constant and hence one will considers $\xi \in TN_\perp$.

**Proposition 3.2.** Let $M = N_\perp \times f N_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,

$$g\left(\nabla_{PX}FX,FZ\right) = \left(Z \ln f\right) \sin^2 \theta \|X\|^2 + \left(1 + \cos^2 \theta\right) g(h(X,PX),FZ),$$

(3.3)

for any $X \in TN_\theta$ and $Z \in TN_\perp$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

**Proof.** For any $X \in TN_\theta$ and $Z \in TN_\perp$, by (2.8), we have

$$g\left(\overline{\nabla}_XFX,FZ\right) = g\left(\overline{\nabla}_XFX,FZ\right) + g\left(\overline{\nabla}_FX,FZ\right).$$

(3.4)

Using (2.5), (2.6), and the covariant derivative property of $\phi$, we obtain

$$g\left(\phi \overline{\nabla}_XFX,FZ\right) = g(h(X,PX),FZ) + g\left(\overline{\nabla}_FX,FZ\right).$$

(3.5)

Then from (2.2), (2.4), and the fact that $\xi$ is a Killing vector field on $\overline{M}$, thus we obtain

$$g\left(\overline{\nabla}_XZ\right) = g(h(X,PX),FZ) + g\left(\overline{\nabla}_FX,FZ\right).$$

(3.6)

Using the property of $\overline{\nabla}$, we get

$$-g\left(X,\overline{\nabla}_XZ\right) = g(h(X,PX),FZ) + g\left(\overline{\nabla}_FX,FZ\right).$$

(3.7)

Then by (2.5) and (3.2), we derive

$$-(Z \ln f) \|X\|^2 = g(h(PX,X),FZ) + g\left(\overline{\nabla}_FX,FZ\right).$$

(3.8)
Interchanging $X$ by $PX$ in (3.8) and using (2.18), (2.19), and the fact that $\xi \in TN_\bot$, we obtain

$$-(Z \ln f) \cos^2 \theta \|X\|^2 = -\cos^2 \theta g(h(X, PX), FZ) + g\left(\nabla^\perp_{PX} FX, FZ\right).$$  \hspace{1cm} (3.9)$$

Thus, the result follows from (3.8) and (3.9).

\textbf{Proposition 3.3.} Let $M = N_\bot \times f N_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then,

$$g\left(\left(\nabla^<_X F\right) X, FZ\right) = \sec^2 \theta g\left(\left(\nabla^<_{PX} F\right) PX, FZ\right)$$  \hspace{1cm} (3.10)$$

for any $X \in TN_\theta$ and $Z \in TN_\bot$, where $N_\theta$ and $N_\bot$ are proper slant and anti-invariant submanifolds of $\tilde{M}$, respectively.

\textbf{Proof.} For any $X \in TN_\theta$ and $Z \in TN_\bot$ by (2.14), we have

$$g\left(\nabla^<_X FX, FZ\right) = g\left(\left(\nabla^<_X F\right) X, FZ\right) + g\left(F \nabla_X X, FZ\right).$$  \hspace{1cm} (3.11)$$

Using (2.20), (2.5), and the fact that $\xi$ is killing vector field, we obtain

$$g\left(\nabla^<_X FX, FZ\right) = g\left(\left(\nabla^<_X F\right) X, FZ\right) - \sin^2 \theta g(X, \nabla_X Z).$$  \hspace{1cm} (3.12)$$

Then from (3.2), we derive

$$g\left(\nabla^<_X FX, FZ\right) = g\left(\left(\nabla^<_X F\right) X, FZ\right) - (Z \ln f) \sin^2 \theta \|X\|^2.$$  \hspace{1cm} (3.13)$$

Now, from (3.8) and (3.13), we obtain

$$g\left(\left(\nabla^<_X F\right) X, FZ\right) = -(Z \ln f) \cos^2 \theta \|X\|^2 - g\left(h(X, PX), FZ\right).$$  \hspace{1cm} (3.14)$$

Interchanging $X$ by $PX$ in (3.14) and then using (2.18), (2.19), and the fact that $\xi \in TN_\bot$, we get

$$g\left(\left(\nabla^<_{PX} F\right) PX, FZ\right) = -(Z \ln f) \cos^4 \theta \|X\|^2 - \cos^2 \theta g\left(h(X, PX), FZ\right).$$  \hspace{1cm} (3.15)$$

From (3.14) and (3.15), we arrive at

$$g\left(\left(\nabla^<_X F\right) X, FZ\right) = \sec^2 \theta g\left(\left(\nabla^<_{PX} F\right) PX, FZ\right).$$  \hspace{1cm} (3.16)$$

Hence, the result is proved. \qed
Lemma 3.4. Let $M = N_f N_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,

$$g(\mathcal{D}_X PX, Z) = g(h(X, Z), FPX) - g(h(PX, Z), FX)$$  \hspace{1cm} (3.17)

for any $X \in TN_\theta$ and $Z \in TN_\perp$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

Proof. For any $X \in TN_\theta$ and $Z \in TN_\perp$ by (2.5), we have

$$g(h(PX, Z), FX) = g(\phi Z PX, FX) = -g(PX, \phi Z FX),$$  \hspace{1cm} (3.18)

Then from (2.8), we derive

$$g(h(PX, Z), FX) = g(PX, \phi Z PX) - g(PX, \phi Z FX).$$  \hspace{1cm} (3.19)

From the covariant derivative property of $\phi$ and (2.5), we obtain

$$g(h(PX, Z), FX) = g(PX, \nabla Z PX) - g\left( PX, \left( \phi Z X \right) \right) - g\left( PX, \phi Z \nabla Z X \right).$$  \hspace{1cm} (3.20)

By (2.2), (2.10), and (3.2), we derive

$$g(h(PX, Z), FX) = (Z \ln f) g(PX, PX) - g(PX, P Z X) + g\left( \phi PX, \phi Z X \right).$$  \hspace{1cm} (3.21)

Using (2.5), (2.8), (2.17)(a), (2.19) and the fact that $\xi \in TN_\perp$, we get

$$g(h(PX, Z), FX) = (Z \ln f) \cos^2 \theta \|X\|^2 + g(PX, P X Z)$$

$$+ g\left( P^2 X, \nabla X \right) + g(h(X, Z), FPX).$$  \hspace{1cm} (3.22)

Thus, by property $(p_3)(i)$, (2.18), and (3.2) and the fact that $\xi \in TN_\perp$, we obtain

$$g(h(PX, Z), FX) = (Z \ln f) \cos^2 \theta \|X\|^2 - g(P X PX, Z)$$

$$- (Z \ln f) \cos^2 \theta \|X\|^2 + g(h(X, Z), FPX).$$  \hspace{1cm} (3.23)

Hence, the above equation takes the form

$$g(P X PX, Z) = g(h(X, Z), FPX) - g(h(PX, Z), FX),$$  \hspace{1cm} (3.24)

which proves our assertion.  \hspace{2cm} \Box
**Theorem 3.5.** Let $M = N_L \times_f N_\theta$ be a warped product submanifold of a nearly cosymplectic manifold $\overline{M}$. Then $M$ is Riemannian product of $N_L$ and $N_\theta$ if and only if $\mathcal{D}_X TX \in TN_\theta$, for any $X \in TN_\theta$, where $N_\theta$ and $N_L$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

**Proof.** If the structure vector field $\xi \in TN_\theta$, then by Theorem 3.1, $M$ is Riemannian product of $N_L$ and $N_\theta$. Now, we consider $\xi \in TN_L$, then for any $X \in TN_\theta$ and $Z \in TN_L$ from (2.5), we have

$$g(h(X, PX), FZ) = g\left(\nabla_{PX} X, \phi Z\right).\quad (3.25)$$

Then by (2.2), we get

$$g(h(X, PX), FZ) = -g\left(\phi \nabla_{PX} X, Z\right).\quad (3.26)$$

Using the covariant derivative formula of $\phi$, we derive

$$g(h(X, PX), FZ) = g\left(\nabla_{PX} \phi X, Z\right) - g\left(\nabla_{PX} \phi X, Z\right).\quad (3.27)$$

Then from (2.10) and the property of $\nabla$, we obtain

$$g(h(X, PX), FZ) = g(\mathcal{D}_{PX} X, Z) + g\left(\nabla_{PX} Z, X\right).\quad (3.28)$$

Thus by (2.5), (2.8), and (2.17)(a), we arrive at

$$g(h(X, PX), FZ) = -g(\mathcal{D}_{PX} X, Z) + g(PX, \nabla_{PX} Z) + g(h(PX, Z), FX).\quad (3.29)$$

Using (3.2) and then (2.19) and the fact that $\xi \in TN_L$, we get

$$g(h(X, PX), FZ) = -g(\mathcal{D}_{PX} X, Z) + (Z \ln f)\cos^2 \theta \|X\|^2$$

$$+ g(h(PX, Z), FX).\quad (3.30)$$

By property $(p_1)(i)$, we derive

$$g(h(X, PX), FZ) = g(PX, \mathcal{D}_{PX} Z) + (Z \ln f)\cos^2 \theta \|X\|^2$$

$$+ g(h(PX, Z), FX).\quad (3.31)$$

Interchanging $X$ by $PX$ in (3.30) and then using (2.18), (2.19), and the fact that $\xi \in TN_L$, we obtain

$$-\cos^2 \theta g(h(X, PX), FZ) = -\cos^2 \theta g(X, \mathcal{D}_{PX} Z) + (Z \ln f)\cos^4 \theta \|X\|^2$$

$$- \cos^2 \theta g(h(X, Z), FX).\quad (3.32)$$
Using the property $(p_3)(i)$ and then $(2.17)(a)$, we arrive at

\[-g(h(X, PX), FZ) = -g(D_X PX, Z) + (Z \ln f) \cos^2 \theta \|X\|^2
- g(h(X, Z), FPX).\]  \hspace{1cm} (3.33)

Then from (3.30) and (3.33), we obtain

\[2(Z \ln f) \cos^2 \theta \|X\|^2 = 2g(D_X PX, Z) + g(h(X, Z), FPX)
- g(h(PX, Z), FX).\]  \hspace{1cm} (3.34)

Thus, by Lemma 3.4, we conclude that

\[(Z \ln f) \cos^2 \theta \|X\|^2 = \frac{3}{2} g(D_X PX, Z).\]  \hspace{1cm} (3.35)

Since $N_\theta$ is proper slant, thus we get $(Z \ln f) = 0$, if and only if $D_X PX$ lies in $TN_\theta$ for all $X \in TN_\theta$ and $Z \in TN_\perp$. This proves the theorem completely.

Now, we discuss the other case, that is, the warped product submanifold $M = N_\theta \times f N_\perp$ of a nearly cosymplectic manifold $\overline{M}$. In this case also, if the structure vector filed $\xi \in TN_\perp$ then the warping function $f$ is constant (by Theorem 3.1), thus we consider $\xi \in TN_\theta$.

**Proposition 3.6.** Let $M = N_\theta \times f N_\perp$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,

\[g\left(\left(\nabla_X F\right)Z, FX\right) + g\left(\left(\nabla_{D_X F}\right)Z, FPX\right) = \sin^2 \theta g(h(X, PX), FZ)
+ \left(1+\cos^2 \theta\right) g(D_X Z, PX) - \cos^2 \theta \eta(X) g(D_\xi Z, PX)
- g(Q_Z X, FX) - g(Q_Z PX, FPX)\]  \hspace{1cm} (3.36)

for any $X \in TN_\theta$ and $Z \in TN_\perp$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

**Proof.** For any $X \in TN_\theta$ and $Z \in TN_\perp$, by (2.2) we have

\[g\left(\phi \nabla_X Z, \phi X\right) = g\left(\nabla_X Z, X\right) - \eta(X)g\left(\nabla_X Z, \xi\right).\]  \hspace{1cm} (3.37)

Using the property of the connection $\nabla$ and the fact that $\xi$ is a Killing vector field, then, from (2.5), we obtain

\[g\left(\phi \nabla_X Z, \phi X\right) = g(\nabla_X Z, X).\]  \hspace{1cm} (3.38)
Thus by (3.2) and the covariant derivative formula of $\phi$, we derive

$$g\left(\nabla_X \phi Z, \phi X\right) - g\left(\left(\nabla_X \phi\right)Z, \phi X\right) = (X \ln f) g(Z, X). \quad (3.39)$$

Then form (2.6), (2.8), (2.10), and by the orthogonality of two distributions, we get

$$-g(A_FZ, PX) + g\left(\nabla_X FZ, FX\right) - g(PXZ, PX) - g(QXZ, FX) = 0. \quad (3.40)$$

Thus, on using (2.7) and (2.17)(b), the above equation takes the form

$$g\left(\nabla_X FZ, FX\right) = g(h(X, PX), FZ) + g(PXZ, PX) - g(QZX, FX). \quad (3.41)$$

Now, for any $X \in TN_\theta$ and $Z \in TN_\perp$ from (2.14), we have

$$g\left(\nabla_X FZ, FX\right) = g\left(\left(\nabla_X F\right)Z, FX\right) + g(F\nabla_X Z, FX). \quad (3.42)$$

Using (3.2), we obtain

$$g\left(\nabla_X FZ, FX\right) = g\left(\left(\nabla_X F\right)Z, FX\right) + (X \ln f) g(FZ, FX). \quad (3.43)$$

By orthogonality of two normal distributions, we get

$$g\left(\nabla_X FZ, FX\right) = g\left(\left(\nabla_X F\right)Z, FX\right). \quad (3.44)$$

Then, from (3.41) and (3.44), we obtain

$$g\left(\left(\nabla_X F\right)Z, FX\right) = g(h(X, PX), FZ) + g(PXZ, PX) - g(QZX, FX). \quad (3.45)$$

Interchanging $X$ by $PX$ in (3.45) and using (2.18) and the fact that $h(X, \xi) = 0$, for any $X$ on a nearly cosymplectic manifold $\overline{M}$, hence we get

$$g\left(\left(\nabla_{PX} F\right)Z, FPX\right) = -\cos^2 \theta g(h(X, PX), FZ) - \cos^2 \theta g(PXZ, X) \quad (3.46)$$

$$+ \cos^2 \theta \eta(X) g(PXZ, \xi) - g(QZPX, FPX).$$

Using property (p3)(i) and (2.17), we derive

$$g\left(\left(\nabla_{PX} F\right)Z, FPX\right) = -\cos^2 \theta g(h(X, PX), FZ) - \cos^2 \theta g(PXZ, Z) \quad (3.47)$$

$$+ \cos^2 \theta \eta(X) g(PXZ, Z) - g(QZPX, FPX).$$
Again, by property $(p_3) (i)$, we obtain
\[
g\left(\left(\overline{\nabla}_PF\right)Z, FPX\right) = -\cos^2\theta g(h(X, PX), FZ) + \cos^2\theta g(\partial_X Z, PX) - \cos^2\eta(X) g(\partial_Z Z, PX) - g(Q_Z PX, FPX). \tag{3.48}
\]

Thus, the result follows from (3.45) and (3.48). \hfill \square

**Theorem 3.7.** Let $M = N_\theta \times_f N_\perp$ be a warped product submanifold of a nearly cosymplectic manifold $\overline{M}$. Then $M$ is Riemannian product of $N_\theta$ and $N_\perp$ if and only if
\[
g(h(X, Z), FZ) = g(h(Z, Z), FX), \tag{3.49}
\]
for any $X \in TN_\theta$ and $Z \in TN_\perp$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

**Proof.** If $\xi \in TN_\perp$, then by Theorem 3.1, $f$ is constant on $M$. Now, we consider $\xi \in TN_\theta$. In this case, for any $X \in TN_\theta$ and $Z \in TN_\perp$ by (2.5), we have
\[
g(h(PX, Z), FZ) = g\left(\overline{\nabla}_Z PX, \phi Z\right). \tag{3.50}
\]
Using (2.2), we get
\[
g(h(PX, Z), FZ) = -g\left(\phi \overline{\nabla}_Z PX, Z\right). \tag{3.51}
\]
Thus, on using the covariant derivative property of $\phi$, we obtain
\[
g(h(PX, Z), FZ) = g\left(\left(\overline{\nabla}_Z \phi\right) PX, Z\right) - g\left(\overline{\nabla}_Z \phi PX, Z\right). \tag{3.52}
\]
Then from (2.8) and (2.10), we get
\[
g(h(PX, Z), FZ) = g(\partial_Z PX, Z) - g\left(\overline{\nabla}_Z P^2 X, Z\right) - g\left(\overline{\nabla}_Z FX, Z\right). \tag{3.53}
\]
Using property $(p_3) (i)$ and the property of the connection $\overline{\nabla}$, we derive
\[
g(h(PX, Z), FZ) = -g(\partial_Z Z, PX) + g\left(P^2 X, \nabla_Z Z\right) + g\left(FPX \overline{\nabla}_Z Z\right). \tag{3.54}
\]
As we have $\partial_Z Z = 0$ from (2.4) and (2.10), then by (2.18) the above equation reduced to
\[
g(h(PX, Z), FZ) = -\cos^2 \theta g\left(X, \overline{\nabla}_Z Z\right) + \cos^2 \theta \eta(X) g(\xi, \overline{\nabla}_Z Z) + g(h(Z, Z), FPX). \tag{3.55}
\]
Since $\xi$ is a Killing vector field on $\overline{M}$, then by (2.5), (3.2), and the property of the connection $\overline{\nabla}$, the above equation takes the form

$$g(h(PX, Z), FZ) = (X \ln f) \cos^2 \theta \|Z\|^2 + g(h(Z, Z), FPX).$$

(3.56)

Interchanging $X$ by $PX$ in (3.56) and using (2.18), we obtain

$$\cos^2 \theta g(h(X, Z), FZ) + \cos^2 \theta \eta(X) g(h(Z, \xi), FZ)$$

$$= -(PX \ln f) \cos^2 \theta \|Z\|^2 + \cos^2 \theta g(h(Z, Z), FX).$$

(3.57)

Since $h(Z, \xi) = 0$, for nearly cosymplectic, then the above equation reduces to

$$(PX \ln f) \|Z\|^2 = g(h(Z, Z), FX) - g(h(X, Z), FZ).$$

(3.58)

Thus, from (3.58), we obtain $(PX \ln f) = 0$ if and only if $g(h(Z, Z), FX) = g(h(X, Z), FZ)$. This proves the theorem completely.  

\[\Box\]

**References**


