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Research Article

Warped Product Pseudo-Slant Submanifolds of a Nearly Cosymplectic Manifold

Siraj Uddin, Bernardine R. Wong, and A. A. Mustafa

Institute of Mathematical Sciences, Faculty of Science, University of Malaya, 50603 Kuala Lumpur, Malaysia

Correspondence should be addressed to Siraj Uddin, siraj.ch@gmail.com

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We study warped product pseudo-slant submanifolds of a nearly cosymplectic manifold. We obtain some characterization results on the existence or nonexistence of warped product pseudo-slant submanifolds of a nearly cosymplectic manifold in terms of the canonical structures P and F.

1. Introduction

To study the manifolds with negative curvature, Bishop and O'Neill [1] introduced the notion of warped product manifolds by homothetically warping the product metric of a product manifold $N_1 \times N_2$ onto the fibers $p \times N_2$ for each $p \in N_1$. Later on, the geometrical aspect of these manifolds has been studied by many researchers (cf. [2–4]). Pseudo-slant submanifolds were introduced by Carriazo [5] as a special case of bislant submanifolds.

Almost contact manifolds with Killing structure tensors were defined in [6] as nearly cosymplectic manifolds, and it was shown that the normal nearly cosymplectic manifolds are cosymplectic (see also [7]). Later on, Blair and Showers [8] studied nearly cosymplectic structure (ϕ, ξ, η, g) on a Riemannian manifold $\overline{M}$ with η closed from the topological viewpoint.

Recently, Sahin [9] studied the warped product hemislant (pseudo-slant) submanifolds of Kaehler manifolds. He proved that the warped product submanifolds of the type $M = N_\perp \times_f N_\theta$ of a Kaehler manifold $\overline{M}$ do not exist and obtained some characterization results on the existence of warped product submanifold $M = N_\theta \times_f N_\perp$, where $N_\perp$ and N_θ are totally real and proper slant submanifolds of a Kaehler manifold $\overline{M}$, respectively. After that, we have extended this study to the more general setting of nearly Kaehler manifolds [4]. The warped product semi-invariant submanifolds of a nearly cosymplectic manifold had been studied in [10].

In this paper, we study warped product pseudo-slant submanifolds of a nearly cosymplectic manifold. We obtain some characterization results of warped product submanifolds of the types $N_\perp \times_f N_\theta$ and $N_\theta \times_f N_\perp$ in terms of the canonical structures P and F, where $N_\perp$ and N_θ are anti-invariant and proper slant submanifolds of a nearly cosymplectic manifold $\overline{M}$, respectively.

2. Preliminaries

A $(2n+1)$-dimensional C^∞ manifold $\overline{M}$ is said to have an *almost contact structure* if there exist on $\overline{M}$ a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying [8]

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold $\overline{M}$ satisfying the following compatibility condition:

$$\eta(X) = g(X,\xi), \qquad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

where X and Y are vector fields on $\overline{M}$ [8].

An almost contact structure (ϕ,ξ,η) is said to be *normal* if the almost complex structure J on the product manifold $\overline{M} \times \mathbb{R}$ given by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right), \tag{2.3}$$

where f is a C^∞-function on $\overline{M} \times \mathbb{R}$ has no torsion, that is, J is integrable, the condition for normality in terms of ϕ,ξ and η is $[\phi,\phi] + 2d\eta \otimes \xi = 0$ on $\overline{M}$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. Finally the *fundamental 2-form* Φ is defined by $\Phi(X,Y) = g(X,\phi Y)$.

An almost contact metric structure (ϕ,ξ,η,g) is said to be *cosymplectic*, if it is normal and both Φ and η are closed [8]. The structure is said to be *nearly cosymplectic* if ϕ is Killing, that is, if

$$\left(\overline{\nabla}_X \phi\right)Y + \left(\overline{\nabla}_Y \phi\right)X = 0, \tag{2.4}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ is the tangent bundle of $\overline{M}$ and $\overline{\nabla}$ denotes the Riemannian connection of the metric g. Equation (2.4) is equivalent to $(\overline{\nabla}_X \phi)X = 0$, for each $X \in T\overline{M}$. The structure is said to be *closely cosymplectic* if ϕ is Killing and η is closed. It is well known that an almost contact metric manifold is *cosymplectic* if and only if $\overline{\nabla}\phi$ vanishes identically, that is, $(\overline{\nabla}_X \phi)Y = 0$ and $\overline{\nabla}_X \xi = 0$.

Proposition 2.1 (see [8]). *On a nearly cosymplectic manifold the vector field ξ is Killing.*

From the above proposition, one has $\overline{\nabla}_X \xi = 0$, for any vector field X tangent to $\overline{M}$, where $\overline{M}$ is a nearly cosymplectic manifold.

Let M be submanifold of an almost contact metric manifold $\overline{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively, then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\overline{M}$. They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\overline{M}$ as well as induced on M.

For any $X \in TM$, one writes

$$\phi X = PX + FX, \tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly for any $N \in T^{\perp}M$, one writes

$$\phi N = BN + CN, \tag{2.9}$$

where BN is the tangential component and CN is the normal component of ϕN.

Now, denote by $\mathcal{P}_X Y$ and $Q_X Y$ the tangential and normal parts of $(\overline{\nabla}_X \phi)Y$, that is,

$$\left(\overline{\nabla}_X \phi\right) Y = \mathcal{P}_X Y + Q_X Y \tag{2.10}$$

for all $X, Y \in TM$. Making use of (2.8), (2.10), and the Gauss and Weingarten formulae, the following equations may easily be obtained:

$$\begin{aligned} \mathcal{P}_X Y &= \left(\overline{\nabla}_X P\right) Y - A_{FY} X - Bh(X, Y), \\ Q_X Y &= \left(\overline{\nabla}_X F\right) Y + h(X, PY) - Ch(X, Y). \end{aligned} \tag{2.11}$$

Similarly, for any $N \in T^{\perp}M$, denoting tangential and normal parts of $(\overline{\nabla}_X \phi)N$ by $\mathcal{P}_X N$ and $Q_X N$, respectively, one obtains

$$\begin{aligned} \mathcal{P}_X N &= \left(\overline{\nabla}_X B\right) N + PA_N X - A_{CN} X, \\ Q_X N &= \left(\overline{\nabla}_X C\right) N + h(BN, X) + FA_N X, \end{aligned} \tag{2.12}$$

where the covariant derivatives of P, F, B, and C are defined by

$$\left(\overline{\nabla}_X P\right)Y = \nabla_X PY - P\nabla_X Y, \tag{2.13}$$

$$\left(\overline{\nabla}_X F\right)Y = \nabla_X^{\perp} FY - F\nabla_X Y, \tag{2.14}$$

$$\left(\overline{\nabla}_X B\right)N = \nabla_X BN - B\nabla_X^{\perp} N, \tag{2.15}$$

$$\left(\overline{\nabla}_X C\right)N = \nabla_X^{\perp} CN - C\nabla_X^{\perp} N, \tag{2.16}$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$.

It is straightforward to verify the following properties of $\mathcal{P}$ and Q, which one enlists here for later use

(p_1) (i) $\mathcal{P}_{X+Y}W = \mathcal{P}_X W + \mathcal{P}_Y W$, (ii) $Q_{X+Y}W = Q_X W + Q_Y W$,

(p_2) (i) $\mathcal{P}_X(Y + W) = \mathcal{P}_X Y + \mathcal{P}_X W$, (ii) $Q_X(Y + W) = Q_X Y + Q_X W$,

(p_3) (i) $g(\mathcal{P}_X Y, W) = -g(Y, \mathcal{P}_X W)$, (ii) $g(Q_X Y, N) = -g(Y, \mathcal{P}_X N)$,

(p_4) $\mathcal{P}_X \phi Y + Q_X \phi Y = -\phi(\mathcal{P}_X Y + Q_X Y)$,

for all $X, Y, W \in TM$ and $N \in T^{\perp}M$.

On a submanifold M of a nearly cosymplectic manifold, by (2.4) and (2.10), one has

$$\text{(a) } \mathcal{P}_X Y + \mathcal{P}_Y X = 0, \qquad \text{(b) } Q_X Y + Q_Y X = 0, \tag{2.17}$$

for any $X, Y \in TM$.

The submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand, M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

One will always consider ξ to be tangent to the submanifold M. There is another class of submanifolds that is called the *slant submanifold*. For each nonzero vector X tangent to M at any $x \in M$, such that X is not proportional to ξ_x, one denotes by $0 \leq \theta(X) \leq \pi/2$, the angle between ϕX and $T_x M$ is called the slant angle. If the slant angle $\theta(X)$ is constant for all $X \in T_x M - \langle \xi_x \rangle$ and $x \in M$, then M is said to be a slant submanifold [11]. Obviously, if $\theta = 0$, then M is an invariant submanifold and if $\theta = \pi/2$, then M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

One recalls the following result for a slant submanifold.

Theorem 2.2 (see [11]). *Let M be a submanifold of an almost contact metric manifold $\overline{M}$, such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.18}$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequence of (2.18):

$$g(PX, PY) = \cos^2\theta\big(g(X, Y) - \eta(Y)\eta(X)\big), \tag{2.19}$$

$$g(FX, FY) = \sin^2\theta\big(g(X, Y) - \eta(Y)\eta(X)\big), \tag{2.20}$$

for all $X, Y \in TM$.

A submanifold M of an almost contact manifold $\overline{M}$ is said to be a *pseudo-slant submanifold* if there exist two orthogonal complementary distributions D_1 and D_2 satisfying:

(i) $TM = D_1 \oplus D_2 \oplus \langle\xi\rangle$,

(ii) D_1 is a slant distribution with slant angle $\theta \neq \pi/2$,

(iii) D_2 is anti-invariant that is, $\phi D_2 \subseteq T^{\perp}M$.

A pseudo-slant submanifold M of an almost contact manifold $\overline{M}$ is *mixed geodesic* if

$$h(X, Z) = 0, \tag{2.21}$$

for any $X \in D_1$ and $Z \in D_2$.

If μ is the invariant subspace of the normal bundle $T^{\perp}M$, then in the case of pseudo-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as follows:

$$T^{\perp}M = FD_1 \oplus FD_2 \oplus \mu. \tag{2.22}$$

3. Warped Product Pseudo-Slant Submanifolds

Bishop and O'Neill [1] introduced the notion of warped product manifolds. These manifolds are the natural generalizations of Riemannian product manifolds. They defined these manifolds as follows Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant. We recall the following general formula on a warped product manifold [1]:

$$\nabla_X Z = \nabla_Z X = \big(X \ln f\big) Z, \tag{3.2}$$

where X is tangential to N_1 and Z is tangential to N_2.

Let $M = N_1 \times_f N_2$ be a warped product manifold. This means that N_1 is totally geodesic and N_2 is a totally umbilical submanifold of M, respectively [1].

Throughout this section, we consider the warped product pseudo-slant submanifolds which are either in the form $N_{\perp} \times_f N_{\theta}$ or $N_{\theta} \times_f N_{\perp}$ in a nearly cosymplectic manifold $\overline{M}$, where N_{θ} and $N_{\perp}$ are proper slant and anti-invariant submanifolds of a nearly cosymplectic

manifold $\overline{M}$, respectively. On a warped product submanifold $M = N_1 \times_f N_2$ of a nearly cosymplectic manifold $\overline{M}$, we have the following result.

Theorem 3.1 (see [10]). *A warped product submanifold $M = N_1 \times_f N_2$ of a nearly cosymplectic manifold $\overline{M}$ is an usual Riemannian product if the structure vector field ξ is tangential to M_2, where M_1 and M_2 are the Riemannian submanifolds of $\overline{M}$.*

Now, one considers the warped product pseudo-slant submanifolds in the form $M = N_\perp \times_f N_\theta$ of a nearly cosymplectic manifold $\overline{M}$. If one considers the structure vector field $\xi \in TN_\theta$ then by Theorem 3.1, the warping function f is constant and hence one will considers $\xi \in TN_\perp$.

Proposition 3.2. *Let $M = N_\perp \times_f N_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,*

$$g\left(\nabla^\perp_{PX} FPX - \nabla^\perp_X FX, FZ\right) = (Z \ln f)\sin^2\theta \|X\|^2 + \left(1 + \cos^2\theta\right) g(h(X, PX), FZ), \tag{3.3}$$

for any $X \in TN_\theta$ and $Z \in TN_\perp$, where N_θ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

Proof. For any $X \in TN_\theta$ and $Z \in TN_\perp$, by (2.8), we have

$$g\left(\overline{\nabla}_X \phi X, FZ\right) = g\left(\overline{\nabla}_X PX, FZ\right) + g\left(\overline{\nabla}_X FX, FZ\right). \tag{3.4}$$

Using (2.5), (2.6), and the covariant derivative property of ϕ, we obtain

$$g\left(\left(\overline{\nabla}_X \phi\right)X, FZ\right) + g\left(\phi \overline{\nabla}_X X, \phi Z\right) = g(h(X, PX), FZ) + g\left(\nabla^\perp_X FX, FZ\right). \tag{3.5}$$

Then from (2.2), (2.4), and the fact that ξ is a Killing vector field on $\overline{M}$, thus we obtain

$$g\left(\overline{\nabla}_X X, Z\right) = g(h(X, PX), FZ) + g\left(\nabla^\perp_X FX, FZ\right). \tag{3.6}$$

Using the property of $\overline{\nabla}$, we get

$$-g\left(X, \overline{\nabla}_X Z\right) = g(h(X, PX), FZ) + g\left(\nabla^\perp_X FX, FZ\right). \tag{3.7}$$

Then by (2.5) and (3.2), we derive

$$-(Z \ln f)\|X\|^2 = g(h(PX, X), FZ) + g\left(\nabla^\perp_X FX, FZ\right). \tag{3.8}$$

Interchanging X by PX in (3.8) and using (2.18), (2.19), and the fact that $\xi \in TN_{\perp}$, we obtain

$$-(Z\ln f)\cos^2\theta\|X\|^2 = -\cos^2\theta g(h(X,PX),FZ) + g\left(\nabla^{\perp}_{PX}FPX,FZ\right). \tag{3.9}$$

Thus, the result follows from (3.8) and (3.9). □

Proposition 3.3. *Let $M = N_{\perp}\times_f N_{\theta}$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,*

$$g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) = sec^2\theta g\left(\left(\overline{\nabla}_{PX}F\right)PX,FZ\right) \tag{3.10}$$

for any $X \in TN_{\theta}$ and $Z \in TN_{\perp}$, where N_{θ} and $N_{\perp}$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

Proof. For any $X \in TN_{\theta}$ and $Z \in TN_{\perp}$ by (2.14), we have

$$g\left(\nabla^{\perp}_X FX,FZ\right) = g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) + g(F\nabla_X X,FZ). \tag{3.11}$$

Using (2.20), (2.5), and the fact that ξ is killing vector field, we obtain

$$g\left(\nabla^{\perp}_X FX,FZ\right) = g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) - \sin^2\theta g(X,\nabla_X Z). \tag{3.12}$$

Then from (3.2), we derive

$$g\left(\nabla^{\perp}_X FX,FZ\right) = g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) - (Z\ln f)\sin^2\theta\|X\|^2. \tag{3.13}$$

Now, from (3.8) and (3.13), we obtain

$$g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) = -(Z\ln f)\cos^2\theta\|X\|^2 - g(h(X,PX),FZ). \tag{3.14}$$

Interchanging X by PX in (3.14) and then using (2.18), (2.19), and the fact that $\xi \in TN_{\perp}$, we get

$$g\left(\left(\overline{\nabla}_{PX} F\right)PX,FZ\right) = -(Z\ln f)\cos^4\theta\|X\|^2 - \cos^2\theta g(h(X,PX),FZ). \tag{3.15}$$

From (3.14) and (3.15), we arrive at

$$g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) = \sec^2\theta g\left(\left(\overline{\nabla}_{PX} F\right)PX,FZ\right). \tag{3.16}$$

Hence, the result is proved. □

Lemma 3.4. *Let $M = N_\perp \times_f N_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,*

$$g(\mathcal{P}_X PX, Z) = g(h(X,Z), FPX) - g(h(PX,Z), FX) \tag{3.17}$$

for any $X \in TN_\theta$ and $Z \in TN_\perp$, where N_θ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

Proof. For any $X \in TN_\theta$ and $Z \in TN_\perp$ by (2.5), we have

$$g(h(PX,Z), FX) = g\left(\overline{\nabla}_Z PX, FX\right) = -g\left(PX, \overline{\nabla}_Z FX\right). \tag{3.18}$$

Then from (2.8), we derive

$$g(h(PX,Z), FX) = g\left(PX, \overline{\nabla}_Z PX\right) - g\left(PX, \overline{\nabla}_Z \phi X\right). \tag{3.19}$$

From the covariant derivative property of ϕ and (2.5), we obtain

$$g(h(PX,Z), FX) = g(PX, \nabla_Z PX) - g\left(PX, \left(\overline{\nabla}_Z \phi\right)X\right) - g\left(PX, \phi\overline{\nabla}_Z X\right). \tag{3.20}$$

By (2.2), (2.10), and (3.2), we derive

$$g(h(PX,Z), FX) = (Z \ln f) g(PX, PX) - g(PX, \mathcal{P}_Z X) + g\left(\phi PX, \overline{\nabla}_Z X\right). \tag{3.21}$$

Using (2.5), (2.8), (2.17)(a), (2.19) and the fact that $\xi \in TN_\perp$, we get

$$\begin{aligned} g(h(PX,Z), FX) = {} & (Z \ln f)\cos^2\theta \|X\|^2 + g(PX, \mathcal{P}_X Z) \\ & + g\left(P^2 X, \nabla_Z X\right) + g(h(X,Z), FPX). \end{aligned} \tag{3.22}$$

Thus, by property (p_3)(i), (2.18), and (3.2) and the fact that $\xi \in TN_\perp$, we obtain

$$\begin{aligned} g(h(PX,Z), FX) = {} & (Z \ln f)\cos^2\theta \|X\|^2 - g(\mathcal{P}_X PX, Z) \\ & - (Z \ln f)\cos^2\theta \|X\|^2 + g(h(X,Z), FPX). \end{aligned} \tag{3.23}$$

Hence, the above equation takes the form

$$g(\mathcal{P}_X PX, Z) = g(h(X,Z), FPX) - g(h(PX,Z), FX), \tag{3.24}$$

which proves our assertion. □

Theorem 3.5. *Let $M = N_\perp \times_f N_\theta$ be a warped product submanifold of a nearly cosymplectic manifold $\overline{M}$. Then M is Riemannian product of $N_\perp$ and N_θ if and only if $\mathcal{P}_X TX \in TN_\theta$, for any $X \in TN_\theta$, where N_θ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.*

Proof. If the structure vector field $\xi \in TN_\theta$, then, by Theorem 3.1, M is Riemannian product of $N_\perp$ and N_θ. Now, we consider $\xi \in TN_\perp$, then for any $X \in TN_\theta$ and $Z \in TN_\perp$ from (2.5), we have

$$g(h(X, PX), FZ) = g\left(\overline{\nabla}_{PX} X, \phi Z\right). \tag{3.25}$$

Then by (2.2), we get

$$g(h(X, PX), FZ) = -g\left(\phi \overline{\nabla}_{PX} X, Z\right). \tag{3.26}$$

Using the covariant derivative formula of ϕ, we derive

$$g(h(X, PX), FZ) = g\left(\left(\overline{\nabla}_{PX} \phi\right) X, Z\right) - g\left(\overline{\nabla}_{PX} \phi X, Z\right). \tag{3.27}$$

Then from (2.10) and the property of $\overline{\nabla}$, we obtain

$$g(h(X, PX), FZ) = g(\mathcal{P}_{PX} X, Z) + g\left(\phi X, \overline{\nabla}_{PX} Z\right). \tag{3.28}$$

Thus by (2.5), (2.8), and (2.17)(a), we arrive at

$$g(h(X, PX), FZ) = -g(\mathcal{P}_X PX, Z) + g(PX, \nabla_{PX} Z) + g(h(PX, Z), FX). \tag{3.29}$$

Using (3.2) and then (2.19) and the fact that $\xi \in TN_\perp$, we get

$$\begin{aligned} g(h(X, PX), FZ) = & -g(\mathcal{P}_X PX, Z) + (Z \ln f) \cos^2\theta \|X\|^2 \\ & + g(h(PX, Z), FX). \end{aligned} \tag{3.30}$$

By property $(p_3)(i)$, we derive

$$\begin{aligned} g(h(X, PX), FZ) = & g(PX, \mathcal{P}_X Z) + (Z \ln f) \cos^2\theta \|X\|^2 \\ & + g(h(PX, Z), FX). \end{aligned} \tag{3.31}$$

Interchanging X by PX in (3.30) and then using (2.18), (2.19), and the fact that $\xi \in TN_\perp$, we obtain

$$\begin{aligned} -\cos^2\theta g(h(X, PX), FZ) = & -\cos^2\theta g(X, \mathcal{P}_{PX} Z) + (Z \ln f) \cos^4\theta \|X\|^2 \\ & - \cos^2\theta g(h(X, Z), FPX). \end{aligned} \tag{3.32}$$

Using the property (p_3)(i) and then (2.17)(a), we arrive at

$$\begin{aligned} -g(h(X,PX),FZ) = & -g(\mathcal{P}_X PX, Z) + (Z\ln f)\cos^2\theta\|X\|^2 \\ & - g(h(X,Z),FPX). \end{aligned} \tag{3.33}$$

Then from (3.30) and (3.33), we obtain

$$\begin{aligned} 2(Z\ \ln f)\cos^2\theta\|X\|^2 = & 2g(\mathcal{P}_X PX, Z) + g(h(X,Z),FPX) \\ & - g(h(PX,Z),FX). \end{aligned} \tag{3.34}$$

Thus, by Lemma 3.4, we conclude that

$$(Z\ \ln f)\cos^2\theta\|X\|^2 = \frac{3}{2}g(\mathcal{P}_X PX, Z). \tag{3.35}$$

Since N_θ is proper slant, thus we get $(Z\ln f) = 0$, if and only if $\mathcal{P}_X PX$ lies in TN_θ for all $X \in TN_\theta$ and $Z \in TN_\perp$. This proves the theorem completely. □

Now, we discuss the other case, that is, the warped product submanifold $M = N_\theta \times_f N_\perp$ of a nearly cosymplectic manifold $\overline{M}$. In this case also, if the structure vector filed $\xi \in TN_\perp$ then the warping function f is constant (by Theorem 3.1), thus we consider $\xi \in TN_\theta$.

Proposition 3.6. *Let $M = N_\theta \times_f N_\perp$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,*

$$\begin{aligned} g\left(\left(\overline{\nabla}_X F\right)Z, FX\right) + g\left(\left(\overline{\nabla}_{PX} F\right)Z, FPX\right) = & \sin^2\theta g(h(X,PX),FZ) \\ & + \left(1+\cos^2\theta\right)g(\mathcal{P}_X Z, PX) - \cos^2\theta\eta(X)g(\mathcal{P}_\xi Z, PX) \\ & - g(Q_Z X, FX) - g(Q_Z PX, FPX) \end{aligned} \tag{3.36}$$

for any $X \in TN_\theta$ and $Z \in TN_\perp$, where N_θ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

Proof. For any $X \in TN_\theta$ and $Z \in TN_\perp$, by (2.2) we have

$$g\left(\phi\overline{\nabla}_X Z, \phi X\right) = g\left(\overline{\nabla}_X Z, X\right) - \eta(X)g\left(\overline{\nabla}_X Z, \xi\right). \tag{3.37}$$

Using the property of the connection $\overline{\nabla}$ and the fact that ξ is a Killing vector field, then, from (2.5), we obtain

$$g\left(\phi\overline{\nabla}_X Z, \phi X\right) = g(\nabla_X Z, X). \tag{3.38}$$

Thus by (3.2) and the covariant derivative formula of ϕ, we derive

$$g\left(\overline{\nabla}_X \phi Z, \phi X\right) - g\left(\left(\overline{\nabla}_X \phi\right) Z, \phi X\right) = (X \ln f) g(Z, X). \tag{3.39}$$

Then form (2.6), (2.8), (2.10), and by the orthogonality of two distributions, we get

$$-g(A_{FZ}X, PX) + g\left(\nabla_X^{\perp} FZ, FX\right) - g(\mathcal{P}_X Z, PX) - g(Q_X Z, FX) = 0. \tag{3.40}$$

Thus, on using (2.7) and (2.17)(b), the above equation takes the form

$$g\left(\nabla_X^{\perp} FZ, FX\right) = g(h(X, PX), FZ) + g(\mathcal{P}_X Z, PX) - g(Q_Z X, FX). \tag{3.41}$$

Now, for any $X \in TN_\theta$ and $Z \in TN_\perp$ from (2.14), we have

$$g\left(\nabla_X^{\perp} FZ, FX\right) = g\left(\left(\overline{\nabla}_X F\right) Z, FX\right) + g(F\nabla_X Z, FX). \tag{3.42}$$

Using (3.2), we obtain

$$g\left(\nabla_X^{\perp} FZ, FX\right) = g\left(\left(\overline{\nabla}_X F\right) Z, FX\right) + (X \ln f) g(FZ, FX). \tag{3.43}$$

By orthogonality of two normal distributions, we get

$$g\left(\nabla_X^{\perp} FZ, FX\right) = g\left(\left(\overline{\nabla}_X F\right) Z, FX\right). \tag{3.44}$$

Then, from (3.41) and (3.44), we obtain

$$g\left(\left(\overline{\nabla}_X F\right) Z, FX\right) = g(h(X, PX), FZ) + g(\mathcal{P}_X Z, PX) - g(Q_Z X, FX). \tag{3.45}$$

Interchanging X by PX in (3.45) and using (2.18) and the fact that $h(X, \xi) = 0$, for any X on a nearly cosymplectic manifold $\overline{M}$, hence we get

$$\begin{aligned} g\left(\left(\overline{\nabla}_{PX} F\right) Z, FPX\right) = & -\cos^2\theta g(h(X, PX), FZ) - \cos^2\theta g(\mathcal{P}_{PX} Z, X) \\ & + \cos^2\theta \eta(X) g(\mathcal{P}_{PX} Z, \xi) - g(Q_Z PX, FPX). \end{aligned} \tag{3.46}$$

Using property (p_3)(i) and (2.17), we derive

$$\begin{aligned} g\left(\left(\overline{\nabla}_{PX} F\right) Z, FPX\right) = & -\cos^2\theta g(h(X, PX), FZ) - \cos^2\theta g(\mathcal{P}_X PX, Z) \\ & + \cos^2\theta \eta(X) g(\mathcal{P}_\xi PX, Z) - g(Q_Z PX, FPX). \end{aligned} \tag{3.47}$$

Again, by property (p_3)(i), we obtain

$$g\left(\left(\overline{\nabla}_{PX}F\right)Z, FPX\right) = -\cos^2\theta g(h(X,PX),FZ) + \cos^2\theta g(\mathcal{P}_X Z, PX) - \cos^2\theta\eta(X)g(\mathcal{P}_\xi Z, PX) - g(Q_Z PX, FPX). \tag{3.48}$$

Thus, the result follows from (3.45) and (3.48). □

Theorem 3.7. *Let $M = N_\theta \times_f N_\perp$ be a warped product submanifold of a nearly cosymplectic manifold $\overline{M}$. Then M is Riemannian product of N_θ and $N_\perp$ if and only if*

$$g(h(X,Z),FZ) = g(h(Z,Z),FX), \tag{3.49}$$

for any $X \in TN_\theta$ and $Z \in TN_\perp$, where N_θ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.

Proof. If $\xi \in TN_\perp$, then by Theorem 3.1, f is constant on M. Now, we consider $\xi \in TN_\theta$. In this case, for any $X \in TN_\theta$ and $Z \in TN_\perp$ by (2.5), we have

$$g(h(PX,Z),FZ) = g\left(\overline{\nabla}_Z PX, \phi Z\right). \tag{3.50}$$

Using (2.2), we get

$$g(h(PX,Z),FZ) = -g\left(\phi\overline{\nabla}_Z PX, Z\right). \tag{3.51}$$

Thus, on using the covariant derivative property of ϕ, we obtain

$$g(h(PX,Z),FZ) = g\left(\left(\overline{\nabla}_Z\phi\right)PX, Z\right) - g\left(\overline{\nabla}_Z\phi PX, Z\right). \tag{3.52}$$

Then from (2.8) and (2.10), we get

$$g(h(PX,Z),FZ) = g(\mathcal{P}_Z PX, Z) - g\left(\overline{\nabla}_Z P^2X, Z\right) - g\left(\overline{\nabla}_Z FPX, Z\right). \tag{3.53}$$

Using property (p_3)(i) and the property of the connection $\overline{\nabla}$, we derive

$$g(h(PX,Z),FZ) = -g(\mathcal{P}_Z Z, PX) + g\left(P^2X, \nabla_Z Z\right) + g\left(FPX\overline{\nabla}_Z Z\right). \tag{3.54}$$

As we have $\mathcal{P}_Z Z = 0$ from (2.4) and (2.10), then by (2.18) the above equation reduced to

$$g(h(PX,Z),FZ) = -\cos^2\theta g\left(X, \overline{\nabla}_Z Z\right) + \cos^2\theta\eta(X)g\left(\xi, \overline{\nabla}_Z Z\right) + g(h(Z,Z),FPX). \tag{3.55}$$

Since ξ is a Killing vector field on $\overline{M}$, then by (2.5), (3.2), and the property of the connection $\overline{\nabla}$, the above equation takes the form

$$g(h(PX,Z),FZ) = (X\ln f)\cos^2\theta\|Z\|^2 + g(h(Z,Z),FPX). \tag{3.56}$$

Interchanging X by PX in (3.56) and using (2.18), we obtain

$$\begin{aligned}\cos^2\theta g(h(X,Z),FZ) + \cos^2\theta\eta(X)g(h(Z,\xi),FZ)\\ = -(PX\ln f)\cos^2\theta\|Z\|^2 + \cos^2\theta g(h(Z,Z),FX).\end{aligned} \tag{3.57}$$

Since $h(Z,\xi) = 0$, for nearly cosymplectic, then the above equation reduces to

$$(PX\ln f)\|Z\|^2 = g(h(Z,Z),FX) - g(h(X,Z),FZ). \tag{3.58}$$

Thus, from (3.58), we obtain $(PX\ln f) = 0$ if and only if $g(h(Z,Z),FX) = g(h(X,Z),FZ)$. This proves the theorem completely. □

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