Research Article

Existence of Solutions for Nonhomogeneous A -Harmonic Equations with Variable Growth

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We study the following nonhomogeneous A -harmonic equations: $d^*A(x, du(x)) + B(x, u(x)) = 0$, $x \in \Omega$, $u(x) = 0$, $x \in \partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded and convex Lipschitz domain, $A(x, du(x))$ and $B(x, u(x))$ satisfy some $p(x)$ -growth conditions, respectively. We obtain the existence of weak solutions for the above equations in subspace $\mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^{l-1})$ of $W_0^{1,p(x)}(\Omega, \Lambda^{l-1})$.

1. Introduction

Spaces of differential forms have been discussed in great details (see [1, 2] and the references therein). The theory of differential forms is an approach to multivariable calculus that is independent of coordinates and provides a better definition for integrals. Differential forms have played an important role in physical laws of thermodynamics, analytical mechanics, and physical theories, in particular Maxwell's theory, and the Yang-Mills theory, the theory of relativity, see for example [3–6].

In recent years, the study of A -harmonic equations for differential forms has developed rapidly. Many interesting results concerning A -harmonic equation have been established recently (see [7–11] and the references therein). In [12], spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are first introduced, and they used them to study the solutions of nonlinear Dirichlet boundary value problems with $p(x)$ -growth conditions. In [13], spaces $L^{p(x)}(\Omega, \Lambda^l, \omega)$ and $W^{1,p(x)}(\Omega, \Lambda^l, \omega)$ are first introduced and used to study the weak solutions of obstacle problems of A -harmonic equations with variable growth for differential forms.

Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex Lipschitz domain. It is our purpose to study the following systems:

$$\begin{aligned} d^* A(x, du(x)) + B(x, u(x)) &= 0, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \quad (1.1)$$

where $u \in \Lambda^{l-1}(\Omega)$, $l = 1, 2, \dots, n$, and $A : \Omega \times \Lambda^l(\Omega) \rightarrow \Lambda^l(\Omega)$, $B : \Omega \times \Lambda^{l-1}(\Omega) \rightarrow \Lambda^{l-1}(\Omega)$ satisfy the following conditions.

- (H1) $A(x, \xi)$ and $B(x, \varsigma)$ are measurable with respect to x for all ξ, ς and continuous with respect to ξ, ς , respectively, for a.e. $x \in \Omega$.
- (H2) $|A(x, \xi)| + |B(x, \varsigma)| \leq C_1 |\xi|^{p(x)-1} + C_2 |\varsigma|^{p(x)-1} + G(x)$, where $G \in L^{p'(x)}(\Omega)$ and $C_1, C_2 \geq 0$ are constants.
- (H3) $\langle A(x, \xi), \xi \rangle \geq a |\xi|^{p(x)} - |h(x)|$, where $a > 0$ is a constant and $h \in L^1(\Omega)$.
- (H4) $\langle B(x, \varsigma), \varsigma \rangle \geq \bar{a} |\varsigma|^{p(x)} - |\bar{h}(x)|$, where $\bar{a} \geq 0$ is a constant and $\bar{h} \in L^1(\Omega)$.
- (H5) For a.e. $x_0 \in \Omega$, the mapping $\xi \rightarrow A(x_0, \xi)$ satisfies

$$\int_D \langle A(x_0, \xi_0 + dv(x)), dv(x) \rangle dx \geq \gamma \int_D |dv(x)|^{p(x)} dx, \quad (1.2)$$

for each $\xi_0 \in \Lambda^l(\Omega)$, $D \subset \Omega$ and $v \in C_0^1(\Omega, \Lambda^{l-1})$, where $\gamma > 0$ is a constant. Here p' is the conjugate function of p . Throughout this paper we suppose (unless declare specially)

$$p \in \mathcal{P}^{\log}(\Omega), \quad 1 < p_* = \operatorname{ess\,inf}_{\Omega} p(x) \leq p(x) \leq \operatorname{ess\,sup}_{\Omega} p(x) = p^* < \infty. \quad (1.3)$$

2. Preliminaries

Let $e_1, e_2, \dots, e_n$ be the standard orthogonal basis of $\mathbb{R}^n$. The space of all l -forms in $\mathbb{R}^n$ is denoted by $\Lambda^l(\mathbb{R}^n)$. The dual basis to $e_1, e_2, \dots, e_n$ is denoted by $e^1, e^2, \dots, e^n$ and referred to as the standard basis for 1-form $\Lambda^1(\mathbb{R}^n)$. The Grassman algebra $\Lambda(\mathbb{R}^n) = \oplus \Lambda^l(\mathbb{R}^n)$ is a graded algebra with respect to the exterior products. The standard ordered basis for $\Lambda(\mathbb{R}^n)$ consists of the forms

$$1, e^1, e^2, \dots, e^n, e^1 \wedge e^2, \dots, e^{n-1} \wedge e^n, \dots, e^1 \wedge e^2 \dots \wedge e^n. \quad (2.1)$$

For $\alpha(x) = \sum \alpha_I(x) e^I \in \Lambda^l(\mathbb{R}^n)$ and $\beta(x) = \sum \beta_I(x) e^I \in \Lambda^l(\mathbb{R}^n)$, the inner product is obtained by $\langle \alpha, \beta \rangle = \sum \alpha_I(x) \beta_I(x)$ with summation over all l -tuples $I = (i_1, \dots, i_l)$ and all integers $l = 0, 1, \dots, n$. The Hodge star operator (see [14]) $\star : \Lambda(\mathbb{R}^n) \rightarrow \Lambda(\mathbb{R}^n)$ is defined by the formulas

$$\star 1 = e^1 \wedge e^2 \dots \wedge e^n, \quad \alpha \wedge \star \beta = \beta \wedge \star \alpha = \langle \alpha, \beta \rangle e^1 \wedge e^2 \dots \wedge e^n. \quad (2.2)$$

Hence, the norm of α is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star \alpha) = \sum \alpha_I(x) \alpha_I(x) \in \Lambda^0(\mathbb{R}^n) = \mathbb{R}$. Notice, the Hodge star operator is an isometric isomorphism operator on $\Lambda(\mathbb{R}^n)$. Moreover,

$$\star : \Lambda^l(\mathbb{R}^n) \longrightarrow \Lambda^{n-l}(\mathbb{R}^n), \quad \star\star = (-1)^{l(n-l)} : \Lambda^l(\mathbb{R}^n) \longrightarrow \Lambda^l(\mathbb{R}^n), \quad (2.3)$$

where I is the identity map.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The coordinate functions $x_1, x_2, \dots, x_n$ in Ω are considered to be differential forms of degree 0. The 1-forms $dx_1, dx_2, \dots, dx_n$ are constant functions from Ω into $\Lambda^1(\mathbb{R}^n)$. The value of dx_i is simply e^i , $i = 1, 2, \dots, n$. Therefore, every l -form $u : \Omega \rightarrow \Lambda^l(\mathbb{R}^n)$ may be written uniquely as

$$u(x) = \sum_I u_I(x) dx_I = \sum_{1 \leq i_1 < \dots < i_l \leq n} u_{i_1, \dots, i_l}(x) dx_{i_1} \wedge \dots \wedge dx_{i_l}, \quad (2.4)$$

where the coefficients $u_{i_1, \dots, i_l}(x)$ are distributions from $\mathfrak{D}'(\Omega)$, dual to the space of smooth functions with compact support on Ω .

We use $\mathfrak{D}'(\Omega, \Lambda^l)$ to denote the space of all differential l -forms. For each form $u(x) \in \mathfrak{D}'(\Omega, \Lambda^l)$, the exterior differential $d : \mathfrak{D}'(\Omega, \Lambda^l) \rightarrow \mathfrak{D}'(\Omega, \Lambda^{l+1})$ is expressed by

$$du(x) = \sum_{k=1}^n \sum_{1 \leq i_1 < \dots < i_l \leq n} \frac{\partial u_{i_1, \dots, i_l}(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \dots \wedge dx_{i_l}. \quad (2.5)$$

For $u \in \mathfrak{D}'(\Omega, \Lambda^l)$, the vector-valued differential form

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right) \quad (2.6)$$

consists of differential forms $\partial u / \partial x_i \in \mathfrak{D}'(\Omega, \Lambda^l)$, where the partial differentiation is applied to the coefficients of u .

The formal adjoint operator, called the Hodge codifferential, is given by

$$d^* = (-1)^{nl-1} \star d \star : \mathfrak{D}'(\Omega, \Lambda^{l+1}) \longrightarrow \mathfrak{D}'(\Omega, \Lambda^l). \quad (2.7)$$

By $C^\infty(\Omega, \Lambda^l)$ denote the space of infinitely differentiable l -forms on Ω and by $C_0^\infty(\Omega, \Lambda^l)$ denote the subspace of $C^\infty(\Omega, \Lambda^l)$ with compact support on Ω .

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \rightarrow (1, \infty)$. For $p \in \mathcal{P}(\Omega)$, we put $p_* = \text{essinf}_\Omega p(x)$ and $p^* = \text{esssup}_\Omega p(x)$. Given $p \in \mathcal{P}(\Omega)$ we define the conjugate function $p' \in \mathcal{P}(\Omega)$ by

$$p'(x) = \frac{p(x)}{p(x) - 1}, \quad \forall x \in \Omega. \quad (2.8)$$

Definition 2.1 (see [15]). A Lebesgue measurable function $p : \Omega \rightarrow \mathbb{R}$ is called globally log-Hölder continuous in Ω if there exist $p_\infty \in \mathbb{R}$ and a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x - y|)}, \quad |p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)} \quad (2.9)$$

hold for all $x, y \in \Omega$. $P^{\log}(\Omega)$ is defined by

$$P^{\log}(\Omega) = \left\{ p \in \mathcal{P}(\Omega) : \frac{1}{p} \text{ is globally log-Hölder continuous} \right\}. \quad (2.10)$$

For a differential l -form $u(x)$ on Ω , $l = 0, 1, \dots, n$, define the functional $\rho_{p(x)}$ by

$$\rho_{p(x), \Lambda^l}(u) = \int_{\Omega} |u(x)|^{p(x)} dx. \quad (2.11)$$

The space $L^{p(x)}(\Omega, \Lambda^l) = \{u \in \Lambda^l(\Omega) : \exists \lambda > 0, \rho_{p(x), \Lambda^l}(\lambda u) < \infty\}$ is a reflexive Banach space endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega, \Lambda^l)} = \inf \left\{ \lambda > 0 : \rho_{p(x), \Lambda^l} \left(\frac{u}{\lambda} \right) \leq 1 \right\}. \quad (2.12)$$

The space $W^{1,p(x)}(\Omega, \Lambda^l) = \{u \in \Lambda^l(\Omega) : u \in L^{p(x)}(\Omega, \Lambda^l) \text{ and } du \in L^{p(x)}(\Omega, \Lambda^{l+1})\}$ is a reflexive Banach space endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega, \Lambda^l)} = \|u\|_{L^{p(x)}(\Omega, \Lambda^l)} + \|du\|_{L^{p(x)}(\Omega, \Lambda^{l+1})}. \quad (2.13)$$

Note that $L^{p(m)}(\Omega, \Lambda^0)$ and $W^{1,p(m)}(\Omega, \Lambda^0)$ are spaces of functions on Ω . In this paper, we denote them by $L^{p(m)}(\Omega)$ and $W^{1,p(m)}(\Omega)$.

Iwaniec and Lutoborski proved the following results in [2].

Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex domain. If $u(x) \in \Lambda^l(\mathbb{R}^n)$ is defined for some $x \in \Omega$, then the value of $u(x)$ at the vectors $\xi_1, \dots, \xi_l \in \mathbb{R}^n$ is denoted by $u(x)(\xi_1, \dots, \xi_l)$. Then to each $y \in \Omega$, there corresponds a linear operator $K_y : L_{\text{loc}}^1(\Omega, \Lambda^l) \rightarrow L_{\text{loc}}^1(\Omega, \Lambda^{l-1})$ defined by

$$K_y u(x)(\xi_1, \xi_2, \dots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty)(x - y, \xi_1, \xi_2, \dots, \xi_{l-1}) dt. \quad (2.14)$$

The homotopy operator $T : L_{\text{loc}}^1(\Omega, \Lambda^l) \rightarrow L_{\text{loc}}^1(\Omega, \Lambda^{l-1})$ is defined by averaging K_y over all points $y \in \Omega$

$$Tu(x) = \int_{\Omega} \varphi(y) K_y u(x) dy, \quad (2.15)$$

where $\varphi \in C_0^\infty(\Omega)$ is normalized so that $\int_\Omega \varphi(y) dy = 1$. Then we have a pointwise estimate

$$|Tu(x)| \leq 2^n \mu(\Omega) \int_\Omega \frac{|u(y)|}{|x-y|^{n-1}} dy, \quad \forall x \in \Omega, \quad (2.16)$$

where

$$\mu(\Omega) = (\text{diam } \Omega)^{n+1} \inf \left\{ \frac{\|\nabla \varphi\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^1(\Omega)}} : \varphi \in C_0^\infty(\Omega) \right\}, \quad (2.17)$$

further infimum is attained at $\varphi(x) = \text{diam}(x, \partial\Omega)$, and the decomposition

$$u = dTu + Tdu \quad (2.18)$$

holds for $u \in L_{\text{loc}}^1(\Omega, \Lambda^l)$.

Definition 2.2. For $u \in L_{\text{loc}}^1(\Omega, \Lambda^l)$, define the l -form $u_\Omega \in \mathfrak{D}'(\Omega, \Lambda^l)$ by

$$u_\Omega = \begin{cases} \frac{1}{\text{meas}(\Omega)} \int_\Omega u(x) dx, & \text{for } l = 0, \\ dTu, & \text{for } l = 1, 2, \dots, n, \end{cases} \quad (2.19)$$

and the Maximal operator is defined by

$$(Mu)(x) = \sup_{r>0} \frac{1}{\text{meas}(B_r(x))} \int_{B_r(x)} |u(y)| dy, \quad (2.20)$$

where $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$.

Lemma 2.3 (see [15]). *Let $p(x)$ satisfies (1.3). Then the inequality*

$$\|(Mu)(x)\|_{L^{p(x)}(\mathbb{R}^n)} \leq C(n, p) \|u(x)\|_{L^{p(x)}(\mathbb{R}^n)} \quad (2.21)$$

holds for every $u \in L^{p(x)}(\mathbb{R}^n)$.

Lemma 2.4 (see [15]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, $x \in \Omega$ and $u \in L_{\text{loc}}^1(\mathbb{R}^n)$. Then*

$$\int_\Omega \frac{|u(y)|}{|x-y|^{n-1}} dy \leq C(n)(\text{diam } \Omega)(Mu)(x). \quad (2.22)$$

Lemma 2.5 (see [15]). *Let Ψ be a Calderón-Zygmund operator with Calderón-Zygmund kernel K on $\mathbb{R}^n \times \mathbb{R}^n$. Then Ψ is bounded on $L^{p(x)}(\mathbb{R}^n)$. Further there exists a constant $C = C(n, p)$ such that*

$$\|\Psi u(x)\|_{L^{p(x)}(\mathbb{R}^n)} \leq C(n, p) \|u(x)\|_{L^{p(x)}(\mathbb{R}^n)} \quad (2.23)$$

holds for every $u \in L^{p(x)}(\mathbb{R}^n)$.

Lemma 2.6. *If $u \in L^{p(x)}(\Omega, \Lambda^l)$, then*

$$\|Tu\|_{L^{p(x)}(\Omega, \Lambda^{l-1})} \leq C(n, p) \mu(\Omega) (\text{diam } \Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.24)$$

Moreover, if $u \in W^{1,p(x)}(\Omega, \Lambda^l)$, then

$$\|u_\Omega\|_{L^{p(x)}(\Omega, \Lambda^l)} \leq C(p) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)} + C(n, p) \mu(\Omega) (\text{diam } \Omega) \|du\|_{L^{p(x)}(\Omega, \Lambda^{l+1})}. \quad (2.25)$$

Proof. First define $u(x) = 0$ if $x \in \mathbb{R}^n \setminus \Omega$. From pointwise estimate (2.16) and Lemma 2.4,

$$|Tu(x)| \leq C(n) \mu(\Omega) (\text{diam } \Omega) M(|u|)(x), \quad \forall x \in \Omega. \quad (2.26)$$

In view of Lemma 2.3, we have

$$\|Tu\|_{L^{p(x)}(\Omega)} \leq C(n, p) \mu(\Omega) (\text{diam } \Omega) \|u\|_{L^{p(x)}(\Omega)}, \quad (2.27)$$

that is to say, (2.24) holds.

From the definition of u_Ω and (2.18), we have $u_\Omega = u - Tdu$. Therefore,

$$\|u_\Omega\|_{L^{p(x)}(\Omega, \Lambda^l)} \leq C(p) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)} + C(n, p) \|Tdu\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.28)$$

Now in (2.24) replace u with du , we obtain (2.25). □

Lemma 2.7. *Let $p(x)$ satisfies (1.3).*

(1) $C_0^\infty(\Omega, \Lambda^l)$ is dense in $L^{p(x)}(\Omega, \Lambda^l)$,

(2) $L^{p(x)}(\Omega, \Lambda^l)$ is separable.

Proof. (1) For any $u(x) = \sum_I u_I(x) dx_I \in L^{p(x)}(\Omega, \Lambda^l)$, since $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$ and $u_I(x) \in L^{p(x)}(\Omega)$ for all I , we can find a sequence $\{u_{I_k}\}_{k=1}^\infty \subset C_0^\infty(\Omega)$ which converges to $u_I(x)$

in $L^{p(x)}(\Omega)$ for each I . Now let $u_k(x) = \sum_I u_{Ik} dx_I$, then the sequence $\{u_k(x)\} \subset C_0^\infty(\Omega, \Lambda^l)$ converges to $u(x)$ in $L^{p(x)}(\Omega, \Lambda^l)$, since

$$\begin{aligned} \int_{\Omega} |u(x) - u_k(x)|^{p(x)} dx &= \int_{\Omega} \left(\left(\sum_I |u_I(x) - u_{Ik}(x)|^2 \right)^{1/2} \right)^{p(x)} dx \\ &\leq \int_{\Omega} \left(\sum_I |u_I(x) - u_{Ik}(x)| \right)^{p(x)} dx \\ &\leq 2^{p^*} \sum_I \int_{\Omega} |u_I(x) - u_{Ik}(x)|^{p(x)} dx. \end{aligned} \quad (2.29)$$

That is to say, $C_0^\infty(\Omega, \Lambda^l)$ is dense in $L^{p(x)}(\Omega, \Lambda^l)$.

(2) Let $u(x) = \sum_I u_I(x) dx_I \in L^{p(x)}(\Omega, \Lambda^l)$. Since $L^{p(x)}(\Omega)$ is separable, there exists a countable dense subset K of $L^{p(x)}(\Omega)$. Then for any $u_I(x)$ above we can extract a sequence $\{u_{Ik}(x)\}$ in K which converges to $u_I(x)$ in $L^{p(x)}(\Omega)$. Similar to (1), the sequence $\{u_k : u_k(x) = \sum_I u_{Ik}(x) dx_I\}$ converges to $u(x)$ in $L^{p(x)}(\Omega, \Lambda^l)$. That is to say, $L^{p(x)}(\Omega, \Lambda^l)$ is separable. $\square$

Let $\mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l) = \{u(x) = \mathfrak{U}(x) - \mathfrak{U}_\Omega(x) : \mathfrak{U} \in W^{1,p(x)}(\Omega, \Lambda^l)\}$. Note that $u \in \mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$ if and only if $u_\Omega = 0$.

Lemma 2.8. *Let $p(x)$ satisfies (1.3). Then $\mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$ is a closed subspace of $W^{1,p(x)}(\Omega, \Lambda^l)$. In particular, it is a reflexive Banach space.*

Proof. Set a sequence $\{u_k(x)\} \subset \mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$ convergent to $u(x)$ in $W^{1,p(x)}(\Omega, \Lambda^l)$, then $(u_k)_\Omega = 0$. By Lemma 2.6, the operator T is continuous on $\mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$. Therefore, $u_\Omega = 0$, we have $u(x) \in \mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$. That is to say, $\mathfrak{K}^{1,p(x)}(\Omega, \Lambda^l)$ is a closed subspace of $W^{1,p(x)}(\Omega, \Lambda^l)$. $\square$

In [2], Iwaniec and Lutoborski obtained

$$\frac{\partial}{\partial x_i}(Tu) = A_i u + S_i u, \quad (2.30)$$

where

$$|A_i u(x)| \leq \frac{2^n \mu(\Omega)}{\text{diam}(\Omega)} \int_{\Omega} \frac{|u(z)|}{|x - z|^{n-1}} dz, \quad (2.31)$$

$$S_i u(x)(\xi) = \int_{\Omega} u(z)(K_i(z, x - z), \xi) dz, \quad (2.32)$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_{l-1})$ and

$$\begin{aligned} K_i(z, x - z) &= \frac{e_i}{|x - z|^n} \int_0^\infty s^{n-1} \varphi \left(z - s \frac{x - z}{|x - z|} \right) ds \\ &\quad - \frac{x - z}{|x - z|^{n+1}} \int_0^\infty s^n \varphi_i \left(z - s \frac{x - z}{|x - z|} \right) ds. \end{aligned} \quad (2.33)$$

Further for each $z \in \Omega$ and $h \in \mathbb{R}^n - \{0\}$, $K_i(z, h)$ satisfies the following properties:

- (i) $K_i(z, h) \leq \mu(\Omega)|h|^{-n}$,
- (ii) $K_i(z, sh) = s^{-n}K_i(z, h)$, $s > 0$,
- (iii) $\int_{|h|=1} K_i(z, h) = 0$ for all $z \in \Omega$.

Let $K_i(z, h) = (K_{i1}, K_{i2}, \dots, K_{in})$. Then $K_{i\alpha}$ satisfies the conditions of Calderón-Zygmund kernel on $\mathbb{R}^n \times \mathbb{R}^n$ for each $\alpha = 1, 2, \dots, n$.

Lemma 2.9. *Let $u \in L^{p(x)}(\Omega, \Lambda^l)$. Then*

$$\|\nabla T u\|_{L^{p(x)}(\Omega)} \leq C(n, p, \Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.34)$$

Proof. By Lemmas 2.3 and 2.4, and (2.31),

$$\|A_i u\|_{L^{p(x)}(\Omega, \Lambda^l)} \leq C(n, p) \mu(\Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.35)$$

Let

$$S_i u(x) = \sum_{1 \leq j_1 < j_2 < \dots < j_{l-1} \leq n} \omega_{j_1, j_2, \dots, j_{l-1}} dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_{l-1}}, \quad (2.36)$$

we can write $u(x)$ as

$$u(x) = \sum_{1 \leq \alpha \leq n, \alpha \neq j_1, j_2, \dots, j_{l-1}} \sum_{1 \leq j_1 < j_2 < \dots < j_{l-1} \leq n} u_{\alpha, j_1, j_2, \dots, j_{l-1}} dx_\alpha \wedge dx_{j_1} \wedge \dots \wedge dx_{j_{l-1}}. \quad (2.37)$$

Hence,

$$\omega_{j_1, j_2, \dots, j_{l-1}}(x) = S_i u(x)(e_{j_1}, e_{j_2}, \dots, e_{j_{l-1}}). \quad (2.38)$$

Taking $\xi = (e_{j_1}, e_{j_2}, \dots, e_{j_{l-1}})$ in (2.32), we obtain

$$\omega_{j_1, j_2, \dots, j_{l-1}}(x) = \int_\Omega \sum_{1 \leq \alpha \leq n, \alpha \neq j_1, \dots, j_{l-1}} K_{i\alpha}(z, x - z) u_{\alpha, j_1, j_2, \dots, j_{l-1}}(z) dz. \quad (2.39)$$

Now define $u(x) = 0$ if $x \in \mathbb{R}^n \setminus \Omega$. Since $K_{i\alpha}$ satisfies the conditions of Calderón-Zygmund kernel on $\mathbb{R}^n \times \mathbb{R}^n$ for each α , in view of Lemma 2.5,

$$\|\omega_{j_1, j_2, \dots, j_{l-1}}\|_{L^{p(x)}(\Omega)} \leq C(n, p) \sum_{1 \leq \alpha \leq n, \alpha \neq j_1, \dots, j_{l-1}} \|u_{\alpha, j_1, j_2, \dots, j_{l-1}}\|_{L^{p(x)}(\Omega)}. \quad (2.40)$$

So that

$$\|S_i u\|_{L^{p(x)}(\Omega, \Lambda^l)} \leq C(n, p) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.41)$$

By (2.30), (2.35), and (2.41), we have

$$\|\nabla T u\|_{L^{p(x)}(\Omega)} \leq C(n, p, \Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.42)$$

□

Now define another norm

$$\|\omega\|_{\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)} = \|\omega\|_{L^{p(x)}(\Omega, \Lambda^l)} + \|\nabla \omega\|_{L^{p(x)}(\Omega)}. \quad (2.43)$$

Remark 2.10. Replacing u with du in (2.34), we get by the definition of u_Ω

$$\begin{aligned} \|\nabla(u - u_\Omega)\|_{L^{p(x)}(\Omega)} &= \|\nabla T du\|_{L^{p(x)}(\Omega)} \\ &\leq C(n, p) \mu(\Omega) \|du\|_{L^{p(x)}(\Omega, \Lambda^l)} \\ &= C(n, p) \mu(\Omega) \|d(u - u_\Omega)\|_{L^{p(x)}(\Omega, \Lambda^l)}. \end{aligned} \quad (2.44)$$

Therefore $\|\cdot\|_{\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)}$ is equivalent to $\|\cdot\|_{\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)}$.

In this paper we also need the following two lemmas.

Lemma 2.11 (see[15]). *Let $p(x)$ satisfies (1.3). Then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact.*

Lemma 2.12 (see[15]). *Suppose that $p \in L^\infty(\Omega)$. Let $\{u_k\}_{k=1}^\infty$ be bounded in $L^{p(x)}(\Omega)$. If $u_k \rightharpoonup u$ a.e. on Ω , then $u_k \rightharpoonup u$ weakly in $L^{p(x)}(\Omega)$.*

Remark 2.13. Let $\mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l)$ be the completion of $C_0^\infty(\Omega, \Lambda^l)$ in $\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)$. Then from Remark 2.10 and Lemma 2.11, the embedding $\mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l) \hookrightarrow L^{p(x)}(\Omega, \Lambda^l)$ is compact.

Remark 2.14. Suppose $p(x)$ satisfies (1.3), Lemma 2.12 also holds on space $L^{p(x)}(\Omega, \Lambda^l)$.

3. Weak Solutions of Dirichlet Problems for the A-Harmonic Equations with Variable Growth

Theorem 3.1. *Under conditions (H1)–(H5), the Dirichlet problem (1.1) has at least one weak solution in $\mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l)$, that is to say, there exists at least one $u = \vartheta - \vartheta_\Omega \in \mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l)$ satisfying*

$$\int_{\Omega} \langle A(x, du(x)), d\varphi(x) \rangle + \langle B(x, u(x)), \varphi(x) \rangle dx = 0, \quad (3.1)$$

for all $\varphi \in W_0^{1,p(x)}(\Omega, \Lambda^{l-1})$. Here, $\vartheta \in W^{1,p(x)}(\Omega, \Lambda^{l-1})$ and $p(x)$ satisfies (1.3).

Let $V = W_0^{1,p(x)}(\Omega, \Lambda^{l-1})$ and $\mathfrak{R}_0 = \mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l)$. For $u \in V$, define $\mathfrak{A} : V \rightarrow V^*$ in the following way: for each $\varphi \in V$

$$(\mathfrak{A}u, \varphi) = \int_{\Omega} \langle A(x, du(x)), d\varphi(x) \rangle + \langle B(x, u(x)), \varphi(x) \rangle dx. \quad (3.2)$$

Now we need only to show that there exists $u \in \mathfrak{R}_0$ such that $(\mathfrak{A}u, \varphi) = 0$ for all $\varphi = \sum \varphi_I(x) dx_I \in V$.

Lemma 3.2. *$\mathfrak{A}$ is strong-weakly continuous on V .*

Proof. Let $\{u_k : u_k(x) = \sum_I u_{kI}(x) dx_I\} \subset V$ be a sequence strongly convergent to an element $u(x) = \sum u_I(x) dx_I \in V$ in V . Let $du_k(x) = \sum_J \omega_{kJ}(x) dx_J$ and $du(x) = \sum_J \omega_J(x) dx_J$. Then

(h₁) $\|u_k\|_V \leq C$ for some constant C ,

(h₂) $\{\omega_{kJ}(x)\}$ is a sequence strongly convergent to $\omega_J(x)$ in $L^{p(x)}(\Omega)$ for each J .

In view of (H2) and (h₁), we know that $A(x, du_k) = \sum A_{kJ}(x) dx_J$ and $B(x, u_k) = \sum B_{kI}(x) dx_I$ are uniformly bounded in $L^{p'(x)}(\Omega, \Lambda^l)$ and $L^{p'(x)}(\Omega, \Lambda^{l-1})$, respectively. Hence, $A_{kJ}(x)$ and $B_{kI}(x)$ are uniformly bounded in $L^{p'(x)}(\Omega)$. On the other hand, by (h₂), there exists a subsequence of $\{\omega_{kJ}(x)\}$ (still denoted by $\{\omega_{kJ}(x)\}$) such that

$$\lim_{k \rightarrow \infty} \omega_{kJ}(x) = \omega_J(x), \quad \text{a.e. } x \in \Omega, \text{ for each } J. \quad (3.3)$$

Then there exists a subsequence of $\{u_k(x)\}$ (still denoted by $\{u_k(x)\}$) such that

$$\lim_{k \rightarrow \infty} u_k(x) = u(x), \quad \lim_{k \rightarrow \infty} du_k(x) = du(x), \quad \text{a.e. } x \in \Omega. \quad (3.4)$$

In view of (H1), we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} A(x, du_k) &= A(x, du), \quad \text{a.e. } x \in \Omega, \\ \lim_{k \rightarrow \infty} B(x, u_k) &= B(x, u), \quad \text{a.e. } x \in \Omega. \end{aligned} \quad (3.5)$$

Let $d\varphi(x) = \sum \varphi_J(x)dx_J$, $A(x, du) = \sum A_J(x)dx_J$, and $B(x, u) = \sum B_I(x)dx_I$, then $\varphi_J(x) \in L^{p(x)}(\Omega)$, in the meantime

$$\lim_{k \rightarrow \infty} A_{kJ}(x) = A_J(x), \quad \text{a.e. } x \in \Omega, \quad (3.6)$$

$$\lim_{k \rightarrow \infty} B_{kI}(x) = B_I(x), \quad \text{a.e. } x \in \Omega, \quad (3.7)$$

for each J and I .

Now by Lemma 2.12, we can show that $\int_{\Omega} A_{kJ}(x)\varphi_J(x)dx \rightarrow \int_{\Omega} A_J(x)\varphi_J(x)dx$ and $\int_{\Omega} B_{kI}(x)\varphi_I(x)dx \rightarrow \int_{\Omega} B_I(x)\varphi_I(x)dx$ as $k \rightarrow \infty$. Therefore,

$$\begin{aligned} (\mathfrak{A}u_k, \varphi) &= \int_{\Omega} \langle A(x, du_k), d\varphi \rangle + \langle B(x, u_k), \varphi \rangle dx \\ &\rightarrow \int_{\Omega} \langle A(x, du), d\varphi \rangle + \langle B(x, u), \varphi \rangle dx \\ &= (\mathfrak{A}u, \varphi), \end{aligned} \quad (3.8)$$

that is to say, $\mathfrak{A}$ is strong-weakly continuous on V . □

Lemma 3.3. $\mathfrak{A}$ is coercive on $\mathfrak{K}_0$, that is,

$$\lim_{\|u\|_{\mathfrak{K}} \rightarrow \infty} \frac{(\mathfrak{A}u, u)}{\|u\|_{\mathfrak{K}}} = +\infty, \quad \forall u \in \mathfrak{K}_0. \quad (3.9)$$

Proof. By (H3) and (H4),

$$\begin{aligned} (\mathfrak{A}u, u) &= \int_{\Omega} \langle A(x, du), du \rangle + \langle B(x, u), u \rangle dx \\ &\geq \int_{\Omega} \left(a|du|^{p(x)} - |h(x)| + \bar{a}|u|^{p(x)} - \left| \bar{h}(x) \right| \right) dx \\ &\geq \int_{\Omega} a|du|^{p(x)} dx - C(h, \bar{h}). \end{aligned} \quad (3.10)$$

By $d\vartheta_{\Omega} = 0$ and Lemma 2.6, we have

$$\|u\|_{L^{p(x)}(\Omega, \Lambda^{l-1})} = \|Td\vartheta\|_{L^{p(x)}(\Omega, \Lambda^{l-1})} \leq 2^n C(n, p) \mu(\Omega) (\text{diam } \Omega) \|du\|_{L^{p(x)}(\Omega, \Lambda^l)}, \quad (3.11)$$

for all $u = \vartheta - \vartheta_{\Omega} \in \mathfrak{K}_0$. Then $\|du\|_{L^{p(x)}(\Omega, \Lambda^l)} \rightarrow \infty$, as $\|u\|_{\mathfrak{K}} \rightarrow \infty$. Taking

$$\delta = \frac{1}{2} \|du\|_{L^{p(x)}(\Omega, \Lambda^l)} > 1, \quad (3.12)$$

we have

$$\begin{aligned} \frac{\int_{\Omega} |du|^{p(x)} dx}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)}} &= \int_{\Omega} \left(\frac{|du|}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)} - \delta} \right)^{p(x)} \frac{(\|du\|_{L^{p(x)}(\Omega, \Lambda^l)} - \delta)^{p(x)}}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)}} dx \\ &\geq \frac{(\|du\|_{L^{p(x)}(\Omega, \Lambda^l)} - \delta)^{p_*}}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)}} = \left(\frac{1}{2}\right)^{p_*} \|du\|_{L^{p(x)}(\Omega, \Lambda^l)}^{p_*-1}. \end{aligned} \quad (3.13)$$

Therefore,

$$\frac{\int_{\Omega} |du|^{p(x)} dx}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)}} \longrightarrow \infty \quad \text{as } \|du\|_{L^{p(x)}(\Omega, \Lambda^l)} \longrightarrow \infty. \quad (3.14)$$

Then it is immediate to obtain that

$$\frac{(\mathfrak{A}u, u)}{\|u\|_{\mathfrak{K}}} \longrightarrow \infty \quad \text{as } \|u\|_{\mathfrak{K}} \longrightarrow \infty. \quad (3.15)$$

That is to say, $\mathfrak{A}$ is coercive on $\mathfrak{K}_0$. □

Lemma 3.4 (see [16]). *Suppose $g = A(x)$ is a mapping from $\mathbb{R}^m$ into itself such that*

$$\lim_{|x| \rightarrow \infty} \frac{A(x) \cdot x}{|x|} = \infty. \quad (3.16)$$

Then the range of A is the whole of $\mathbb{R}^m$.

Lemma 3.5. *There exists a sequence $\{u_k\} \subset \mathfrak{K}_0$ and $u_0 \in \mathfrak{K}_0$, such that*

$$(\mathfrak{A}u_k, u_k - u_0) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (3.17)$$

Proof. By Lemmas 2.7 and 2.8, we can choose a Schauder basis $\{\omega_s\}$ of $\mathfrak{K}_0$ such that the union of subspace finitely generated from ω_s is dense in $\mathfrak{K}_0$. Let $\mathfrak{K}_0^k$ be the subspace of $\mathfrak{K}_0$ generated by $\omega_1, \omega_2, \dots, \omega_k$. Since $\mathfrak{K}_0^k$ is topologically isomorphic to $\mathbb{R}^k$. By By Lemmas 3.3, and 3.4, there exists $u_k \in \mathfrak{K}_0^k$ such that

$$(\mathfrak{A}u_k, \omega) = 0 \quad \forall \omega \in \mathfrak{K}_0^k. \quad (3.18)$$

By Lemma 3.3 again, we know that $\|u_k\|_{\mathfrak{K}} \leq C$, where C is independent of k . Since $\mathfrak{K}_0$ is reflexive, by Remark 2.14 and (H1), we can extract a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } \mathfrak{K}_0, \quad \mathfrak{A}u_k \rightharpoonup \xi \quad \text{weakly}^* \text{ in } \mathfrak{K}_0^*, \quad (\xi, \omega) = 0, \quad (3.19)$$

where ω is in a dense subset of $\mathfrak{K}_0$. For fixed ξ , by the continuity of $(\xi, \cdot)$, we get $(\xi, \omega) = 0$ for all $\omega \in \mathfrak{K}_0$. For $(\mathfrak{A}u_k, u_k - u_0)$, we have

$$(\mathfrak{A}u_k, u_k - u_0) = (\mathfrak{A}u_k, u_k) - (\mathfrak{A}u_k, u_0) = -(\mathfrak{A}u_k, u_0) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (3.20)$$

This completes the proof of Lemma 3.5. $\square$

Set $v_k = u_k - u_0 = \sum v_{kI} dx_I$. Then

$$v_k \rightharpoonup 0 \quad \text{weakly in } \mathfrak{K}_0 \quad \text{as } k \longrightarrow \infty. \quad (3.21)$$

Consider $(\mathfrak{A}u_k, u_k - u_0)$ once more, then

$$(\mathfrak{A}u_k, u_k - u_0) = \int_{\Omega} \langle A(x, du_0 + dv_k), dv_k \rangle + \langle B(x, u_0 + v_k), v_k \rangle dx \longrightarrow 0, \quad (3.22)$$

as $k \rightarrow \infty$. By Remark 2.13, we get

$$v_k \longrightarrow 0 \quad \text{strongly in } L^{p(x)}(\Omega, \Lambda^{l-1}). \quad (3.23)$$

In view of (3.23) and (H2), it is immediate that

$$\int_{\Omega} \langle B(x, u_0 + v_k), v_k \rangle dx \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \quad (3.24)$$

that is to say,

$$\int_{\Omega} \langle A(x, du_0 + dv_k), dv_k \rangle dx \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (3.25)$$

Now if we can prove that there exists a subsequence of $\{v_k\}$ which is strongly convergent in $\mathfrak{K}_0$, then from the strong-weakly continuity of $\mathfrak{A}$, we get $\mathfrak{A}u_k \rightharpoonup \mathfrak{A}u_0 = \xi$ weakly in $\mathfrak{K}_0$ as $k \rightarrow \infty$ and u_0 will be a weak solution of (1.1). We need the following lemmas.

Definition 3.6. Let Ω be an open subset of $\mathbb{R}^n$ provided with the Lebesgue measure. The mapping $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is said Carathéodory function if for almost all $x \in \Omega$, $f(x, \cdot)$ is continuous on $\mathbb{R}^N$, for all $\xi \in \mathbb{R}^N$ is measurable on Ω .

Lemma 3.7 (see[17]). *A mapping $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Carathéodory function if and only if for all compact sets $K \subset \Omega$ and all $\varepsilon > 0$, there exists a compact subset $K_\varepsilon \subset K$ such that $\text{meas}(K - K_\varepsilon) < \varepsilon$ for with the restriction of f to $K_\varepsilon \times \mathbb{R}^N$ is continuous.*

Lemma 3.8 (see[15]). Let $\{f_k\}$ be a sequence of bounded function in $L^1(\mathbb{R}^n)$. For each $\varepsilon > 0$ there exists $(A_\varepsilon, \delta, N)$ (where A_ε is measurable and $\text{meas}(A_\varepsilon) < \varepsilon, \delta > 0, N$ is an infinite subset of natural numbers set $\mathbb{N}$) such that for each $k \in N$,

$$\int_B |f_k(x)| dx < \varepsilon, \quad (3.26)$$

where B and A_ε are disjoint and $\text{meas}(B) < \delta$.

Definition 3.9. For $u \in C_0^1(\mathbb{R}^n)$, define

$$(M^*u)(x) = (Mu)(x) + \sum_{\alpha=1}^n \left(M \frac{\partial u}{\partial x_\alpha} \right)(x). \quad (3.27)$$

Lemma 3.10 (see[18]). If $u \in C_0^\infty(\mathbb{R}^n)$, then $M^*u \in C^0(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$,

$$|u(x)| + \sum_{\alpha=1}^n \left| \frac{\partial u}{\partial x_\alpha}(x) \right| \leq (M^*u)(x). \quad (3.28)$$

Furthermore, if $p > 1$, then

$$\|M^*u\|_{L^p(\mathbb{R}^n)} \leq C(n, p) \|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad (3.29)$$

and if $p = 1$, then

$$\text{meas}(\{x \in \mathbb{R}^n : (M^*u)(x) > \lambda\}) \leq \frac{C(n)}{\lambda} \|u\|_{W^{1,1}(\mathbb{R}^n)}, \quad (3.30)$$

for all $\lambda > 0$.

Lemma 3.11 (see[19]). Let $u \in C_0^\infty(\mathbb{R}^n)$ and $\lambda > 0$. Set

$$H^\lambda = \{x \in \mathbb{R}^n : (M^*u)(x) < \lambda\}. \quad (3.31)$$

Then for all $x, y \in H^\lambda$, we have

$$|u(y) - u(x)| \leq C(n)\lambda|y - x|. \quad (3.32)$$

Lemma 3.12 (see[16]). Let X be a metric space, E be a subspace of X , and k be a positive number. Then any k -Lipchitz mapping from E into $\mathbb{R}$ can be extended to a k -Lipchitz mapping from X into $\mathbb{R}$.

Proof of Theorem 3.1. We need only to show that there exists subsequence of $\{v_k\}$ which is strongly convergent in $\mathfrak{K}_0$.

For each measurable set $S \subset \Omega$, define

$$F(v, S) = \int_S \langle A(x, du_0 + dv), dv \rangle dx, \quad (3.33)$$

where $v \in \mathfrak{K}_0$. Similar to the proof of Lemma 3.2, $F(\cdot, S)$ is strongly continuous on $\mathfrak{K}_0$. Since $C_0^\infty(\Omega, \Lambda^{l-1})$ is dense in $\mathfrak{K}_0$, there exists $h_k \in C_0^\infty(\Omega, \Lambda^{l-1})$ such that

$$\|h_k - v_k\|_{\mathfrak{K}} < \frac{1}{k}, \quad |F(h_k, \Omega) - F(v_k, \Omega)| < \frac{1}{k}. \quad (3.34)$$

So we can suppose that $\{v_k\} \subset C_0^\infty(\Omega, \Lambda^{l-1})$ is bounded in $\mathfrak{K}_0$.

Next define

$$v_k(x) = 0 \quad \text{when } x \in \mathbb{R}^n \setminus \Omega. \quad (3.35)$$

In this way, we extend the domain of v_k to $\mathbb{R}^n$ and $\text{supp } v_k \subset \Omega$.

Let $\beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous increasing function satisfying $\beta(0) = 0$ and for each measurable set $D \subset \Omega$,

$$\int_D \left(|G(x)|^{p'(x)} + |h(x)| + (C_1 + 1)|du_0|^{p(x)} \right) dx \leq \beta(\text{meas}(D)), \quad (3.36)$$

where C_1 is the constant in (H2).

Let $\{\varepsilon_j\}$ be a positive decreasing sequence with $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$. For ε_1 , by Lemma 3.8, we get a subsequence $\{k_1\}$ of $\{k\}$, a set $A_{\varepsilon_1} \subset \Omega$ satisfying $\text{meas}(A_{\varepsilon_1}) < \varepsilon_1$, and a real number $\delta_1 > 0$ such that

$$\int_B (M^* v_{k_1 I})^{p(x)} dx < \varepsilon_1, \quad (3.37)$$

for each k_1 , I and $B \subset \Omega \setminus A_{\varepsilon_1}$ satisfying $\text{meas}(B) < \delta_1$. By Lemma 3.10, we can choose $\lambda > 1$ so large that for all I and k_1 ,

$$\text{meas}(\{x \in \mathbb{R}^n : (M^* v_{k_1 I})(x) \geq \lambda\}) \leq \min\{\varepsilon_1, \delta_1\}. \quad (3.38)$$

For each I and k_1 , define

$$H_{k_1 I}^\lambda = \{x \in \mathbb{R}^n : (M^* v_{k_1 I})(x) < \lambda\}, \quad H_{k_1}^\lambda = \bigcap_I H_{k_1 I}^\lambda. \quad (3.39)$$

In view of Lemma 3.11, we have

$$\frac{|v_{k_1 I}(y) - v_{k_1 I}(x)|}{|y - x|} \leq C(n)\lambda \quad \forall x, y \in H_{k_1}^\lambda \text{ and } I. \quad (3.40)$$

Form Lemma 3.12, there exists a Lipschitz function $g_{k_1 I}$ which extends $v_{k_1 I}$ outside $H_{k_1}^\lambda$ and Lipschitz constant of $g_{k_1 I}$ is no more than $C(n)\lambda$. As $H_{k_1}^\lambda$ is an open set, we have $g_{k_1 I} = v_{k_1 I}$ and $\nabla g_{k_1 I}(x) = \nabla v_{k_1 I}(x)$ for all $x \in H_{k_1}^\lambda$, and $\|\nabla g_{k_1 I}\|_{L^\infty(\mathbb{R}^n)} \leq C(n)\lambda$. We can further suppose that

$$\|g_{k_1 I}\|_{L^\infty(\mathbb{R}^n)} \leq \|v_{k_1 I}\|_{L^\infty(H_{k_1}^\lambda)} \leq \lambda, \quad \|g_{k_1 I}\|_{W^{1,\infty}(\Omega)} \leq C(n)\lambda. \quad (3.41)$$

By the uniformly boundedness of $\{\|g_{k_1 I}\|_{W^{1,\infty}(\Omega)}\}$, there exists a subsequence of $\{g_{k_1 I}\}$ (still denoted by $\{g_{k_1 I}\}$) such that

$$g_{k_1 I} \rightharpoonup \omega_I \text{ weakly}^* \text{ in } W^{1,\infty}(\Omega) \quad \text{as } k_1 \rightarrow \infty \forall I. \quad (3.42)$$

Set $\omega = \sum_I \omega_I dx_I$ and $g_{k_1} = \sum_I g_{k_1 I} dx_I$. We have

$$F(v_{k_1}, \Omega) = F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) - F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda) + F(v_{k_1}, A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda)). \quad (3.43)$$

Next we estimate $F(v_{k_1}, \Omega)$ in four steps.

(1) *The estimate of $F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda)$ and $F(v_{k_1}, A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda))$.* Since

$$\text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda\right) \leq \sum_I \text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1 I}^\lambda\right) \leq C_n^{l-1} \min\{\varepsilon_1, \delta_1\}, \quad (3.44)$$

where $C_n^{l-1} = n(n-1) \cdots (n-l+2)/(l-1)(l-2) \cdots 1$, from (H2), (H3), and the choose of A_{ε_1} , we have

$$\begin{aligned} & \left| F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda) \right| \\ & \leq \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda} \left(C_1 |du_0 + dg_{k_1}|^{p(x)-1} |dg_{k_1}| + |G(x)| |dg_{k_1}| \right) dx \\ & \leq \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda} \left(C_1 2^{p^*-1} \left(|du_0|^{p(x)} + |dg_{k_1}|^{p(x)} \right) + C_1 |dg_{k_1}|^{p(x)} + |G(x)|^{p'(x)} + |dg_{k_1}|^{p(x)} \right) dx \\ & \leq 2^{p^*-1} \beta \left(\text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda\right) \right) + 2^{p^*} (C_1 + 1) \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda} |dg_{k_1}|^{p(x)} dx \\ & \leq 2^{p^*-1} \beta \left(\text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda\right) \right) + 2^{p^*} (C_1 + 1) \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda} \left(\sum_I |\nabla g_{k_1 I}| \right)^{p(x)} dx \end{aligned}$$

$$\begin{aligned}
&\leq 2^{p^*-1}\beta\left(C_n^{l-1}\varepsilon_1\right) + 2^{p^*}C(C_1, n, l) \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^l} \lambda^{p(x)} dx \\
&\leq 2^{p^*-1}\beta\left(C_n^{l-1}\varepsilon_1\right) + 2^{p^*}C(C_1, n, l) \sum_I \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1 I}^l} (M^* v_{k_1 I})^{p(x)} dx \\
&\leq 2^{p^*-1}\beta\left(C_n^{l-1}\varepsilon_1\right) + 2^{p^*}C(C_1, n, l)\varepsilon_1 \leq O(\varepsilon_1),
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
&F\left(v_{k_1}, A_{\varepsilon_1} \cup \left(\Omega \setminus H_{k_1}^l\right)\right) \\
&= \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} \langle A(x, du_0 + dv_{k_1}), du_0 + dv_{k_1} \rangle - \langle A(x, du_0 + dv_{k_1}), du_0 \rangle dx \\
&\geq \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} \left(a|du_0 + dv_{k_1}|^{p(x)} - h(x) \right) - \left(C_1|du_0 + dv_{k_1}|^{p(x)-1}|du_0| + |G(x)||du_0| \right) dx \\
&\geq \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} \left(\left(a2^{-(p^*-1)} - C_1\mu 2^{p^*-1} \right) |dv_{k_1}|^{p(x)} - |h(x)| - |G(x)|^{p'(x)} \right) \\
&\quad - \left(-a2^{-(p^*-1)} + C_1\mu 2^{p^*-1} + C_1C(\mu) + 1 \right) |du_0|^{p(x)} dx \\
&\geq \left(a2^{-(p^*-1)} - C_1\mu 2^{p^*-1} \right) \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} |dv_{k_1}|^{p(x)} dx - C(a, p, C_1, \mu) \beta\left(\text{meas}\left(A_{\varepsilon_1} \cup \left(\Omega \setminus H_{k_1}^l\right)\right)\right) \\
&\geq a2^{-p^*} \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1),
\end{aligned} \tag{3.46}$$

where $\mu > 0$ is small enough.

From (3.43)–(3.46), we get

$$F(v_{k_1}, \Omega) \geq F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) + a2^{-p^*} \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \tag{3.47}$$

(2) *The estimate of $F(g_{k_1}, \Omega \setminus A_{\varepsilon_1})$.* Set $f_{k_1 I} = g_{k_1 I} - \omega_I$, where ω_I is defined by (3.42). Then

$$\begin{aligned}
&f_{k_1 I} \rightharpoonup 0 \text{ weakly}^* \text{ in } W^{1,\infty}(\Omega) \text{ as } k_1 \longrightarrow \infty \quad \forall I, \\
&\|f_{k_1 I}\|_{L^\infty(\Omega)} \leq 2\lambda, \quad \|df_{k_1 I}\|_{L^\infty(\Omega, \Lambda^l)} \leq 2C(n)\lambda.
\end{aligned} \tag{3.48}$$

Let $G = \bigcup_I G_I$ with $G_I = \{x \in \Omega : \omega_I(x) \neq 0\}$. According to Acerbi and Fusco [19], we have $\text{meas}(G) \leq (C_n^{l-1} + 1)\varepsilon_1$ where $C_n^{l-1} = n(n-1) \cdots (n-l+2)/(l-1)(l-2) \cdots 1$, and set $f_{k_1} = \sum_I f_{k_1 I} dx_I$, then

$$\begin{aligned} F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) &= F(f_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus G) \\ &\quad + F(v_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap H_{k_1}^\lambda \cap G) \\ &\quad + F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap (G \setminus H_{k_1}^\lambda)). \end{aligned} \quad (3.49)$$

Define

$$\begin{aligned} \Omega_1^{\varepsilon_1, k_1} &= A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda), & \Omega_2^{\varepsilon_1} &= (\Omega \setminus A_{\varepsilon_1}) \setminus G, \\ \Omega_3^{\varepsilon_1, k_1} &= (\Omega \setminus A_{\varepsilon_1}) \cap H_{k_1}^\lambda \cap G, & \Omega_4^{\varepsilon_1, k_1} &= (\Omega \setminus A_{\varepsilon_1}) \cap (G \setminus H_{k_1}^\lambda). \end{aligned} \quad (3.50)$$

Similar to the proof of (3.46), we get

$$F(v_{k_1}, \Omega_3^{\varepsilon_1, k_1}) \geq a2^{-p^*} \int_{\Omega_3^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \quad (3.51)$$

Since on $\Omega_4^{\varepsilon_1, k_1}$ we have

$$\int_{\Omega_4^{\varepsilon_1, k_1}} |dg_{k_1}|^{p(x)} dx \leq C(n, p)(C_n^{l-1} + 1)\varepsilon_1, \quad (3.52)$$

then similar to the proof of (3.45), we get

$$\left| F(g_{k_1}, \Omega_4^{\varepsilon_1, k_1}) \right| \leq O(\varepsilon_1). \quad (3.53)$$

By (3.49)–(3.53), we have

$$F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) \geq F(f_{k_1}, \Omega_2^{\varepsilon_1}) + a2^{-p^*} \int_{\Omega_3^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \quad (3.54)$$

Thus, we have

$$F(v_{k_1}, \Omega) \geq F(f_{k_1}, \Omega_2^{\varepsilon_1}) + a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1), \quad (3.55)$$

where $\Omega_5^{\varepsilon_1, k_1} = \Omega_1^{\varepsilon_1, k_1} \cup \Omega_3^{\varepsilon_1, k_1}$.

Choose an open set $\Omega' \subset \Omega$ which contains $\Omega_2^{\varepsilon_1}$ such that

$$|F(f_{k_1}, \Omega') - F(f_{k_1}, \Omega_2^{\varepsilon_1})| < \varepsilon_1. \quad (3.56)$$

From (3.55), we get

$$F(v_{k_1}, \Omega) \geq F(f_{k_1}, \Omega') + a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \quad (3.57)$$

Approximate Ω' by hypercubes with edges parallel to coordinate axes, that is, construct

$$\begin{aligned} H_j &\subset \Omega', \\ \text{meas}(\Omega' \setminus H_j) &\longrightarrow 0 \text{ as } j \longrightarrow \infty, \\ H_j &= \bigcup_{s=1}^{h_j} D_{j,s}, \\ \text{meas}(D_{j,s}) &= 1/2^{nj}, \quad 1 \leq s \leq h_j. \end{aligned} \quad (3.58)$$

Let $j > 0$ be large enough such that for all $k_1 > 0$, we have

$$\begin{aligned} |F(f_{k_1}, \Omega') - F(f_{k_1}, H_j)| &< \varepsilon_1, \quad \int_{\Omega' \setminus H_j} |df_{k_1}|^{p(x)} dx < \varepsilon_1, \\ \text{meas}(\Omega' \setminus H_j) &< \min\{\varepsilon_1, \delta_1\}. \end{aligned} \quad (3.59)$$

Thus,

$$F(v_{k_1}, \Omega) \geq F(f_{k_1}, H_j) + a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1) - 2\varepsilon_1. \quad (3.60)$$

(3) *The estimate of $F(f_{k_1}, H_j)$.* Let $\alpha > 0$ be large enough such that for $E = \{x \in \Omega' : \eta(x) \leq \alpha\}$. Then

$$\text{meas}(\Omega' \setminus E) < \frac{\varepsilon_1}{N}, \quad \int_{\Omega' \setminus E} \eta(x) dx < \varepsilon_1, \quad (3.61)$$

where $\|df_{k_1}\|_{L^\infty(\Omega, \Lambda')} \leq 2C_n^{l-1} C(n)\lambda = N$ and $\eta(x) = |G(x)|^{p'(x)} + 2^{p^*-1}(C_1 + 1)|du_0|^{p(x)}$.

For $x \in \Omega, \xi \in \Lambda^l(\Omega)$, define

$$\varphi(x, \xi) = \langle A(x, du_0(x) + \xi), \xi \rangle. \quad (3.62)$$

By Lemma 3.7 and (H1), there exists a compact subset $K \subset H_j$ such that $\psi(x, \xi)$ is continuous on $K \times \Lambda^l(\Omega)$ and $\text{meas}(H_j \setminus K) < \varepsilon_1/(\alpha + N)$. Hence, $\psi(x, \xi)$ is uniformly continuous on bounded subsets of $K \times \Lambda^l(\Omega)$.

Divide each $D_{j,s}$ into 2^{nm} hypercubes $Q_{t,j,s}^m$ with edge length 2^{-jm} , $1 \leq t \leq 2^{nm}$. For all j, s, m, t , take $x_{t,j,s}^m \in Q_{t,j,s}^m \cap K \cap E$ (if this set is empty, take $x_{t,j,s}^m \in Q_{t,j,s}^m$) such that

$$\eta(x_{t,j,s}^m) \text{meas}(Q_{t,j,s}^m) \leq \int_{Q_{t,j,s}^m} \eta(x) dx. \quad (3.63)$$

Then

$$\begin{aligned} F(f_{k_1}, H_j) &= F(f_{k_1}, H_j \cap K \cap E) + F(f_{k_1}, H_j \setminus E) + F(f_{k_1}, (H_j \cap E) \setminus K) \\ &\geq F(f_{k_1}, H_j \cap K \cap E) - \int_{H_j \setminus E} \eta(x) dx - \int_{(H_j \cap E) \setminus K} \eta(x) dx \\ &\quad - 2^{p^*}(C_1 + 1) \left(\int_{H_j \setminus E} |df_{k_1}|^{p(x)} dx + \int_{(H_j \cap E) \setminus K} |df_{k_1}|^{p(x)} dx \right) \\ &= F(f_{k_1}, H_j \cap K \cap E) - O(\varepsilon_1) \\ &= b_{k_1}^{m,j} + c_{k_1}^{m,j} + d_{k_1}^{m,j} - O(\varepsilon_1), \end{aligned} \quad (3.64)$$

where

$$\begin{aligned} b_{k_1}^{m,j} &= \sum_{t,s} \int_{Q_{t,j,s}^m \cap K \cap E} \left(\psi(x, df_{k_1}(x)) - \psi(x_{t,j,s}^m, df_{k_1}(x)) \right) dx, \\ c_{k_1}^{m,j} &= \sum_{t,s} \int_{Q_{t,j,s}^m} \psi(x_{t,j,s}^m, df_{k_1}(x)) dx, \\ d_{k_1}^{m,j} &= - \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} \psi(x_{t,j,s}^m, df_{k_1}(x)) dx. \end{aligned} \quad (3.65)$$

By (3.25), we have

$$\lim_{k_1 \rightarrow \infty} F(v_{k_1}, \Omega) = 0. \quad (3.66)$$

Note that if $Q_{t,j,s}^m \cap K \cap E$ is an empty set, then

$$\int_{Q_{t,j,s}^m \cap K \cap E} \left[\psi(x, df_{k_1}(x)) - \psi(x_{t,j,s}^m, df_{k_1}(x)) \right] dx = 0. \quad (3.67)$$

Now we only consider $Q_{t,j,s}^m$ which satisfies $Q_{t,j,s}^m \cap K \cap E \neq \emptyset$. Since $du_0(x)$ is uniformly continuous on H_j , then by the uniform continuity of ψ on bounded subsets of $K \times \Lambda^l(\Omega)$, we obtain that for $x \in Q_{t,j,s}^m$, there exists a constant $L > 0$ such that

$$\begin{aligned} & \left| \psi(x, df_{k_1}(x)) - \psi(x_{t,j,s}^m, df_{k_1}(x)) \right| \\ &= \left| \left\langle A(x, du_0(x) + df_{k_1}(x)) - A(x_{t,j,s}^m, du_0(x_{t,j,s}^m) + df_{k_1}(x)), df_{k_1}(x) \right\rangle \right| \\ &< \frac{1}{\text{meas}(H_j)} \varepsilon_1 \end{aligned} \quad (3.68)$$

holds for all $m > L$ and each k_1 . Therefore, $|b_{k_1}^{m,j}| < \varepsilon_1$ for all k_1 .

$$\begin{aligned} |d_{k_1}^{m,j}| &\leq \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} \left| \psi(x_{t,j,s}^m, df_{k_1}(x)) \right| dx \\ &= \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} \left\langle A(x_{t,j,s}^m, du_0(x_{t,j,s}^m) + df_{k_1}(x)), df_{k_1}(x) \right\rangle dx \\ &\leq \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} C_1 \left| du_0(x_{t,j,s}^m) + df_{k_1}(x) \right|^{p(x)-1} |df_{k_1}(x)| + \left| G(x_{t,j,s}^m) \right| |df_{k_1}(x)| dx \\ &\leq \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} \left(\eta(x_{t,j,s}^m) + 2^{p^*} (C_1 + 1) N \right) dx \\ &\leq \int_{(H_j \cap E) \setminus K} \left(\eta(x_{t,j,s}^m) + 2^{p^*} (C_1 + 1) N \right) dx + C(C_1, p) \sum_{t,s} \int_{Q_{t,j,s}^m \setminus E} \left(\eta(x_{t,j,s}^m) + N \right) dx \\ &\leq C(\alpha, N, C_1, p) \text{meas}((H_j \cap E) \setminus K) + C(C_1, p) \int_{H_j \setminus E} [\eta(x) + N] dx \\ &\leq C(\alpha, N, C_1, p) \varepsilon_1 \leq O(\varepsilon_1). \end{aligned} \quad (3.69)$$

Now we suppose that m is large enough that $|b_{k_1}^{m,j}| < \varepsilon_1$ for each $k_1 > 0$ and there exists $\bar{k}_1 > 0$ such that $F(v_{k_1}, \Omega) < \varepsilon_1$ for $k_1 > \bar{k}_1$. Therefore, from (3.25), (3.60), and (3.64), we have

$$\begin{aligned} \varepsilon_1 &\geq F(v_{k_1}, \Omega) \\ &\geq c_{k_1}^{m,j} + a 2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1) - 3\varepsilon_1 - C(C_1, p) \varepsilon_1 \\ &= c_{k_1}^{m,j} + a 2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \end{aligned} \quad (3.70)$$

(4) The estimate of $c_{k_1}^{m,j}$. By $f_{k_1 I} \rightharpoonup 0$ weakly* in $W^{1,\infty}(\Omega)$ as $k_1 \rightarrow \infty$, we obtain $\|f_{k_1 I}\|_{L^\infty(\Omega)} \rightarrow 0$ as $k_1 \rightarrow \infty$ for each I . Then

$$R_{t,s,j}^{k_1,m} = \|f_{k_1}\|_{L^\infty(Q_{t,s,j}^m)} \rightarrow 0 \quad \text{as } k_1 \rightarrow \infty \text{ for fixed } m. \quad (3.71)$$

Define a hypercube $E_{t,s,j}^{k_1,m}$ contained in $Q_{t,s,j}^m$ with edge length $1/2^{jm} - 2R_{t,s,j}^{k_1,m}$ such that $\text{dist}(\partial Q_{t,s,j}^m, E_{t,s,j}^{k_1,m}) = R_{t,s,j}^{k_1,m}$.

Next define

$$\begin{aligned} \varphi_{k_1}(x) &= 0, \quad x \in \partial Q_{t,s,j}^m, \\ \varphi_{k_1}(x) &= f_{k_1}(x), \quad x \in E_{t,s,j}^m. \end{aligned} \quad (3.72)$$

Since $\varphi_{k_1 I}$ is a Lipschitz mapping on set $E_{t,s,j}^m \cup \partial Q_{t,s,j}^m$ and its Lipschitz constant is no more than $2C(n)\lambda$, by Lemma 3.12, $\varphi_{k_1 I}$ can be extended to the whole $Q_{t,s,j}^m$, where it is also a Lipschitz mapping with the same Lipschitz constant. We still denote the extension by $\varphi_{k_1 I}$ and suppose that it is defined on the whole H_j . Then by [20]

$$\nabla \varphi_{k_1 I} - \nabla f_{k_1 I} \rightarrow 0 \quad \text{a.e. on } H_j. \quad (3.73)$$

Thus, there exists a $\overline{k_1} > \overline{k_1}$ such that for all $k_1 > \overline{k_1}$, we have

$$\begin{aligned} \int_{H_j} |d\varphi_{k_1} - df_{k_1}|^{p(x)} dx &\leq \frac{\varepsilon_1}{2}, \\ \sum_{t,s} \left| \int_{Q_{t,j,s}^m} \varphi(x_{t,j,s}^m, df_{k_1}(x)) - \varphi(x_{t,j,s}^m, d\varphi_{k_1}(x)) dx \right| &\leq \frac{\varepsilon_1}{2}. \end{aligned} \quad (3.74)$$

In view of (H5), we obtain that

$$\begin{aligned} c_{k_1}^{m,j} &= \sum_{t,s} \int_{Q_{t,j,s}^m} \varphi(x_{t,j,s}^m, df_{k_1}(x)) dx \\ &\geq \sum_{t,s} \int_{Q_{t,j,s}^m} \varphi(x_{t,j,s}^m, d\varphi_{k_1}(x)) dx - \frac{\varepsilon_1}{2} \\ &= \sum_{t,s} \int_{Q_{t,j,s}^m} \langle A(x_{t,j,s}^m, du_0(x_{t,j,s}^m) + d\varphi_{k_1}(x)), d\varphi_{k_1}(x) \rangle dx - \frac{\varepsilon_1}{2} \\ &\geq \gamma \sum_{t,s} \int_{Q_{t,j,s}^m} |d\varphi_{k_1}|^{p(x)} dx - \frac{\varepsilon_1}{2} \\ &\geq \frac{\gamma}{2^{p^*-1}} \int_{H_j} |df_{k_1}|^{p(x)} dx - \frac{(\gamma+1)\varepsilon_1}{2}. \end{aligned} \quad (3.75)$$

Thus in (3.70) for $k_1 > \overline{\overline{k_1}}$, we obtain the estimate of $F(v_{k_1}, \Omega)$ from the four steps above

$$\begin{aligned} \varepsilon_1 &\geq F(v_{k_1}, \Omega) \\ &\geq a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx + \frac{\gamma}{2^{p^*-1}} \int_{H_j} |df_{k_1}|^{p(x)} dx - \frac{(\gamma+1)\varepsilon_1}{2} - O(\varepsilon_1). \end{aligned} \quad (3.76)$$

Let $K(\varepsilon_1) = (\gamma+1)\varepsilon_1 / (2 + o(\varepsilon_1)) / \min\{a2^{-p^*}, \gamma/2^{p^*-1}\}$. Then

$$\int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx + \int_{H_j} |df_{k_1}|^{p(x)} dx \leq K(\varepsilon_1), \quad \text{for } k_1 > \overline{\overline{k_1}}. \quad (3.77)$$

Form (3.59) and (3.77), we deduce that

$$\int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx \leq K(\varepsilon_1), \quad \int_{\Omega'} |df_{k_1}|^{p(x)} dx \leq K(\varepsilon_1) + \varepsilon_1. \quad (3.78)$$

According to the definition of $\Omega_2^{\varepsilon_1}$, we have

$$\int_{\Omega_{\varepsilon_1}^2} |dg_{k_1}|^{p(x)} dx \leq K(\varepsilon_1) + \varepsilon_1. \quad (3.79)$$

Since $dg_{k_1}(x) = dv_{k_1}(x)$ for each $x \in H_{k_1}^\lambda$, we get

$$\int_{\Omega_{\varepsilon_1}^2 \cap H_{k_1}^\lambda} |dv_{k_1}|^{p(x)} dx \leq K(\varepsilon_1) + \varepsilon_1. \quad (3.80)$$

By the definitions of $\Omega_2^{\varepsilon_1}$ and $\Omega_5^{\varepsilon_1, k_1}$, it is immediate that

$$(\Omega_2^{\varepsilon_1} \cap H_{k_1}^\lambda) \cup \Omega_5^{\varepsilon_1, k_1} = \Omega, \quad (3.81)$$

which implies that

$$\int_{\Omega} |dv_{k_1}|^{p(x)} dx \leq 2K(\varepsilon_1) + \varepsilon_1 \leq O(\varepsilon_1). \quad (3.82)$$

For $\varepsilon_2 > 0$ and the sequence $\{v_{k_1}\}$, repeating the above arguments we can extract a subsequence $\{v_{k_2}\}$ of $\{v_{k_1}\}$ such that

$$\int_{\Omega} |dv_{k_2}|^{p(x)} dx \leq O(\varepsilon_2), \quad (3.83)$$

whenever $k_2 > \overline{\overline{k_2}}$ for some $\overline{\overline{k_2}}$. If $\{v_{k_n}\}$ has been obtained, repeating the above process, we can extract a subsequence $\{k_{n+1}\}$ of $\{k_n\}$ such that

$$\int_{\Omega} |dv_{k_{n+1}}|^{p(x)} dx \leq O(\varepsilon_{n+1}), \quad (3.84)$$

whenever $k_{n+1} > \overline{\overline{k_{n+1}}}$ for some $\overline{\overline{k_{n+1}}}$. Finally, by a diagonal argument we get a subsequence $\{v_{k_i}\}_{i=1}^{\infty}$ which satisfies

$$\int_{\Omega} |dv_{k_i}|^{p(x)} dx \rightarrow 0 \quad \text{as } i \rightarrow \infty. \quad (3.85)$$

Therefore,

$$\|dv_{k_i}\|_{L^{p(x)}(\Omega, \Lambda^l)} \rightarrow 0 \quad \text{as } i \rightarrow \infty, \quad (3.86)$$

and by (3.23), $\{v_{k_i}\}_{i=1}^{\infty}$ strongly converges to zero in $\mathfrak{K}_0$ as $i \rightarrow \infty$. This completes the proof of Theorem 3.1. $\square$

4. Applications

In this section, we explore applications of our results developed in this paper.

Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex Lipschitz domain. Suppose that maps $A : \Omega \times \Lambda^l(\Omega) \rightarrow \Lambda^l(\Omega)$ and $B : \Omega \times \Lambda^{l-1}(\Omega) \rightarrow \Lambda^{l-1}(\Omega)$, where $l = 1, 2, \dots, n$.

Example 4.1. If $p(x)$ satisfies (1.3), let $l = 1$, $A(x, \xi) = \xi|\xi|^{p(x)-2}$ and $B(x, \varsigma) = \varsigma|\varsigma|^{p(x)-2} - f(x)$, where $f(x) \in L^{p'(x)}(\Omega)$. Then A, B satisfy the required conditions, and (1.1) reduce to the following $p(x)$ -Laplacian equations:

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) + |u|^{p(x)-2} u &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega. \end{aligned} \quad (4.1)$$

Now by Theorem 3.1, we deduce that the $p(x)$ -Laplacian equations (4.1) have at least one weak solution in $\mathfrak{K}^{1,p(x)}(\Omega)$ with $u = 0$ on $\partial\Omega$.

Example 4.2. If $l = 1$, $A(x, \xi) = \sum_{i,j} A_{ij}(x) \xi_j dx_i$, $B(x, \varsigma) = B(x) \varsigma - f(x)$, where $f(x) \in L^2(\Omega)$, and $A_{ij}(x), B(x)$ satisfy the following conditions:

$$A_{ij}(x) = A_{ji}(x), \quad \wedge |\xi|^2 \geq \sum_{i,j=1}^n A_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2, \quad \lambda \leq B(x) \leq \wedge, \quad (4.2)$$

for some constants $\lambda, \Lambda > 0$. Then A, B satisfy the required conditions, and (1.1) reduce to the following Divergence form equations:

$$\sum_{i,j=1}^n \nabla_j (A_{ij}(x) \nabla_i u(x)) + B(x)u(x) = f(x), \quad x \in \Omega, \quad (4.3)$$

$$u(x) = 0, \quad x \in \partial\Omega, \quad (4.4)$$

where $\nabla_i = (\partial/\partial x_i)$. Now by Theorem 3.1, we deduce that the divergence form (4.3) have at least one weak solution $u(x)$ in $\mathcal{R}^{1,2}(\Omega)$ with $u = 0$ on $\partial\Omega$. The comparison principles, the maximum principles, and the existence of weak solutions for divergence form equation (4.3) can be found in [21].

Example 4.3. If $p(x)$ satisfies (1.3), let $A(x, \xi) = \xi|\xi|^{p(x)-2}$ and $B(x, \varsigma) = \varsigma|\varsigma|^{p(x)-2} - f(x)$, where $f(x) \in L^{p'(x)}(\Omega, \wedge^{l-1})$. Then A, B satisfy the required conditions, and (1.1) reduce to the following $p(x)$ -harmonic equations for differential forms:

$$d^* (du|du|^{p(x)-2}) + u|u|^{p(x)-2} = f(x), \quad x \in \Omega, \quad (4.5)$$

$$u(x) = 0, \quad x \in \partial\Omega. \quad (4.6)$$

Now by Theorem 3.1, we deduce that (4.5) have at least one weak solution $u(x)$ in $\mathcal{R}_0^{1,p(x)}(\Omega, \wedge^{l-1})$. If $p(x)$ is a constant q and $1 < q < \infty$, the equation (4.5) is called nonhomogeneous q -harmonic equation. In [2], Iwaniec and Lutoborski studied the L^q theory of weak solution for homogeneous q -harmonic equations.

References

- [1] V. M. Gol'dshtein, V. I. Kuz'minov, and I. A. Shvedov, "Dual spaces to spaces of differential forms," *Akademiya Nauk SSSR*, vol. 27, no. 1, pp. 35–44, 1986.
- [2] T. Iwaniec and A. Lutoborski, "Integral estimates for null Lagrangians," *Archive for Rational Mechanics and Analysis*, vol. 125, no. 1, pp. 25–79, 1993.
- [3] F. L. Teixeira and W. C. Chew, "Differential forms, metrics, and the reflectionless absorption of electromagnetic waves," *Journal of Electromagnetic Waves and Applications*, vol. 13, no. 5, pp. 665–686, 1999.
- [4] F. L. Teixeira, "Differential form approach to the analysis of electromagnetic cloaking and masking," *Microwave and Optical Technology Letters*, vol. 49, no. 8, pp. 2051–2053, 2007.
- [5] P. Morando, "Liouville condition, Nambu mechanics, and differential forms," *Journal of Physics A*, vol. 29, no. 13, pp. L329–L331, 1996.
- [6] H. C. J. Sealey, "Some conditions ensuring the vanishing of harmonic differential forms with applications to harmonic maps and Yang-Mills theory," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 91, no. 3, pp. 441–452, 1982.
- [7] S. Ding and C. A. Nolder, "Weighted Poincaré inequalities for solutions to A -harmonic equations," *Illinois Journal of Mathematics*, vol. 46, no. 1, pp. 199–205, 2002.
- [8] R. P. Agarwal and S. Ding, "Advances in differential forms and the A -harmonic equation," *Mathematical and Computer Modelling*, vol. 37, no. 12–13, pp. 1393–1426, 2003.
- [9] Y. Xing and S. Ding, "Caccioppoli inequalities with Orlicz norms for solutions of harmonic equations and applications," *Nonlinearity*, vol. 23, no. 5, pp. 1109–1119, 2010.
- [10] Y. Xing and S. Ding, "Norms of the composition of the maximal and projection operators," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 12, pp. 4614–4624, 2010.
- [11] S. Ding and B. Liu, "Global estimates for singular integrals of the composite operator," *Illinois Journal of Mathematics*, vol. 53, no. 4, pp. 1173–1185, 2009.

- [12] O. Kováčik and J. Rákosník, "On spaces $L^{p(x)}$ and $W^{k,p(x)}$," *Czechoslovak Mathematical Journal*, vol. 41(116), no. 4, pp. 592–618, 1991.
- [13] Y. Fu and L. Guo, "Weighted variable exponent spaces of differential forms and their applications".
- [14] G. de Rham, *Differentiable Manifolds*, vol. 266 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1984.
- [15] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents*, vol. 2017 of *Lecture Notes in Mathematics*, Springer, Heidelberg, Germany, 2011.
- [16] C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer, 1966.
- [17] I. Ekeland and R. T'emam, *Convex Analysis and Variational Problems*, North-Holland, 1976.
- [18] F. C. Liu, "A Luzin type property of Sobolev functions," *Indiana University Mathematics Journal*, vol. 26, no. 4, pp. 645–651, 1977.
- [19] E. Acerbi and N. Fusco, "Semicontinuity problems in the calculus of variations," *Archive for Rational Mechanics and Analysis*, vol. 86, no. 2, pp. 125–145, 1984.
- [20] B. Dacorogna, *Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functionals*, vol. 922 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1982.
- [21] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.