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Research Article

On Some Lacunary Almost Convergent Double Sequence Spaces and Banach Limits

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The object of this paper is to introduce some new sequence spaces related with the concept of lacunary strong almost convergence for double sequences and also to characterize these spaces through sublinear functionals that both dominate and generate Banach limits and to establish some inclusion relations.

1. Introduction and Preliminaries

Let w_2 be the set of all real or complex double sequences. We mean the convergence in the Pringsheim sense, that is, a double sequence $x = (x_{i,j})_{i,j=0}^{\infty}$ has a Pringsheim limit λ (denoted by $P - \lim x = \lambda$) provided that given $\varepsilon > 0$ and there exists $N \in \mathbb{N}$ such that $|x_{i,j} - \lambda| < \varepsilon$ whenever $i, j \geq N$ [1]. We denote by c_2, the space of P-convergent sequences. A double sequence $x = (x_{i,j})$ is bounded if $\|x\| = \sup_{i,j\geq 0}|x_{i,j}| < \infty$. Let l_2^{∞} and c_2^{∞} be the set of all real or complex bounded double sequences and the set of bounded and convergent double sequences, respectively. Moricz and Rhoades [2] defined the almost convergence of double sequences that $x = (x_{i,j})$ is said to be almost convergent to a number λ if

$$\lim_{p,q\to\infty} \sup_{m,n\geq 0} \left| \frac{1}{(p+1)(q+1)} \sum_{i=m}^{m+p} \sum_{j=n}^{n+q} x_{i,j} - \lambda \right| = 0, \tag{1.1}$$

that is, the average value of $(x_{i,j})$ taken over any rectangle

$$D = \{(i,j) : m \leq i \leq m+p,\ n \leq j \leq n+q\}, \tag{1.2}$$

tends to λ as both p and q tend to ∞ and this convergence is uniform in m and n. We denote the space of almost convergent double sequences by f_2, as

$$f_2 = \left\{ x = (x_{i,j}) : \lim_{k,l\to\infty} \left| t_{klpq}(x) - \lambda \right| = 0, \text{ uniformly in } p,q \right\}, \tag{1.3}$$

where

$$t_{klpq}(x) = \frac{1}{(k+1)(l+1)} \sum_{i=p}^{p+k} \sum_{j=q}^{q+l} x_{i,j}. \tag{1.4}$$

The notion of almost convergence for single sequences was introduced by Lorentz [3] and for double sequences by Moricz and Rhoades [2] and some further studies are in [4–14].

A double sequence x is called strongly almost convergent to a number λ if

$$\lim_{k,l\to\infty} \frac{1}{(k+1)(l+1)} \sum_{i=p}^{p+k} \sum_{j=q}^{q+l} \left| x_{i,j} - \lambda e \right| = 0, \quad \text{uniformly in } p,q. \tag{1.5}$$

By $[f_2]$, we denote the space of all strongly almost convergent double sequences. It is easy to see that the inclusions $c_2^\infty \subset [f_2] \subset f_2 \subset l_2^\infty$ strictly hold. As in the case of single sequences, every almost convergent double sequence is bounded. But a convergent double sequence need not be bounded. Thus, a convergent double sequence need not be almost convergent. However every bounded convergent double sequence is almost convergent.

The notion of strong almost convergence for single sequences has been introduced by Maddox [15, 16] and for double sequences by Başarir [17].

A linear functional L on l_2^∞ is said to be Banach limit if it has the following properties [7],

(1) $L(x) \geq 0$ if $x \geq 0$ (i.e., $x_{i,j} \geq 0$ for all i, j),

(2) $L(e) = 1$, where $e = (e_{i,j})$ with $e_{i,j} = 1$ for all i, j and

(3) $L(x) = L(S_{10}x) = L(S_{01}x) = L(S_{11}x)$ where the shift operators $S_{10}x, S_{01}x, S_{11}x$ are defined by $S_{10}x = (x_{i+1,j}), S_{01}x = (x_{i,j+1}), S_{11}x = (x_{i+1,j+1})$.

Let B_2 be the set of all Banach limits on l_2^∞. A double sequence $x = (x_{i,j})$ is said to be almost convergent to a number λ if $L(x) = \lambda$ for all $L \in B_2$. If φ is any sublinear functional on l_2^∞, then we write $\{l_2^\infty, \varphi\}$ to denote the set of all linear functionals F on l_2^∞, such that $F \leq \varphi$, that is, $F(x) \leq \varphi(x)$, $\forall x \in l_2^\infty$. A sublinear functional φ is said to generate Banach limits if $F \in \{l_2^\infty, \varphi\}$ implies that F is a Banach limit; φ is said to dominate Banach limits if F is a Banach limit implies that $F \in \{l_2^\infty, \varphi\}$. Then if φ both generates and dominates Banach limits, then $\{l_2^\infty, \varphi\}$ is the set of all Banach limits.

Using the notations for single sequences, we present the notations for double-lacunary sequences that can be seen in [10]. The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called a double-lacunary if there exist two increasing sequences of nonnegative integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $l_0 = 0$, $h_s = l_s - l_{s-1} \to \infty$ as $s \to \infty$. Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r h_s$, $\theta_{r,s}$ is determined by $I_{r,s} = \{(i,j) : k_{r-1} < i \leq k_r \text{ and } l_{s-1} < j \leq l_s\}$. Also $\overline{h}_{r,s} = k_r l_s - k_{r-1} l_{s-1}$ and $\theta = \overline{\theta}_{r,s}$ is determined by $\overline{I}_{r,s} = \{(i,j) : k_{r-1} < i \leq k_r \cup l_{s-1} < j \leq l_s\}/(I^1 \cup I^2)$, where

$I^1 = \{(i,j) : k_{r-1} < i \leq k_r \text{ and } l_s < j < \infty.\}$ and $I^2 = \{(i,j) : l_{s-1} < j \leq l_s \text{ and } k_r < i < \infty.\}$ with $q_r = k_r/k_{r-1}$, $q_s = l_s/l_{s-1}$ and $q_{r,s} = q_r q_s$.

Das and Mishra [18] introduced the space of lacunary almost convergent sequences by combining the space of lacunary convergent sequences and the space of almost convergent sequences. Savaş and Patterson [10] extended the notions of lacunary almost convergence and lacunary strongly almost convergence to double-lacunary P-convergence and double-lacunary strongly almost P-convergence. They also established multidimensional analogues of Das and Patel's results.

We will use the following definition which may be called convergence in Pringsheim's sense with a bound:

$$(x_{i,j} - L) = O(1), \quad (i,j \longrightarrow \infty), \tag{1.6}$$

and also we will use the following definition which may be called convergence in Pringsheim's sense as follows:

$$(x_{i,j} - L) = o(1), \quad (i,j \longrightarrow \infty). \tag{1.7}$$

The following sequence spaces were introduced and examined by Başarir [19]:

$$\begin{aligned} w_\theta &= \left\{ x : \lim_r \sup_i \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(x - s) = 0, \text{ for some } s \right\}, \\ [w]_\theta &= \left\{ x : \lim_r \sup_i \frac{1}{h_r} \sum_{k \in I_r} |t_{ki}(x - s)| = 0, \text{ for some } s \right\}, \\ [w_1]_\theta &= \left\{ x : \lim_r \sup_i \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(|x - s|) = 0, \text{ for some } s \right\}, \end{aligned} \tag{1.8}$$

with respect to sublinear functionals on l_∞ (the set of all real or complex bounded single sequences) by

$$\begin{aligned} \phi_\theta(x) &= \overline{\lim_r} \sup_i \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(x), \\ \psi_\theta(x) &= \overline{\lim_r} \sup_i \frac{1}{h_r} \sum_{k \in I_r} |t_{ki}(x)|, \\ \zeta_\theta(x) &= \overline{\lim_r} \sup_i \frac{1}{h_r} \sum_{k \in I_r} t_{ki}(|x|), \end{aligned} \tag{1.9}$$

where $t_{ki}(x) = (1/k) \sum_{j=i}^{i+k-1} x_j$ and $|x| = (|x_j|)_{j=1}^{\infty}$.

It can be easily seen that each of the above functionals are finite, well defined, and sublinear on l_∞. There is a very close connection among these sequence spaces with the sublinear functionals which were given by Başarir [19]. Recently Mursaleen and

Mohiuddine [7] generalized the sequence spaces which were studied by Das and Sahoo [20] for single sequences, to the double sequences as follows:

$$\begin{aligned}
w_2 &= \left\{ x = (x_{i,j}) : \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m} \sum_{l=0}^{n} t_{klpq}(x - \lambda e) \longrightarrow 0, \right. \\
&\qquad \left. \text{as } m,n \longrightarrow \infty, \text{ uniformly in } p,q, \text{ for some} \right\}, \\
[w_2] &= \left\{ x = (x_{i,j}) : \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m} \sum_{l=0}^{n} \left| t_{klpq}(x - \lambda e) \right| \longrightarrow 0, \right. \\
&\qquad \left. \text{as } m,n \longrightarrow \infty, \text{ uniformly in } p,q, \text{ for some } \lambda \right\}, \\
[w]_2 &= \left\{ x = (x_{i,j}) : \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m} \sum_{l=0}^{n} t_{klpq}(|x - \lambda e|) \longrightarrow 0, \right. \\
&\qquad \left. \text{as } m,n \longrightarrow \infty, \text{ uniformly in } p,q, \text{ for some } \lambda \right\}
\end{aligned} \tag{1.10}$$

by using (1.4).

The object of the present paper is to determine some new sublinear functionals involving double-lacunary sequence that both dominates and generates Banach limits. We also extend the sequence spaces which were introduced for single sequences by Başarir [19] to the double sequences with respect to these sublinear functionals. Furthermore, we present some inclusion relations with these new sequence spaces between the sequence spaces which were introduced by Mursaleen and Mohiuddine [7], earlier.

2. Sublinear Functionals and Double-Lacunary Sequence Spaces

In this section, we introduce the following sequence spaces:

$$\begin{aligned}
w_\theta^2 &= \left\{ x = (x_{i,j}) : P - \lim_{r,s\to\infty} \sup_{p,q} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in \overline{I}_{r,s}} t_{klpq}(x - \lambda e) = 0, \text{ for some } \lambda \right\}, \\
\left[w_\theta^2\right] &= \left\{ x = (x_{i,j}) : P - \lim_{r,s\to\infty} \sup_{p,q} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in \overline{I}_{r,s}} \left| t_{klpq}(x - \lambda e) \right| = 0, \text{ for some } \lambda \right\}, \\
[w]_\theta^2 &= \left\{ x = (x_{i,j}) : P - \lim_{r,s\to\infty} \sup_{p,q} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in \overline{I}_{r,s}} t_{klpq}(|x - \lambda e|) = 0, \text{ for some } \lambda \right\}, \\
W(\theta,2) &= \left\{ x = (x_{i,j}) : P - \lim_{r,s\to\infty} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in \overline{I}_{r,s}} t_{kl00}(x - \lambda e) = 0, \text{ for some } \lambda \right\},
\end{aligned}$$

$$[W(\theta,2)] = \left\{ x = (x_{i,j}) : P - \lim_{r,s\to\infty} \sup_{p,q} \frac{1}{h_{r,s}} \sum_{(k,l)\in \bar{I}_{r,s}} |t_{kl00}(x-\lambda e)| = 0, \text{ for some } \lambda \right\},$$

$$W[\theta,2] = \left\{ x = (x_{i,j}) : P - \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{(k,l)\in \bar{I}_{r,s}} t_{kl00}(|x-\lambda e|) = 0, \text{ for some } \lambda \right\}. \tag{2.1}$$

It may be noted that almost convergent double sequences are necessarily bounded but the sequence spaces w_θ^2 and $[w_\theta^2]$ may contain unbounded sequences. Now we define the following functionals on l_2^∞ for a double-lacunary sequence $\theta = (\bar{\theta}_{r,s})$ by,

$$\begin{aligned}
\phi_\theta^2(x) &= \overline{\lim_{r,s\to\infty}} \sup_{p,q} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l)\in \bar{I}_{r,s}} t_{klpq}(x), \\
\psi_\theta^2(x) &= \overline{\lim_{r,s\to\infty}} \sup_{p,q} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l)\in \bar{I}_{r,s}} \left|t_{klpq}(x)\right|, \\
\varphi_\theta^2(x) &= \overline{\lim_{r,s\to\infty}} \sup_{p,q} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l)\in \bar{I}_{r,s}} t_{klpq}(|x|), \\
\zeta_2(x) &= \overline{\lim_{k,l\to\infty}} \sup_{p,q} t_{klpq}(x), \\
\eta_2(x) &= \overline{\lim_{k,l\to\infty}} \sup_{p,q} t_{klpq}(|x|).
\end{aligned} \tag{2.2}$$

It is easy to see that each of the above functionals are finite, well defined, and sublinear on l_2^∞.

Throughout the paper we will write $\lim_{r,s}$ for $P-\lim_{r,s\to\infty}$ and by this notation we shall mean the convergence in the Pringsheim sense. In the following theorem, we demonstrate that $\{l_2^\infty, \phi_\theta^2\}$ is the set of all Banach limits on l_2^∞ and characterize the space $w_\theta^2 \cap l_2^\infty$ in terms of the sublinear functional ϕ_θ^2.

Theorem 2.1. *One has the following.*

(1) The sublinear functional ϕ_θ^2 both dominates and generates Banach limits, that is, $\phi_\theta^2(x) = \zeta_2(x)$, for all $x = (x_{i,j}) \in l_2^\infty$.

(2)

$$\begin{aligned}
f_2 &= \left\{ x = (x_{i,j}) \in l_2^\infty : \phi_\theta^2(x) = -\phi_\theta^2(-x) \right\} \\
&= \left\{ x = (x_{i,j}) \in l_2^\infty : \frac{1}{\bar{h}_{r,s}} \sum_{(k,l)\in \bar{I}_{r,s}} t_{klpq}(x) \longrightarrow \lambda, \text{ as } r,s \longrightarrow \infty, \text{ uniformly in } p,q. \right\} \\
&= w_\theta^2 \cap l_2^\infty.
\end{aligned} \tag{2.3}$$

Proof. (1) From the definition of ζ_2, for given $\varepsilon > 0$ there exist k_0, l_0 such that

$$t_{klpq}(x) < \zeta_2(x) + \varepsilon, \tag{2.4}$$

for $k \geq k_0,\ l \geq l_0$ and for all p, q. This implies that

$$\phi_\theta^2(x) < \zeta_2(x) + \varepsilon, \tag{2.5}$$

for all $x = (x_{i,j}) \in l_2^\infty$. Since ε is arbitrary, so that $\phi_\theta^2(x) \leq \zeta_2(x)$, for all $x = (x_{i,j}) \in l_2^\infty$ and hence

$$\left\{l_2^\infty, \phi_\theta^2\right\} \subset \left\{l_2^\infty, \zeta_2\right\} = B_2, \tag{2.6}$$

that is, ϕ_θ^2 generates Banach limits.

Conversely, suppose that $L \in B_2$. As L is the shift invariant, that is, $L(S_{11}x) = L(x) = L(S_{10}x) = L(S_{01}x)$ and using the properties of $L \in B_2$, we obtain

$$\begin{aligned} L(x) &= L\left(\frac{1}{(k+1)(l+1)}\sum_{i=p}^{p+k}\sum_{j=q}^{q+l} x_{i,j}\right) = L\left(t_{klpq}(x)\right) \\ &= \frac{1}{\overline{h}_{r,s}} L\left(\sum_{(k,l)\in\overline{I}_{r,s}} t_{klpq}(x)\right) \leq \sup_{p,q} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} t_{klpq}(x). \end{aligned} \tag{2.7}$$

It follows from the definition of ϕ_θ^2, that for given $\varepsilon > 0$ there exist r_0, s_0 such that

$$\frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} t_{klpq}(x) < \phi_\theta^2(x) + \varepsilon, \tag{2.8}$$

for $r \geq r_0,\ s \geq s_0$ and for all p, q. Hence by (2.8) and properties (1) and (2) of Banach limits, we have

$$L\left(\frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} t_{klpq}(x)\right) < L\left(\left(\phi_\theta^2(x) + \varepsilon\right)e\right) = \phi_\theta^2(x) + \varepsilon, \tag{2.9}$$

for $r \geq r_0,\ s \geq s_0$ and for all p, q; where $e = (e_{i,j})$ with $e_{i,j} = 1$ for all $i,\ j$. Since ε is arbitrary, it follows from (2.7) and (2.9) that

$$L(x) \leq \phi_\theta^2(x), \quad \forall x = \left(x_{i,j}\right) \in l_2^\infty. \tag{2.10}$$

Hence

$$B_2 \subset \left\{l_2^\infty, \phi_\theta^2\right\}. \tag{2.11}$$

That is, ϕ_θ^2 dominates Banach limits. Combining (2.6) and (2.11), we get

$$\{l_2^\infty, \zeta_2\} = \left\{l_2^\infty, \phi_\theta^2\right\}, \tag{2.12}$$

this implies that ϕ_θ^2 dominates and generates Banach limits and $\phi_\theta^2(x) = \zeta_2(x)$ for all $x \in l_2^\infty$.

(2) As a consequence of Hahn-Banach theorem, $\{l_2^\infty, \phi_\theta^2\}$ is non empty and a linear functional $F \in \{l_2^\infty, \phi_\theta^2\}$ is not necessarily uniquely defined at any particular value of x. This is evident in the manner the linear functionals are constructed. But in order that all the functionals $\{l_2^\infty, \phi_\theta^2\}$ coincide at $x = (x_{i,j})$, it is necessary and sufficient that

$$\phi_\theta^2(x) = -\phi_\theta^2(-x), \tag{2.13}$$

we have

$$\limsup_{r,s} \sup_{p,q} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} t_{klpq}(x) = \liminf_{r,s} \inf_{p,q} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} t_{klpq}(x). \tag{2.14}$$

But (2.14) holds if and only if

$$\frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} t_{klpq}(x) \longrightarrow \lambda, \quad \text{as } r,s \longrightarrow \infty, \text{ uniformly in } p,q. \tag{2.15}$$

Hence, $x = (x_{i,j}) \in w_\theta^2 \cap l_2^\infty$. But (2.13) is equivalent to $\zeta_2(x) = -\zeta_2(-x)$, this holds if and only if $x = (x_{i,j}) \in f_2$. This completes the proof of the theorem. □

If $F(x - \lambda e) = 0$ for all $F \in \{l_2^\infty, \psi_\theta^2\}$, then we say that $x = (x_{i,j})$ is ψ_θ^2-convergent to λ. Similarly we define the φ_θ^2-convergent sequences. In the following theorem we characterize the spaces $[w_\theta^2] \cap l_2^\infty$ and $[w]_\theta^2 \cap l_2^\infty$ in terms of the sublinear functionals.

Theorem 2.2. *One has the following:*

(1) $[w_\theta^2] \cap l_2^\infty = \{x = (x_{i,j}) : \psi_\theta^2(x - \lambda e) = 0, \textit{for some } \lambda\} = \{x = (x_{i,j}) : F(x - \lambda e) = 0, \textit{for all } F \in \{l_2^\infty, \psi_\theta^2\}, \textit{for some } \lambda\}$

(2) $[w]_\theta^2 \cap l_2^\infty = \{x = (x_{i,j}) : \varphi_\theta^2(x - \lambda e) = 0, \textit{for some } \lambda\} = \{x = (x_{i,j}) : F(x - \lambda e) = 0, \textit{for all } F \in \{l_2^\infty, \varphi_\theta^2\}, \textit{for some } \lambda\}$.

Proof. (1) It can be easily verified that $x = (x_{i,j}) \in [w_\theta^2] \cap l_2^\infty$ if and only if

$$\psi_\theta^2(x - \lambda e) = -\psi_\theta^2(\lambda e - x). \tag{2.16}$$

Since $\psi_\theta^2(x) = -\psi_\theta^2(-x)$ then (2.16) reduces to

$$\psi_\theta^2(x - \lambda e) = 0. \tag{2.17}$$

Now if $F \in \{l_2^\infty, \psi_\theta^2\}$ then from (2.17) and linearity of F, we have

$$F(x - \lambda e) = 0. \tag{2.18}$$

Conversely, suppose that $F(x - \lambda e) = 0$ for all $F \in \{l_2^\infty, \psi_\theta^2\}$ and hence by Hahn-Banach theorem, there exists $F_0 \in \{l_2^\infty, \psi_\theta^2\}$ such that $F_0(x) = \psi_\theta^2(x)$. Hence

$$0 = F_0(x - \lambda e) = \psi_\theta^2(x - \lambda e). \tag{2.19}$$

(2) The proof is similar to the proof of (1), above. □

3. Inclusion Relations

We establish here some inclusion relations between the sequence spaces defined in Section 2.

Theorem 3.1. *We have the following proper inclusions and the limit is preserved in each case.*

(1) $[f_2] \subset [w]_\theta^2 \subset [w_\theta^2] \subset w_\theta^2 \subset W(\theta, 2)$.

(2) $[w]_\theta^2 \subset [w_\theta^2] \subset [W(\theta, 2)] \subset W(\theta, 2)$.

(3) $[w]_\theta^2 \subset W[\theta, 2] \subset [W(\theta, 2)] \subset W(\theta, 2)$.

Proof. (1) Let $x \in [f_2]$ with $[f_2] - \lim x = \lambda$. Then

$$t_{klpq}(|x - \lambda e|) \longrightarrow 0, \quad \text{as } k, l \longrightarrow \infty, \text{ uniformly in } p, q. \tag{3.1}$$

This implies that

$$\frac{1}{\overline{h}_{r,s}} \sum_{(k,l) \in \overline{I}_{r,s}} t_{klpq}(|x - \lambda e|) \longrightarrow 0, \quad \text{as } k, l \longrightarrow \infty, \text{ uniformly in } p,\ q. \tag{3.2}$$

This proves that $x \in [w]_\theta^2$ and $[f_2] - \lim x = [w]_\theta^2 - \lim x = \lambda$. Since

$$\begin{aligned} \frac{1}{\overline{h}_{r,s}} \left| \sum_{(k,l) \in \overline{I}_{r,s}} t_{klpq}(x - \lambda e) \right| &\leq \frac{1}{\overline{h}_{r,s}} \sum_{(k,l) \in \overline{I}_{r,s}} \left| t_{klpq}(x - \lambda e) \right| \\ &\leq \frac{1}{\overline{h}_{r,s}} \sum_{(k,l) \in \overline{I}_{r,s}} t_{klpq}(|x - \lambda e|), \end{aligned} \tag{3.3}$$

this implies that $[w]_\theta^2 \subset [w_\theta^2] \subset w_\theta^2$ and $[w]_\theta^2 - \lim x = [w_\theta^2] - \lim x = w_\theta^2 - \lim x = \lambda$. Since

$$\frac{1}{\overline{h}_{r,s}} \sum_{(k,l) \in \overline{I}_{r,s}} t_{klpq}(x - \lambda e) \tag{3.4}$$

converges uniformly in p, q as $r, s \to \infty$, implies the convergence for $p = 0 = q$. It follows that $w_\theta^2 \subset W(\theta, 2)$ and $w_\theta^2 - \lim x = W(\theta, 2) - \lim x = \lambda$. This completes the proof of (1).

It is easy to see the proof of (2) and (3). So we omit them. □

Theorem 3.2. *One has the following proper inclusions;*

$$[f_2] \subset \left([w]_\theta^2 \cap l_2^\infty\right) \subset \left(\left[w_\theta^2\right] \cap l_2^\infty\right) \subset f_2. \tag{3.5}$$

Proof. The proof of the theorem is similar as in [7, Theorem 4.2]. So we omit it. □

Prior to giving Lemmas 3.3 and 3.5, we need the following notations used in [10]:

$$\begin{gathered}
I_C^1 = \{(i,j) : q \le j \le q+n,\ p+m < i < \infty\}, \qquad I_C^2 = \{(i,j) : q+n < j < \infty,\ p \le i \le p+m\}, \\
C_{p,q}^{m,n} = \left\{\begin{matrix}(i,j) : p \le i \le p+m \text{ or} \\ q \le j \le q+n\end{matrix}\right\} \setminus \left(I_C^1 \cup I_C^2\right), \\
I_D^1 = \{(i,j) : q + (y+1)h_s - 1 < j < \infty,\ p + xh_r \le i \le p + (x+1)h_r - 1\}, \\
I_D^2 = \{(i,j) : q + yh_s \le j \le q + (y+1)h_s - 1,\ p + (x+1)h_r - 1 < i < \infty\}, \\
D_{p,q}^{x,y} = \left\{\begin{matrix}(i,j) : p + xh_r \le i \le p + (x+1)h_r - 1 \text{ or} \\ q + yh_s \le j \le q + (y+1)h_s - 1\end{matrix}\right\} \setminus \left(I_D^1 \cup I_D^2\right).
\end{gathered} \tag{3.6}$$

Lemma 3.3. *Suppose $\varepsilon > 0$ there exist m_0, n_0, p_0, and q_0 such that*

$$\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l) \in C_{0,0}^{m,n}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j) \in C_{p,q}^{p+k,q+l}} \left|x_{i,j} - \lambda e\right| \right) < \varepsilon, \tag{3.7}$$

for $m \ge m_0$, $n \ge n_0$ and $p \ge p_0$, $q \ge q_0$. Then $x \in [w]_2$.

Proof. Let $\varepsilon > 0$ be given. Choose m_0^1, n_0^1, p_0 and q_0 such that

$$\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l) \in C_{0,0}^{m,n}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j) \in C_{p,q}^{p+k,q+l}} \left|x_{i,j} - \lambda e\right| \right) < \frac{\varepsilon}{6} \tag{3.8}$$

for $m \ge m_0^1$, $n \ge n_0^1$, $p \ge p_0$, $q \ge q_0$. We need only to show that given $\varepsilon > 0$ there exist m_0^2 and n_0^2 such that

$$\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l) \in C_{0,0}^{m,n}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j) \in C_{p,q}^{p+k,q+l}} \left|x_{i,j} - \lambda e\right| \right) < \varepsilon, \tag{3.9}$$

for $m \geq m_0^2$, $n \geq n_0^2$ and $0 \leq p \leq p_0$, $0 \leq q \leq q_0$. If we take $m_0 = \max\{m_0^1, m_0^2\}$ and $n_0 = \max\{n_0^1, n_0^2\}$, then (3.9) holds for $m \geq m_0$, $n \geq n_0$ and for all p and q, which gives the result. Once p_0 and q_0 have been chosen, they are fixed, so

$$\sum_{(k,l)\in C_{0,0}^{p_0,q_0}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p_0,q_0}} \left|x_{i,j} - \lambda e\right| \right) = M, \tag{3.10}$$

is finite. Now taking $0 \leq p \leq p_0$, $0 \leq q \leq q_0$ and $m \geq p_0$, $n \geq q_0$ we have from (3.8) and (3.10)

$$\begin{aligned}
&\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{m,n}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p+k,q+l}} \left|x_{i,j} - \lambda e\right| \right) \\
&\quad = \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{p_0,q_0}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p_0,q_0}} \left|x_{i,j} - \lambda e\right| \right) \\
&\qquad + \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{p_0,q_0}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p_0+1,q_0+1}^{p+k,q+l}} \left|x_{i,j} - \lambda e\right| \right) \\
&\qquad + \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{p_0+1,q_0+1}^{m,n}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p_0,q_0}} \left|x_{i,j} - \lambda e\right| \right) \\
&\qquad + \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(i,j)\in C_{p_0+1,q_0+1}^{m,n}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p_0+1,q_0+1}^{p+k,q+l}} \left|x_{i,j} - \lambda e\right| \right) \\
&\quad \leq \frac{M}{\left|C_{0,0}^{m,n}\right|} + 3 \cdot \frac{\varepsilon}{6} = \frac{M}{\left|C_{0,0}^{m,n}\right|} + \frac{\varepsilon}{2}.
\end{aligned} \tag{3.11}$$

Therefore taking m and n sufficiently large, we can make $M/|C_{0,0}^{m,n}| + \varepsilon/2 < \varepsilon$ which gives (3.9) and hence the result. □

Theorem 3.4. *We have* $[w]_\theta^2 = [w]_2$ *for every* $\overline{\theta}_{r,s}$.

Proof. Let $x \in [w]_\theta^2$; then given $\varepsilon > 0$ there exist r_0, s_0 and λ such that

$$\frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in C_{0,0}^{h_r-1,h_s-1}} t_{klpq}(|x - \lambda e|) < \varepsilon, \tag{3.12}$$

for $r \geq r_0$, $s \geq s_0$, $p = k_{r-1} + 1 + \alpha$ and $q = l_{s-1} + 1 + \beta$ where $\alpha, \beta \geq 0$. Let $m \geq h_r$ such that $m = \delta_1 h_r + \theta_1 - 1$ where δ_1 is an integer. Also let $n \geq h_s$ such that $n = \delta_2 h_s + \theta_2 - 1$ where δ_2 is an integer and $1 \leq \theta_1 \leq h_r$, $1 \leq \theta_2 \leq h_s$. Since $m \geq h_r$ for $\delta_1 \geq 1$ and $n \geq h_s$ for $\delta_2 \geq 1$ we have,

$$\begin{aligned}
&\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{m,n}} t_{klpq}(|x-\lambda e|) \\
&\quad \leq \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{(\delta_1+1)h_r-1,(\delta_2+1)h_s-1}} t_{klpq}(|x-\lambda e|) \\
&\quad = \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{x,y=0,0}^{\delta_1,\delta_2} \sum_{(k,l)\in D_{0,0}^{x,y}} t_{klpq}(|x-\lambda e|) \\
&\quad \leq \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{x,y=0,0}^{\delta_1,\delta_2} \left(\overline{h}_{r,s}\cdot\varepsilon\right) \leq \frac{(\delta_1+1)(\delta_2+1)}{\left|C_{0,0}^{m,n}\right|}\left(\overline{h}_{r,s}\cdot\varepsilon\right) = o(1),
\end{aligned} \tag{3.13}$$

which gives the result. Therefore by Lemma 3.3, $[w]_\theta^2 \subset [w]_2$. It is clear that $[w]_2 \subset [w]_\theta^2$ for every $\overline{\theta}_{r,s}$. This completes the proof. □

Lemma 3.5. *Suppose, for a given $\varepsilon > 0$, there exist m_0, n_0, p_0, and q_0 such that*

$$\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{m,n}} \left| \frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p+k,q+l}} (x_{i,j} - \lambda e) \right| < \varepsilon, \tag{3.14}$$

for all $m \geq m_0$, $n \geq n_0$ and $p \geq p_0$, $q \geq q_0$. Then $x \in [w_2]$.

Proof. Let $\varepsilon > 0$ be given and choose m_0, n_0, p_0 and q_0 such that

$$\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{m,n}} \left| \frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p+k,q+l}} (x_{i,j} - \lambda e) \right| < \frac{\varepsilon}{4} \tag{3.15}$$

for all $m \geq m_0$, $n \geq n_0$, $p \geq p_0$, and $q \geq q_0$. As in Lemma 3.3, it is enough to show that there exist m_0^1 and n_0^1 such that for $m \geq m_0^1$, $n \geq n_0^1$ implies

$$\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{m,n}} \left| \frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p+k,q+l}} (x_{i,j} - \lambda e) \right| < \varepsilon, \tag{3.16}$$

for all p and q with $0 \le p \le p_0$ and $0 \le q \le q_0$. Since p_0 and q_0 are fixed,

$$\sum_{(k,l)\in C_{0,0}^{p_0,q_0}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p_0,q_0}} \left|x_{i,j} - \lambda e\right| \right) = M. \tag{3.17}$$

Now, let $0 \le p \le p_0$, $0 \le q \le q_0$, and $m \ge p_0$, $n \ge q_0$ and consider the following;

$$\begin{aligned}
&\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{m,n}} \left| \frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p+k,q+l}} (x_{i,j} - \lambda e) \right| \\
&\quad \le \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{p_0,q_0}} \left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p_0,q_0}} \left|x_{i,j} - \lambda e\right| \right) \\
&\qquad + \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{p_0,q_0}} \left| \frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p_0+1,q_0+1}^{p+k,q+l}} (x_{i,j} - \lambda e) \right| \\
&\qquad + \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{p_0+1,q_0+1}^{m,n}} \left| \frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p,q}^{p+k,q+l}} (x_{i,j} - \lambda e) \right| \\
&\quad \le \frac{M}{\left|C_{0,0}^{m,n}\right|} + \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{p_0,q_0}} \left| \frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p_0+1,q_0+1}^{p+k,q+l}} (x_{i,j} - \lambda e) \right| \\
&\qquad + \frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{p_0+1,q_0+1}^{m,n}} \left| t_{klpq}(x - \lambda e) \right|.
\end{aligned} \tag{3.18}$$

Let $k - p_0 \ge m_0^1$, then $k + p - p_0 \ge m_0^1$ for $0 \le p \le p_0$. Also if we let $l - q_0 \ge n_0^1$, then $l + q - q_0 \ge n_0^1$ for $0 \le q \le q_0$. Therefore from (3.15)

$$\begin{aligned}
&\frac{1}{\left|C_{0,0}^{m,n}\right|} \sum_{(k,l)\in C_{0,0}^{p_0,q_0}} \left| \frac{1}{\left|C_{p,q}^{p+k,q+l}\right|} \sum_{(i,j)\in C_{p_0+1,q_0+1}^{p+k,q+l}} (x_{i,j} - \lambda e) \right| \\
&\quad < \frac{1}{\left|C_{0,0}^{p_0,q_0}\right|} \sum_{(k,l)\in C_{0,0}^{p_0,q_0}} \left| \frac{1}{\left|C_{p_0+1,q_0+1}^{p_0+(p+k-p_0),q_0+(q+l-q_0)}\right|} \sum_{(i,j)\in C_{p_0+1,q_0+1}^{p_0+(p+k-p_0),q_0+(q+l-q_0)}} (x_{i,j} - \lambda e) \right| \\
&\quad < \frac{\varepsilon}{4}.
\end{aligned} \tag{3.19}$$

From (3.18) and (3.19)

$$\frac{1}{\left|C_{0,0}^{m,n}\right|}\sum_{(k,l)\in C_{0,0}^{m,n}}\left|\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|}\sum_{(i,j)\in C_{p,q}^{p+k,q+l}}(x_{i,j}-\lambda e)\right|\leq\frac{M}{\left|C_{0,0}^{m,n}\right|}+2\frac{\varepsilon}{4}<\varepsilon, \tag{3.20}$$

for sufficiently large values of m and n. Hence the result. □

Theorem 3.6. *For every $\overline{\theta}_{r,s}$, One has $[w_\theta^2]\cap l_2^\infty=[w_2]\cap l_2^\infty$.*

Proof. Let $x\in[w_\theta^2]\cap l_2^\infty$. For $\varepsilon>0$, there exist r_0, s_0, p_0 and q_0 such that

$$\frac{1}{\overline{h}_{r,s}}\sum_{(k,l)\in C_{0,0}^{h_r-1,h_s-1}}\left|t_{klpq}(x-\lambda e)\right|<\frac{\varepsilon}{2} \tag{3.21}$$

for $r\geq r_0$, $s\geq s_0$, $p\geq p_0$ and $q\geq q_0$ with $p=k_{r-1}+1+\alpha$ where $\alpha\geq 0$, $q=l_{s-1}+1+\beta$ and $\beta\geq 0$. Let $m\geq h_r$ and $n\geq h_s$ where $m\geq 1$ and $n\geq 1$. Then

$$\begin{aligned}
&\frac{1}{\left|C_{0,0}^{m,n}\right|}\sum_{(k,l)\in C_{0,0}^{m,n}}\left|\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|}\sum_{(i,j)\in C_{p,q}^{p+k,q+l}}(x_{i,j}-\lambda e)\right| \\
&\quad=\frac{1}{\left|C_{0,0}^{m,n}\right|}\sum_{(k,l)\in C_{0,0}^{\delta_1 h_r-1,\delta_2 h_s-1}}\left|t_{klpq}(x-\lambda e)\right|+\frac{1}{\left|C_{0,0}^{m,n}\right|}\sum_{(k,l)\in C_{\delta_1 h_r,\delta_2 h_s}^{m,n}}\left|t_{klpq}(x-\lambda e)\right| \\
&\quad\leq\frac{1}{\left|C_{0,0}^{m,n}\right|}\sum_{x,y=0,0}^{\delta_1-1,\delta_2-1}\sum_{(k,l)\in D_{0,0}^{x,y}}\left|t_{klpq}(x-\lambda e)\right| \\
&\qquad+\frac{1}{\left|C_{0,0}^{m,n}\right|}\sum_{(k,l)\in C_{\delta_1 h_r,\delta_2 h_s}^{m,n}}\left(\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|}\sum_{(i,j)\in C_{p,q}^{p+k,q+l}}\left|x_{i,j}-\lambda e\right|\right).
\end{aligned} \tag{3.22}$$

Since $(x_{i,j})\in l_2^\infty$ for all i and j, there exists M such that $|x_{i,j}-\lambda e|\leq M$. From (3.21) and (3.22), we have the following:

$$\frac{1}{\left|C_{0,0}^{m,n}\right|}\sum_{(k,l)\in C_{0,0}^{m,n}}\left|\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|}\sum_{(i,j)\in C_{p,q}^{p+k,q+l}}(x_{i,j}-\lambda e)\right|\leq\frac{\delta_1\delta_2}{\left|C_{0,0}^{m,n}\right|}\left(\overline{h}_{r,s}\frac{\varepsilon}{2}\right)+\frac{M\overline{h}_{r,s}}{\left|C_{0,0}^{m,n}\right|}. \tag{3.23}$$

Thus for m and n sufficiently large, we have the following:

$$\frac{1}{\left|C_{0,0}^{m,n}\right|}\sum_{(k,l)\in C_{0,0}^{m,n}}\left|\frac{1}{\left|C_{p,q}^{p+k,q+l}\right|}\sum_{(i,j)\in C_{p,q}^{p+k,q+l}}(x_{i,j}-\lambda e)\right|<\varepsilon, \tag{3.24}$$

for $r \geq r_0$, $s \geq s_0$ and $p \geq p_0$, $q \geq q_0$. Thus by Lemma 3.5, we have $[w_\theta^2] \cap l_2^\infty \subset [w_2] \cap l_2^\infty$. It is clear that $[w_2] \cap l_2^\infty \subset [w_\theta^2] \cap l_2^\infty$. This completes the proof of the theorem. □

Corollary 3.7. $[f_2] \subset [w]_\theta^2 \subset ([w_\theta^2] \cap l_2^\infty) \subset (w_\theta^2 \cap l_2^\infty) = f_2$.

Proof. It is easy to see by combining Theorem 3.4, Theorem 3.6 with [7, Theorems 4.1 and 3.1(ii)]. So we omit it. □

A paranormed space (X, g) is a topological linear space with the topology given by the paranorm g. It may be recalled that a paranorm g is a real subadditive function on X such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, that is, $\mu_n \to \mu$, $x_n \to x$ imply that $(\mu_n x_n) \to (\mu x)$ where μ_n, μ are scalars and x_n, $x \in X$.

Let $u = (u_{k,l})$ be a bounded double sequence of positive real numbers, that is, $u_{k,l} > 0$ for all k, l with $\sup_{k,l} u_{k,l} = H < \infty$. Let

$$\left[w_\theta^2(u)\right] = \left\{ x = (x_{i,j}) : \lim_{r,s} \sup_{p,q} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in \overline{I}_{r,s}} \left| t_{klpq}(x - \lambda e) \right|^{u_{k,l}} = 0 \text{ for some } \lambda \right\}. \tag{3.25}$$

If $u = (u_{k,l})$ is constant we write $[w_\theta^2]_u$ in place of $[w_\theta^2(u)]$. If we take $u = (u_{k,l})$ with $u_{k,l} = 1$ for all k and l, then $[w_\theta^2(u)]$ is reduced to $[w_\theta^2]$ which is defined in Section 2.

Theorem 3.8. *Let $u = (u_{k,l})$ be a bounded sequence of positive real numbers with $\sup_{k,l} u_{k,l} = H < \infty$. Then $[w_\theta^2(u)]$ is a complete linear topological space paranormed by*

$$g(x) = \sup_{r,s,p,q} \left(\frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in \overline{I}_{r,s}} \left| t_{klpq}(x) \right|^{u_{k,l}} \right)^{1/M}, \tag{3.26}$$

where $M = \max(1, H)$. In the case u is constant, $[w_\theta^2]_u$ is a Banach space if $u \geq 1$ and is a p-normed space if $0 < u < 1$.

Proof. It is easy to see that $[w_\theta^2(u)]$ is a linear space with coordinatewise addition and scalar multiplication. Clearly $g(\theta) = 0$, $g(x) = g(-x)$ and g is subadditive. To prove the continuity of multiplication, assume that $x \in [w_\theta^2(u)]$. Since $u = (u_{k,l})$ is bounded and positive there exists a constant $\delta > 0$ such that $u_{k,l} \geq \delta$ for all k, l. Now for $|\mu| \leq 1$, $|\mu|^{u_{k,l}} \leq |\mu|^\delta$ and hence $g(\mu x) \leq |\mu|^{\delta/M} g(x)$. This proves the fact that g is a paranorm on $[w_\theta^2(u)]$.

To prove that $[w_\theta^2(u)]$ is complete, assume that (x^m) is a Cauchy sequence in $[w_\theta^2(u)]$, that is, $g(x^m - x^n) \to 0$ as $m, n \to \infty$. Since

$$\frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in \overline{I}_{r,s}} \left| t_{klpq}(x^m - x^n) \right|^{u_{k,l}} \leq \left[g(x^m - x^n) \right]^M, \tag{3.27}$$

it follows that $|t_{klpq}(x^m - x^n)|^{u_{k,l}} = 0(1)$ as $m, n \to \infty$ for each k, l, p, and q. In particular

$$t_{00pq}(x^m - x^n) = \left| x_{p,q}^m - x_{p,q}^n \right| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty, \text{ for each fixed } p \text{ and } q. \tag{3.28}$$

Hence, (x^m) is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$). Since $\mathbb{R}$ (or $\mathbb{C}$) is complete, there exists $x \in \mathbb{R}$ (or $\mathbb{C}$) such that $x^m \to x$ coordinate wise as $m \to \infty$. It follows from (3.27) that given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\left(\frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} \left| t_{klpq}(x^m - x^n) \right|^{u_{k,l}} \right)^{1/M} < \varepsilon, \tag{3.29}$$

for $m, n > m_0$. Now making $n \to \infty$ and then taking supremum with respect to p and q in (3.29) we obtain $g(x^m - x) \leq \varepsilon$ for $m > m_0$. This proves that $x^m \to x$ and $x \in [w_\theta^2(u)]$. Hence $[w_\theta^2(u)]$ is complete. When u is constant, it is easy to derive the rest of the theorem. □

Theorem 3.9. *Let $0 < \rho_{k,l} \leq \sigma_{k,l} < \infty$ for each k and l. Then $[w_\theta^2(\rho)] \subset [w_\theta^2(\sigma)]$.*

Proof. Let $x \in [w_\theta^2(\rho)]$. By the definition of $[w_\theta^2(\rho)]$, that for given $\varepsilon > 0$ there exist r_0, s_0 such that

$$\frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} \left| t_{klpq}(x - \lambda e) \right|^{\rho_{k,l}} < \varepsilon, \tag{3.30}$$

for $r > r_0$, $s > s_0$ and for all p, q. Since $1/\overline{h}_{r,s} \to 0$ as $r, s \to \infty$, then

$$\sum_{(k,l)\in\overline{I}_{r,s}} \left| t_{klpq}(x - \lambda e) \right|^{\rho_{k,l}} < \infty, \tag{3.31}$$

for $r > r_0$, $s > s_0$ and for all p, q. This implies that

$$\left| t_{klpq}(x - \lambda e) \right| < 1, \tag{3.32}$$

for sufficiently large values of k, l and for all p, q. Then we get,

$$\frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} \left| t_{klpq}(x - \lambda e) \right|^{\sigma_{k,l}} \leq \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} \left| t_{klpq}(x - \lambda e) \right|^{\rho_{k,l}} = o(1), \tag{3.33}$$

as r, $s \to \infty$ and for all p, q. Hence,

$$\lim_{r,s} \sup_{p,q} \frac{1}{\overline{h}_{r,s}} \sum_{(k,l)\in\overline{I}_{r,s}} \left| t_{klpq}(x - \lambda e) \right|^{\rho_{k,l}} = 0, \tag{3.34}$$

is obtained and consequently we have $x \in [w_\theta^2(\rho)]$. This completes the proof. □

Theorem 3.10. *One has the following.*

(1) *Let* $0 < \inf_{k,l} u_{k,l} \leq u_{k,l} \leq 1$ *for each* k *and* l. *Then* $[w_\theta^2(u)] \subset [w_\theta^2]$.

(2) *Let* $1 \leq u_{k,l} \leq \sup_{k,l} u_{k,l} < \infty$ *for each* k *and* l. *Then* $[w_\theta^2] \subset [w_\theta^2(u)]$.

Proof. (1) It is clear from the above theorem. If we take $\rho_{k,l} = u_{k,l}$ and $\sigma_{k,l} = 1$ for each k and l, then we have $[w_\theta^2(u)] \subset [w_\theta^2]$.

(2) From the above theorem, if we take $\rho_{k,l} = 1$ and $\sigma_{k,l} = u_{k,l}$ for each k and l, then we have $[w_\theta^2] \subset [w_\theta^2(u)]$.

This completes the proof. □

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