Research Article

Sharp Bounds for Seiffert Mean in Terms of Contraharmonic Mean

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We find the greatest value \( \alpha \) and the least value \( \beta \) in \((1/2, 1)\) such that the double inequality

\[
C(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < T(a,b) < C(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)
\]

holds for all \( a, b > 0 \) with \( a \neq b \). Here, \( T(a,b) = (a - b)/[2 \arctan((a - b)/(a + b))] \) and \( C(a,b) = (a^2 + b^2)/(a + b) \) are the Seiffert and contraharmonic means of \( a \) and \( b \), respectively.

1. Introduction

For \( a, b > 0 \) with \( a \neq b \), the Seiffert mean \( T(a,b) \) and contraharmonic mean \( C(a,b) \) are defined by

\[
T(a,b) = \frac{a - b}{2 \arctan((a - b)/(a + b))}, \tag{1.1}
\]

\[
C(a,b) = \frac{a^2 + b^2}{a + b}, \tag{1.2}
\]

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities and properties for these means can be found in the literature [1–12].

Let \( A(a,b) = (a + b)/2 \), \( G(a,b) = \sqrt{ab} \), \( S(a,b) = \sqrt{(a^2 + b^2)/2} \), and let \( M_p(a,b) = ((a^p + b^p)/2)^{1/p} \ (p \neq 0) \) and \( M_0(a,b) = \sqrt{ab} \) be the arithmetic, geometric, square root, and \( p \)th power means of two positive numbers \( a \) and \( b \), respectively. Then it is well known that
$M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and the inequalities

$$G(a,b) = M_0(a,b) < A(a,b) = M_1(a,b) < S(a,b) = M_2(a,b) < C(a,b) \quad \text{(1.3)}$$

hold for all $a,b > 0$ with $a \neq b$.

Seiffert [12] proved that the double inequality

$$A(a,b) = M_1(a,b) < T(a,b) < M_2(a,b) = S(a,b) \quad \text{(1.4)}$$

holds for all $a,b > 0$ with $a \neq b$.

Hasto [13] proved that the function $T(1,x)/M_p(1,x)$ is increasing in $(0,\infty)$ if $p \leq 1$.

In [14], the authors found the greatest value $p$ and the least value $q$ such that the double inequality $H_p(a,b) < T(a,b) < H_q(a,b)$ holds for all $a,b > 0$ with $a \neq b$. Here, $H_k(a,b) = ((a^k + (ab)^k/2 + b^k)/3)^{1/k}$ $(k \neq 0)$, and $H_0(a,b) = \sqrt{ab}$ is the $k$th power-type Heron mean of $a$ and $b$.

Wang et al. [15] answered the question: what are the best possible parameters $\alpha$ and $\beta$ such that the double inequality $L_1(a,b) < T(a,b) < L_p(a,b)$ holds for all $a,b > 0$ with $a \neq b$, where $L_r(a,b) = (a^{r+1} + b^{r+1})/(a^r + b^r)$ is the $r$th Lehmer mean of $a$ and $b$.

In [16, 17], the authors proved that the inequalities

$$a_1T(a,b) + (1 - a_1)G(a,b) < A(a,b) < \beta_1T(a,b) + (1 - \beta_1)G(a,b),$$
$$a_2S(a,b) + (1 - a_2)A(a,b) < T(a,b) < \beta_2S(a,b) + (1 - \beta_2)A(a,b),$$

$$S^{\alpha_1}(a,b)A^{1-\alpha_1}(a,b) < T(a,b) < S^{\beta_3}(a,b)A^{1-\beta_3}(a,b) \quad \text{(1.5)}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $a_1 \leq 3/5$, $\beta_1 \geq \pi/4$, $a_2 \leq (4 - \pi)/[(\sqrt{2} - 1)\pi]$, $\beta_2 \geq 2/3$, $\alpha_3 \leq 2/3$ and $\beta_3 \geq 4 - 2\log \pi/\log 2$.

For fixed $a,b > 0$ with $a \neq b$, let $x \in [1/2,1]$ and

$$J(x) = C(xa + (1-x)b,xb + (1-x)a). \quad \text{(1.6)}$$

Then it is not difficult to verify that $J(x)$ is continuous and strictly increasing in $[1/2,1]$. Note that $J(1/2) = A(a,b) < T(a,b)$ and $J(1) = C(a,b) > T(a,b)$. Therefore, it is natural to ask what are the greatest value $\alpha$ and the least value $\beta$ in $(1/2,1)$ such that the double inequality

$$C(aa + (1 - a)b,ab + (1 - a)a) < T(a,b) < C(\beta a + (1 - \beta)b,\beta b + (1 - \beta)a) \quad \text{(1.7)}$$

holds for all $a,b > 0$ with $a \neq b$. The main purpose of this paper is to answer this question. Our main result is the following Theorem 1.1.

**Theorem 1.1.** If $\alpha, \beta \in (1/2,1)$, then the double inequality

$$C(aa + (1 - a)b,ab + (1 - a)a) < T(a,b) < C(\beta a + (1 - \beta)b,\beta b + (1 - \beta)a) \quad \text{(1.8)}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 + \sqrt{4/\pi - 1})/2$ and $\beta \geq (3 + \sqrt{3})/6$. 

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let \( \lambda = (1 + \sqrt{3}/\pi - 1)/2 \) and \( \mu = (3 + \sqrt{3})/6 \). We first prove that the inequalities

\[
T(a, b) > C(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a),
\]

\[
T(a, b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a)
\]

hold for all \( a, b > 0 \) with \( a \neq b \).

From (1.1) and (1.2) we clearly see that both \( T(a, b) \) and \( C(a, b) \) are symmetric and homogenous of degree 1. Without loss of generality, we assume that \( a > b \). Let \( t = a/b > 1 \) and \( p \in (1/2, 1) \), then from (1.1) and (1.2) one has

\[
C(pa + (1 - p)b, pb + (1 - p)a) - T(a, b)
\]

\[
= b \frac{[pt + (1 - p)]^2 + [(1 - p)t + p]^2}{2(t + 1) \arctan((t - 1)/(t + 1))}
\]

\[
\times \left\{ 2 \arctan\left(\frac{t - 1}{t + 1}\right) - \frac{t^2 - 1}{[pt + (1 - p)]^2 + [(1 - p)t + p]^2} \right\}
\]

Let

\[
f(t) = 2 \arctan\left(\frac{t - 1}{t + 1}\right) - \frac{t^2 - 1}{[pt + (1 - p)]^2 + [(1 - p)t + p]^2}.
\]

Then simple computations lead to

\[
f(1) = 0,
\]

\[
\lim_{t \to +\infty} f(t) = \frac{\pi}{2} - \frac{1}{p^2 + (1 - p)^2},
\]

\[
f'(t) = \frac{2f_1(t)}{\left\{ [pt + (1 - p)]^2 + [(1 - p)t + p]^2 \right\}^2 (t^2 + 1)},
\]

where

\[
f_1(t) = (4p^4 - 8p^3 + 10p^2 - 6p + 1)t^4 - 2(2p - 1)^2(2p^2 - 2p + 1)t^3
\]

\[
+ 2(12p^4 - 24p^3 + 18p^2 - 6p + 1)t^2
\]

\[
- 2(2p - 1)^2(2p^2 - 2p + 1)t + 4p^4 - 8p^3 + 10p^2 - 6p + 1,
\]

\[
f_1(1) = 0.
\]
Let \( f_2(t) = f'_1(t)/2, f_3(t) = f'_2(t)/2, f_4(t) = f'_3(t)/3 \). Then from (2.8) we get

\[
\begin{align*}
    f_2(t) &= 2(4p^4 - 8p^3 + 10p^2 - 6p + 1)t^3 - 3(2p - 1)^2(2p^2 - 2p + 1)t^2 \\
    &\quad + 12p^4 - 24p^3 + 18p^2 - 6p + 1, \\
    f_2(1) &= 0, \\
    f_3(t) &= 3(4p^4 - 8p^3 + 10p^2 - 6p + 1)t^2 - 3(2p - 1)^2(2p^2 - 2p + 1)t \\
    &\quad + 12p^4 - 24p^3 + 18p^2 - 6p + 1, \\
    f_3(1) &= 6p^2 - 6p + 1, \\
    f_4(t) &= 2(4p^4 - 8p^3 + 10p^2 - 6p + 1)t - (2p - 1)^2(2p^2 - 2p + 1), \\
    f_4(1) &= 6p^2 - 6p + 1.
\end{align*}
\]

We divide the proof into two cases.

Case 1 \((p = \lambda = (1 + \sqrt{4/\pi - 1})/2)\). Then (2.6), (2.13), and (2.15) lead to

\[
\begin{align*}
    \lim_{t \to +\infty} f(t) &= 0, \\
    f_3(1) &= -\frac{2(\pi - 3)}{\pi} < 0, \\
    f_4(1) &= -\frac{2(\pi - 3)}{\pi} < 0.
\end{align*}
\]

Note that

\[
4p^4 - 8p^3 + 10p^2 - 6p + 1 = \frac{4 + 2\pi - \pi^2}{\pi^2} > 0.
\]

It follows from (2.8), (2.10), (2.12), (2.14), and (2.19) that

\[
\begin{align*}
    \lim_{t \to +\infty} f_1(t) &= +\infty, \\
    \lim_{t \to +\infty} f_2(t) &= +\infty, \\
    \lim_{t \to +\infty} f_3(t) &= +\infty, \\
    \lim_{t \to +\infty} f_4(t) &= +\infty.
\end{align*}
\]

From (2.14) and inequality (2.19), we clearly see that \( f_4(t) \) is strictly increasing in \([1, +\infty)\). Then (2.18) and (2.23) lead to the conclusion that there exists \( t_0 > 1 \) such that \( f_4(t) < 0 \) for \( t \in [1, t_0) \) and \( f_4(t) > 0 \) for \( t \in (t_0, +\infty) \). Hence, \( f_3(t) \) is strictly decreasing in \([1, t_0]\) and strictly increasing in \([t_0, +\infty)\).
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It follows from (2.17) and (2.22) together with the piecewise monotonicity of $f_3(t)$ that there exists $t_1 > t_0 > 1$ such that $f_2(t)$ is strictly decreasing in $[1, t_1]$ and strictly increasing in $[t_1, +\infty)$.

From (2.11) and (2.21) together with the piecewise monotonicity of $f_2(t)$, we conclude that there exists $t_2 > t_1 > 1$ such that $f_1(t)$ is strictly decreasing in $[1, t_2]$ and strictly increasing in $[t_2, +\infty)$.

Equations (2.7), (2.9), and (2.20) together with the piecewise monotonicity of $f_1(t)$ imply that there exists $t_3 > t_2 > 1$ such that $f(t)$ is strictly decreasing in $[1, t_3]$ and strictly increasing in $[t_3, +\infty)$.

Therefore, inequality (2.1) follows from (2.3)–(2.5) and (2.16) together with the piecewise monotonicity of $f(t)$.

**Case 2** ($p = \mu = (3 + \sqrt{3})/6$). Then (2.8) leads to

$$f_1(t) = \frac{(t - 1)^4}{9} > 0 \quad (2.24)$$

for $t > 1$.

Inequality (2.24) and (2.7) imply that $f(t)$ is strictly increasing in $[1, +\infty)$. Therefore, inequality (2.2) follows from (2.3)–(2.5) together with the monotonicity of $f(t)$.

From inequalities (2.1) and (2.2) together with the monotonicity of $J(x) = C(xa + (1 - x)b, xb + (1 - x)a)$ in $[1/2, 1]$, we know that inequality (1.8) holds for all $a \leq (1 + \sqrt{4/\pi - 1})/2$, $\beta \geq (3 + \sqrt{3})/6$, and all $a, b > 0$ with $a \neq b$.

Next, we prove that $\lambda = (1 + \sqrt{4/\pi - 1})/2$ is the best possible parameter in $[1/2, 1]$ such that inequality (2.1) holds for all $a, b > 0$ with $a \neq b$.

For any $1 > p > \lambda = (1 + \sqrt{4/\pi - 1})/2$, from (2.6) one has

$$\lim_{t \to +\infty} f(t) = \frac{\pi}{2} - \frac{1}{p^2 + (1 - p)^2} > 0. \quad (2.25)$$

Equations (2.3) and (2.4) together with inequality (2.25) imply that for any $1 > p > \lambda = (1 + \sqrt{4/\pi - 1})/2$ there exists $T_0 = T_0(p) > 1$ such that

$$C(pa + (1 - p)b, pb + (1 - p)a) > T(a, b) \quad (2.26)$$

for $a/b \in (T_0, +\infty)$.

Finally, we prove that $\mu = (3 + \sqrt{3})/6$ is the best possible parameter such that inequality (2.2) holds for all $a, b > 0$ with $a \neq b$.

For any $1/2 < p < \mu = (3 + \sqrt{3})/6$, from (2.13) one has

$$f_3(1) = 6p^2 - 6p + 1 < 0. \quad (2.27)$$

From inequality (2.27) and the continuity of $f_3(t)$, we know that there exists $\delta = \delta(p) > 0$ such that

$$f_3(t) < 0 \quad (2.28)$$

for $t \in (1, 1 + \delta)$.
Equations (2.3)–(2.5), (2.7), (2.9), and (2.11) together with inequality (2.28) imply that for any $1/2 < p < \mu = (3 + \sqrt{3})/6$ there exists $\delta = \delta(p) > 0$ such that

$$T(a, b) > C(pa + (1 - p)b, pb + (1 - p)a)$$

(2.29)

for $a/b \in (1, 1 + \delta)$.

\[ \square \]

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**References**


