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*Research Article*

# On a Stability of Logarithmic-Type Functional Equation in Schwartz Distributions

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We prove the Hyers-Ulam stability of the logarithmic functional equation of Heuvers and Kannappan $f(x+y)-g(xy)-h(1/x+1/y)=0,\ x,y>0$, in both classical and distributional senses. As a classical sense, the Hyers-Ulam stability of the inequality $|f(x+y)-g(xy)-h(1/x+1/y)|\leq \epsilon,\ x,y>0$ will be proved, where $f,g,h:\mathbb{R}_+\to\mathbb{C}$. As a distributional analogue of the above inequality, the stability of inequality $\|u\circ(x+y)-v\circ(xy)-w\circ(1/x+1/y)\|\leq\epsilon$ will be proved, where $u,v,w\in\mathfrak{D}'(\mathbb{R}_+)$ and $\circ$ denotes the pullback of distributions.

## 1. Introduction

The stability problems of functional equations have originated with Ulam in 1940 (see [1]). One of the first assertions to be obtained is the following result, essentially due to Hyers [2], that gives an answer for the question of Ulam.

**Theorem 1.1.** *Suppose that $G$ is an additive semigroup, $B$ is a Banach space, $f:G\to B$, $\epsilon\geq 0$, and*

$$\|f(x+y)-f(x)-f(y)\|\leq\epsilon \tag{1.1}$$

*for all $x,y\in G$. Then there exists a unique function $A:G\to B$ satisfying*

$$A(x+y)=A(x)+A(y) \tag{1.2}$$

*for which*

$$\|f(x) - A(x)\| \leq \epsilon \tag{1.3}$$

*for all* $x \in G$.

As a direct consequence of the result, we obtain the stability of the logarithmic functional equation (see also the result of Forti [3]) as follows:

$$L(xy) - L(x) - L(y) = 0, \quad x, y > 0. \tag{1.4}$$

**Theorem 1.2.** *Let* $\mathbb{R}_+$ *be the set of positive real numbers. Suppose that* $f : \mathbb{R}_+ \to C$, $\epsilon \geq 0$, *and*

$$|f(xy) - f(x) - f(y)| \leq \epsilon \tag{1.5}$$

*for all* $x, y > 0$. *Then there exists a unique function* $L : \mathbb{R}_+ \to \mathbb{C}$ *satisfying* (1.4) *for which*

$$|f(x) - L(x)| \leq \epsilon \tag{1.6}$$

*for all* $x > 0$.

We call the function $L$ satisfying (1.4) *logarithmic function*. The logarithmic functional equation has been modified in various forms [4, 5] and Heuvers and Kannappan introduced the functional equation [6] as follows:

$$f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) = 0, \quad x, y > 0 \tag{1.7}$$

for $f, g, h : \mathbb{R}_+ \to \mathbb{R}$. In particular, it is shown that the general solution of (1.7) has the form

$$\begin{aligned} f(x) &= c_1 + c_2 + L(x), \\ g(x) &= c_1 + L(x), \\ h(x) &= c_2 + L(x), \end{aligned} \tag{1.8}$$

where $L : \mathbb{R}_+ \to \mathbb{R}$ is a logarithmic function.

In 1950, Schwartz introduced the theory of distributions in his monograph *Théorie des distributions* [7]. In this book Schwartz systematizes the theory of generalized functions, basing it on the theory of linear topological spaces, relating all the earlier approaches, and obtaining many important results. After his elegant theory appeared, many important concepts and results on the classical spaces of functions have been generalized to the space of distributions.

Making use of differentiation of distributions, several authors have dealt with functional equations in the spaces of Schwartz distributions, converting given functional equations to differential equations, and finding the solutions in the space of distributions

(see [8–11]). However, when we try to consider the Hyers-Ulam stability problems of functional equations, the differentiation is not available for solving them in both the space of infinitely differentiable functions and the space of distributions. In the paper [12], using convolutional approach we initiated the following distributional version of the well-known Hyers-Ulam stability problem for the Cauchy functional equation:

$$\|u \circ A - u \circ P_1 - u \circ P_2\| \leq \epsilon, \tag{1.9}$$

where $\circ$ is the pullback. See Section 3 for the pullback and see below the definition of the norm $\|\cdot\|$ in (1.9). Using the heat kernel

$$E_t(x) = (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right), \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.10}$$

we proved the stability problems (1.9) in the space of tempered distributions [7] by converting the inequality (1.9) to the classical stability problems

$$\left|U(x+y, t+s) - U(x,t) - U(y,s)\right| \leq \epsilon \tag{1.11}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where $U$ is an infinitely differentiable function in $\mathbb{R}^n \times (0,\infty)$ given by $U(x,t) = u * E_t(x)$. We also refer the reader to [13] for the stability of Pexider equations in the space of tempered distributions. In [14] we extend the stability problems in the space of tempered distributions to the space of distributions. Instead of the heat kernel, using the regularizing function $\delta_t(x) := t^{-n}\delta(x/t)$, $x \in \mathbb{R}^n$, $t > 0$, where

$$\delta(x) = \begin{cases} qe^{-\sqrt{(1-|x|^2)^{-1}}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{1.12}$$

$$q = \left(\int_{|x|<1} e^{-\sqrt{(1-|x|^2)^{-1}}} dx\right)^{-1}, \tag{1.13}$$

we prove that the unknown distributions in the functional inequalities are tempered distributions and then use the same method as in [12, 13].

In this paper, developing the previous method in [12–14], we consider a distributional version of the Hyers-Ulam stability of (1.7) in the space of distributions as

$$\left\|u \circ (x+y) - v \circ (xy) - w \circ \left(\frac{1}{x} + \frac{1}{y}\right)\right\| \leq \epsilon, \tag{1.14}$$

where $u, v, w \in \mathcal{D}'(\mathbb{R}_+)$, $\circ$ denotes the pullback of distributions and the inequality $\|\cdot\| \leq \epsilon$ in (1.14) means $|\langle \cdot, \varphi\rangle| \leq \epsilon\|\varphi\|_{L^1}$ for all test functions $\varphi(x,y)$ defined on $\mathbb{R}_+^2$. Since the tempered distributions are defined in whole real line or whole space $\mathbb{R}^n$, the methods used in [14] are not available for the inequality (1.14). For the proof of the above problem, we need some technical method than those employed in [12–14]. Indeed, we will show a method to control

a functional inequality satisfied in a subset of $\mathbb{R}^2$. As a direct consequence of the result, we obtain the Hyers-Ulam stability of (1.7) in $L^\infty$-sense, that is, the Hyers-Ulam stability of the inequality

$$\left\| f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) \right\|_{L^\infty} \leq \epsilon \tag{1.15}$$

will be obtained. Finally, we also find locally integrable solutions of (1.7) as a consequence of the stability of the inequality (1.15).

## 2. Stability in Classical Sense

In this section, we prove the Hyers-Ulam stability of the functional inequality

$$\left| f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) \right| \leq \epsilon, \quad x, y > 0, \tag{2.1}$$

where $f, g, h : \mathbb{R}_+ \to \mathbb{R}$ and $\epsilon \geq 0$.

**Theorem 2.1.** *Suppose that $f, g, h : \mathbb{R}_+ \to \mathbb{C}$ satisfy* (2.1). *Then there exists a logarithmic function $L : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$\begin{aligned} |f(x) - L(x) - f(1)| &\leq 4\epsilon, \\ |g(x) - L(x) - g(1)| &\leq 4\epsilon, \\ |h(x) - L(x) - h(1)| &\leq 4\epsilon \end{aligned} \tag{2.2}$$

*for all $x > 0$.*

*Proof.* Let $xy = t$, $1/x + 1/y = s$. Then we have

$$|f(ts) - g(t) - h(s)| \leq \epsilon \tag{2.3}$$

for all $t > 0$, $s > 0$ such that $ts^2 \geq 4$. For given $t, s > 0$, choose a large $u > 0$ so that $ts^2u^2 \geq 4$, $tsu^2 \geq 4$, $su^2 \geq 4$, $s^2u^2 \geq 2$. Then in view of (2.3), we have

$$|f(tsu) - g(t) - h(su)| \leq \epsilon, \tag{2.4}$$

$$|f(tsu) - g(ts) - h(u)| \leq \epsilon, \tag{2.5}$$

$$|f(su) - g(s) - h(u)| \leq \epsilon, \tag{2.6}$$

$$|f(su) - g(1) - h(su)| \leq \epsilon. \tag{2.7}$$

From (2.4)–(2.7), using the triangle inequality we have

$$|g(ts) - g(t) - g(s) + g(1)| \leq 4\epsilon \tag{2.8}$$

for all $t, s > 0$. Changing the roles of $g$ and $h$ in (2.3), we can show that

$$|h(ts) - h(t) - h(s) + h(1)| \leq 4\epsilon \tag{2.9}$$

for all $t, s > 0$. Now we prove that

$$\left|f(ts) - f(t) - f(s) + f(1)\right| \leq 4\epsilon \tag{2.10}$$

for all $t, s > 0$. Replacing $t$ by $u$ and $s$ by $s/u$ in (2.3), we have

$$\left|f(s) - g(u) - h\left(\frac{s}{u}\right)\right| \leq \epsilon, \quad 4u \leq s^2. \tag{2.11}$$

Similarly, we have

$$\left|f(ts) - g(u) - h\left(\frac{ts}{u}\right)\right| \leq \epsilon, \quad 4u \leq t^2 s^2, \tag{2.12}$$

$$\left|f(t) - g\left(\frac{u}{s}\right) - h\left(\frac{ts}{u}\right)\right| \leq \epsilon, \quad 4u \leq t^2 s, \tag{2.13}$$

$$\left|f(1) - g\left(\frac{u}{s}\right) - h\left(\frac{s}{u}\right)\right| \leq \epsilon, \quad 4u \leq s. \tag{2.14}$$

For given $t, s > 0$, let $u = 1/4 \min\{s^2, t^2 s^2, t^2 s, s\}$. Then, from (2.11)–(2.14), using the triangle inequality we get the inequality (2.10).

Now by Theorem 1.2, there exist functions $L_j : \mathbb{R}_+ \to \mathbb{R}$, $j = 1, 2, 3$, satisfying the logarithmic functional equation

$$L_j(ts) = L_j(t) + L_j(s), \quad j = 1, 2, 3, \tag{2.15}$$

for which

$$\left|f(t) - L_1(t) - f(1)\right| \leq 4\epsilon, \tag{2.16}$$

$$\left|g(t) - L_2(t) - g(1)\right| \leq 4\epsilon, \tag{2.17}$$

$$|h(t) - L_3(t) - h(1)| \leq 4\epsilon. \tag{2.18}$$

Now we show that $L_1 = L_2 = L_3$. From (2.3), we have

$$\left|f(t) - g(t) - h(1)\right| \leq \epsilon, \quad t \geq 2, \tag{2.19}$$

$$\left|f(t) - h(t) - g(1)\right| \leq \epsilon, \quad t \geq 4. \tag{2.20}$$

From (2.16), (2.17), and (2.19), using the triangle inequality we have

$$|L_1(t) - L_2(t)| \le 9\epsilon + \left|f(1) - g(1) - h(1)\right| := M, \quad t \ge 2. \tag{2.21}$$

In view of (2.15) and (2.21), we have

$$|L_1(t) - L_2(t)| = \frac{1}{|n|}|L_1(t^n) - L_2(t^n)| \le \frac{1}{|n|}M \tag{2.22}$$

for all $t > 0$, $t \neq 1$ and $t^n \ge 2$. Letting $n \to \infty$ for $t > 1$ and letting $n \to -\infty$ for $0 < t < 1$, we have $L_1(t) = L_2(t)$ for $t \neq 1$. Since $L_1(1) = L_2(1) = 0$, we have $L_1(t) = L_2(t)$ for all $t > 0$. Similarly we can show that $L_1 = L_3$. This completes the proof. □

Letting $g = h = f$ in Theorem 2.1, in view of the inequalities (2.4), (2.5), and (2.6), using the triangle inequality we have

$$\left|f(ts) - f(t) - f(s)\right| \le 3\epsilon \tag{2.23}$$

for all $t, s > 0$. Thus by Theorem 1.2 we have the following.

**Theorem 2.2.** *Let $f : \mathbb{R}_+ \to \mathbb{R}$ satisfy the inequality*

$$\left|f(x+y) - f(xy) - f\left(\frac{1}{x} + \frac{1}{y}\right)\right| \le \epsilon \tag{2.24}$$

*for all $x, y > 0$. Then there exists a logarithmic function $L : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$\left|f(x) - L(x)\right| \le 3\epsilon \tag{2.25}$$

*for all $x > 0$.*

## 3. Schwartz Distributions

Let $\Omega$ be an open subset of $\mathbb{R}^n$. We briefly introduce the space $\mathfrak{D}'(\Omega)$ of distributions. We denote by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, \ldots, n$.

*Definition 3.1.* Let $C_c^\infty(\Omega)$ be the set of all infinitely differentiable functions on $\Omega$ with compact supports. A distribution $u$ is a linear form on $C_c^\infty(\Omega)$ such that for every compact set $K \subset \Omega$ there exist constants $C > 0$ and $k \in \mathbb{N}_0$ for which

$$\left|\langle u, \varphi\rangle\right| \le C \sum_{|\alpha| \le k} \sup\left|\partial^\alpha \varphi\right| \tag{3.1}$$

holds for all $\varphi \in C_c^\infty(\Omega)$ with supports contained in $K$. The set of all distributions is denoted by $\mathfrak{D}'(\Omega)$.

Let $\Omega_j$ be open subsets of $\mathbb{R}^{n_j}$ for $j = 1, 2$, with $n_1 \ge n_2$.

*Definition 3.2.* Let $u_j \in \mathfrak{D}'(\Omega_j)$ and $\lambda : \Omega_1 \to \Omega_2$ be a smooth function such that for each $x \in \Omega_1$ the derivative $\lambda'(x)$ is surjective, that is, the Jacobian matrix $\nabla\lambda$ of $\lambda$ has rank $n_2$. Then there exists a unique continuous linear map $\lambda^* : \mathfrak{D}'(\Omega_2) \to \mathfrak{D}'(\Omega_1)$ such that $\lambda^* u = u \circ \lambda$ when $u$ is a continuous function. We call $\lambda^* u$ the pullback of $u$ by $\lambda$ and is usually denoted by $u \circ \lambda$.

In particular, if $S(x,y) = x + y,\ P_1(x,y) = x,\ P_2(x,y) = y$, the pullbacks $u \circ S$, $u \circ P_1$, $u \circ P_2$ can be written as

$$\langle u \circ S, \varphi(x,y)\rangle = \left\langle u, \int \varphi(x-y,y)dy \right\rangle, \tag{3.2}$$

$$\langle u \circ P_1, \varphi(x,y)\rangle = \left\langle u, \int \varphi(x,y)dy \right\rangle, \tag{3.3}$$

$$\langle u \circ P_2, \varphi(x,y)\rangle = \left\langle u, \int \varphi(x,y)dx \right\rangle \tag{3.4}$$

for all test functions $\varphi \in C_c^\infty(\Omega)$.

Also, if $\lambda$ is a diffeomorphism (a bijection with $\lambda,\ \lambda^{-1}$ smooth functions), the pullback $u \circ \lambda$ can be written as

$$\langle u \circ \lambda, \varphi\rangle = \left\langle u, \left(\varphi \circ \lambda^{-1}\right)(x)\left|\nabla\lambda^{-1}(x)\right|\right\rangle. \tag{3.5}$$

For more details of distributions we refer the reader to [7, 15].

## 4. Stability in Schwartz Distributions

We employ a function $\delta$ on $\mathbb{R}^n$ defined by

$$\delta(x) = \begin{cases} qe^{-\sqrt{(1-|x|^2)^{-1}}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{4.1}$$

where

$$q = \left(\int_{|x|<1} e^{-\sqrt{(1-|x|^2)^{-1}}} dx\right)^{-1}. \tag{4.2}$$

It is easy to see that $\delta(x)$ is an infinitely differentiable function with support $\{x : |x| \leq 1\}$. Let $\delta_t(x) := t^{-n}\delta(x/t),\ t > 0$ and $u \in \mathfrak{D}'(\mathbb{R}^n)$. Then for each $t > 0,\ u * \delta_t(x) = \langle u_y, \delta_t(x-y)\rangle$ is a smooth function of $x \in \mathbb{R}^n$ and $u * \delta_t(x) \to u$ as $t \to 0^+$ in the sense that

$$\lim_{t\to 0^+} \int (u * \delta_t)(x)\varphi(x)dx = \langle u, \varphi\rangle \tag{4.3}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Here after we denote by $S, P, P_1, P_2 : \mathbb{R}^2 \to \mathbb{R}$, $R : \mathbb{R}_+^2 \to \mathbb{R}$, $E : \mathbb{R} \to \mathbb{R}^+$ by

$$S(x,y) = x + y, \qquad P(x,y) = xy, \qquad P_1(x,y) = x, \qquad P_2(x,y) = y,$$
$$R(x,y) = \frac{1}{x} + \frac{1}{y}, \qquad E(x) = 2^x. \tag{4.4}$$

Now we are in a position to prove the Hyers-Ulam stability of the inequality

$$\|u \circ S - v \circ P - w \circ R\| \le \epsilon. \tag{4.5}$$

Recall that the inequality $\|\cdot\| \le \epsilon$ in (4.5) means that $|\langle \cdot, \varphi \rangle| \le \epsilon \|\varphi\|_{L^1}$ for all test functions $\varphi(x,y)$ defined on $\mathbb{R}_+^2$.

**Theorem 4.1.** *Let $u, v, w \in \mathfrak{D}'(\mathbb{R}_+)$ satisfy* (4.5). *Then there exist $a, c_1, c_2, c_3 \in \mathbb{C}$ such that*

$$\begin{aligned} \|u - c_1 - a \ln x\| &\le 4\epsilon, \\ \|v - c_2 - a \ln x\| &\le 4\epsilon, \\ \|w - c_3 - a \ln x\| &\le 4\epsilon. \end{aligned} \tag{4.6}$$

*Proof.* Let $U = \{(x,y) : x + 2y > 2\}$, $V = \{(x,y) : x > y > 0\}$ and define $J : U \to V$ by

$$J(x,y) = \left( \frac{2^{x+y} + \sqrt{4^{x+y} - 2^{x+2}}}{2}, \frac{2^{x+y} - \sqrt{4^{x+y} - 2^{x+2}}}{2} \right). \tag{4.7}$$

Then $J$ is a diffeomorphism with $J^{-1} : V \to U$, $J^{-1}(x,y) = (\log_2 xy, \log_2 (x+y)/xy)$. Taking pullback by $J$ in (4.5) and using (3.5), we have

$$\|u \circ E \circ S - v \circ E \circ P_1 - w \circ E \circ P_2\| \le \epsilon \tag{4.8}$$

in $U$. Thus it follows that

$$\|\tilde{u} \circ S - \tilde{v} \circ P_1 - \tilde{w} \circ P_2\| \le \epsilon \tag{4.9}$$

in $U$, where $\tilde{u} = u \circ E$, $\tilde{v} = v \circ E$, $\tilde{w} = w \circ E$. Denoting by $(\delta_t \otimes \delta_s)(x,y) = \delta_t(x)\delta_s(y)$ and convolving $\delta_t \otimes \delta_s$ in the left hand side of (4.9) we have, in view of (3.2),

$$\begin{aligned} [(\tilde{u} \circ S) * (\delta_t \otimes \delta_s)](x,y) &= \langle \tilde{u} \circ (\xi + \eta), \delta_t(x - \xi)\delta_s(y - \eta) \rangle \\ &= \left\langle \tilde{u}_\xi, \int \delta_t(x - \xi + \eta)\delta_s(y - \eta)\, d\eta \right\rangle \\ &= \langle \tilde{u}_\xi, \delta_t * \delta_s(x + y - \xi) \rangle \\ &= \tilde{u} * \delta_t * \delta_s(x + y). \end{aligned} \tag{4.10}$$

Similarly we have, in view of (3.3) and (3.4),

$$\begin{aligned}[(\tilde{v}\circ P_1)*(\delta_t\otimes\delta_s)](x,y) &= \tilde{v}*\delta_t(x),\\ [(\tilde{w}\circ P_2)*(\delta_t\otimes\delta_s)](x,y) &= \tilde{w}*\delta_s(y).\end{aligned} \tag{4.11}$$

Thus the inequality (4.9) is converted to the classical stability problem

$$\left|\tilde{u}*\delta_t*\delta_s(x+y)-\tilde{v}*\delta_t(x)-\tilde{w}*\delta_s(y)\right|\le\epsilon\|\delta_t\otimes\delta_s\|_{L^1}=\epsilon \tag{4.12}$$

for all $x+2y\ge 5$ and $0<t<1$, $0<s<1$. From now on, we assume that $0<t<1$, $0<s<1$. From the inequality (4.12), we have

$$\left|\tilde{u}*\delta_t*\delta_s(x+y+z)-\tilde{v}*\delta_t(x+y)-\tilde{w}*\delta_s(z)\right|\le\epsilon \tag{4.13}$$

for $x+y+2z\ge 5$,

$$\left|\tilde{u}*\delta_t*\delta_s(x+y+z)-\tilde{v}*\delta_t(x)-\tilde{w}*\delta_s(y+z)\right|\le\epsilon \tag{4.14}$$

for $x+2y+2z\ge 5$,

$$\left|\tilde{u}*\delta_t*\delta_s(y+z)-\tilde{v}*\delta_t(y)-\tilde{w}*\delta_s(z)\right|\le\epsilon \tag{4.15}$$

for $y+2z\ge 5$, and

$$\left|\tilde{u}*\delta_t*\delta_s(y+z)-\tilde{v}*\delta_t(0)-\tilde{w}*\delta_s(y+z)\right|\le\epsilon \tag{4.16}$$

for $2y+2z\ge 5$.

For given $x,y\in\mathbb{R}$, choose $z\ge(1/2)(5+|x|+2|y|)$. Then in view of (4.13)–(4.16), using triangle inequality, we have

$$\left|\tilde{v}*\delta_t(x+y)-\tilde{v}*\delta_t(x)-\tilde{v}*\delta_t(y)+\tilde{v}*\delta_t(0)\right|\le 4\epsilon \tag{4.17}$$

for all $x,y\in\mathbb{R}$. Replacing $(x,t)$ by $(y,s)$, $(y,s)$ by $(x,t)$ in (4.12) and changing the positions of $\tilde{v}$ and $\tilde{w}$, we have

$$\left|\tilde{w}*\delta_t(x+y)-\tilde{w}*\delta_t(x)-\tilde{w}*\delta_t(y)+\tilde{w}*\delta_t(0)\right|\le 4\epsilon \tag{4.18}$$

for all $x,y\in\mathbb{R}$. Now we prove that

$$\left|\tilde{u}*\delta_t(x+y)-\tilde{u}*\delta_t(x)-\tilde{u}*\delta_t(y)+\tilde{u}*\delta_t(0)\right|\le 4\epsilon \tag{4.19}$$

for all $x, y \in \mathbb{R}$. From the inequality (4.12), we have

$$\begin{gathered}
\left|\tilde{u} * \delta_t * \delta_s(x+y) - \tilde{v} * \delta_t(z) - \tilde{w} * \delta_s(x+y-z)\right| \le \epsilon, \\
\left|\tilde{u} * \delta_t * \delta_s(x) - \tilde{v} * \delta_t(z-y) - \tilde{w} * \delta_s(x+y-z)\right| \le \epsilon, \\
\left|\tilde{u} * \delta_t * \delta_s(y) - \tilde{v} * \delta_t(z) - \tilde{w} * \delta_s(y-z)\right| \le \epsilon, \\
\left|\tilde{u} * \delta_t * \delta_s(0) - \tilde{v} * \delta_t(z-y) - \tilde{w} * \delta_s(y-z)\right| \le \epsilon
\end{gathered} \tag{4.20}$$

for all $x, y, z$ such that $2x + 2y - z \ge 5$, $2x + y - z \ge 5$, $2y - z \ge 5$, and $y - z \ge 5$. For given $x, y \in \mathbb{R}$, choose $z \le -5 - 2|x| - 2|y|$. Then in view of (4.20), using triangle inequality, we have

$$\left|\tilde{u} * \delta_t * \delta_s(x+y) - \tilde{u} * \delta_t * \delta_s(x) - \tilde{u} * \delta_t * \delta_s(y) + \tilde{u} * \delta_t * \delta_s(0)\right| \le 4\epsilon. \tag{4.21}$$

Letting $s \to 0^+$ in (4.21), we get the inequality (4.19).

Now in view of (4.17), (4.18), and (4.19), it follows from Theorem 1.1 that for each $0 < t < 1$, there exist functions $A_j(\cdot, t)$, $j = 1, 2, 3$, satisfying

$$A_j(x+y, t) = A_j(x, t) + A_j(y, t), \quad x, y \in \mathbb{R}, \tag{4.22}$$

for which

$$|\tilde{u} * \delta_t(x) - A_1(x, t) - \tilde{u} * \delta_t(0)| \le 4\epsilon, \tag{4.23}$$

$$|\tilde{v} * \delta_t(x) - A_2(x, t) - \tilde{v} * \delta_t(0)| \le 4\epsilon, \tag{4.24}$$

$$|\tilde{w} * \delta_t(x) - A_3(x, t) - \tilde{w} * \delta_t(0)| \le 4\epsilon \tag{4.25}$$

for all $x \in \mathbb{R}$.

Now we prove that $A_1 = A_2 = A_3$. From (4.12), using the triangle inequality, we have

$$|\tilde{v} * \delta_t(x)| \le \epsilon + \left|\tilde{u} * \delta_t * \delta_s(x+y)\right| + \left|\tilde{w} * \delta_s(y)\right| \tag{4.26}$$

for all $x + 2y \ge 5$. Since $(\tilde{u} * \delta_t * \delta_s)(x) \to (\tilde{u} * \delta_s)(x)$ as $t \to 0^+$, in view of (4.26) it is easy to see that

$$g(x) := \limsup_{t \to 0^+} \tilde{v} * \delta_t(x) \tag{4.27}$$

exists for all $x \in \mathbb{R}$. Similarly, we can show that

$$h(x) := \limsup_{s \to 0^+} \tilde{w} * \delta_s(x) \tag{4.28}$$

exists for all $x \in \mathbb{R}$. Putting $y = 0$ in (4.12) and letting $s \to 0^+$ so that $\widetilde{w} * \delta_s(0) \to h(0)$, we have

$$|\widetilde{u} * \delta_t(x) - \widetilde{v} * \delta_t(x) - h(0)| \leq \epsilon \tag{4.29}$$

for all $x \geq 5$. Similarly, we have

$$\left|\widetilde{u} * \delta_t(x) - \widetilde{w} * \delta_t(x) - g(0)\right| \leq \epsilon \tag{4.30}$$

for all $x \geq 5/2$. Using (4.23), (4.24), (4.29), and the triangle inequality, we have

$$|A_1(x,t) - A_2(x,t)| \leq 9\epsilon + |\widetilde{u} * \delta_t(0) - \widetilde{v} * \delta_t(0) - h(0)| := M(t) \tag{4.31}$$

for all $x \geq 5$. From (4.22) and (4.31), we have

$$|A_1(x,t) - A_2(x,t)| = \frac{1}{|k|}|A_1(kx,t) - A_2(kx,t)| \leq \frac{1}{|k|}M(t) \tag{4.32}$$

for all $x \in \mathbb{R}$, $x \neq 0$ and all integers $k$ with $kx \geq 5$. Letting $k \to \infty$ if $x > 0$ and letting $k \to -\infty$ if $x < 0$ in (4.32) we have $A_1(x,t) = A_2(x,t)$ for $x \neq 0$, which implies $A_1 = A_2$ since $A_1(0,t) = A_2(0,t) = 0$. Similarly, using (4.23), (4.25), and (4.30) we can show that $A_1 = A_3$.

Finally we prove that $A_1$ is independent of $t$. Fixing $x \in \mathbb{R}$ and letting $t \to 0^+$ so that $\widetilde{v} * \delta_t(x) \to g(x)$ in (4.12), we have

$$\left|\widetilde{u} * \delta_s(x+y) - g(x) - \widetilde{w} * \delta_s(y)\right| \leq \epsilon \tag{4.33}$$

for all $x + 2y \geq 5$. The same substitution as the inequalities (4.13)–(4.16) gives

$$\left|g(x+y) - g(x) - g(y) + g(0)\right| \leq 4\epsilon. \tag{4.34}$$

Using the stability Theorem [2], we obtain that there exists a unique function $A$ satisfying the Cauchy functional equation

$$A(x+y) - A(x) - A(y) = 0 \tag{4.35}$$

for which

$$\left|g(x) - A(x) - g(0)\right| \leq 4\epsilon. \tag{4.36}$$

Now we show that $A_1(x,t) = A(x)$ for all $x \in \mathbb{R}$ and $0 < t < 1$. Putting $y = 0$ in (4.33), we have

$$\left|\widetilde{u} * \delta_s(x) - g(x) - \widetilde{w} * \delta_s(0)\right| \leq \epsilon \tag{4.37}$$

for all $x \geq 5$. From (4.23), (4.36), and (4.37), using the triangle inequality, we have

$$|A_1(x,t) - A(x)| \leq 9\epsilon + \left|\tilde{u} * \delta_t(0) - \tilde{w} * \delta_t(0) - g(0)\right| \tag{4.38}$$

for all $x \geq 5$. From (4.38), using the method of proving $A_1 = A_2$, we can show that $A_1(x,t) = A(x)$ for all $x \in \mathbb{R}$ and $t$. Thus we have $A_1 = A_2 = A_3 = A$.

Letting $t \to 0^+$ in (4.24) so that $\tilde{v} * \delta_t(0) \to g(0)$, we have

$$\|\tilde{v} - A(x) - c_2\| \leq 4\epsilon \tag{4.39}$$

for some $c_2 \in \mathbb{C}$. Similarly, letting $t \to 0^+$ in (4.25) so that $\tilde{w} * \delta_t(0) \to h(0)$, we have

$$\|\tilde{w} - A(x) - c_3\| \leq 4\epsilon \tag{4.40}$$

for some $c_3 \in \mathbb{C}$. Now we prove the inequality

$$\|\tilde{u} - A(x) - c_1\| \leq 4\epsilon \tag{4.41}$$

for some $c_1 \in \mathbb{C}$. Putting $x = -5$, $y = 5$ in (4.33) and using the triangle inequality, we have

$$|\tilde{u} * \delta_s(0)| \leq \epsilon + \left|g(-5) + \tilde{w} * \delta_s(5)\right|. \tag{4.42}$$

From (4.42), there exists a sequence $s_n \to 0^+$ such that $\tilde{u} * \delta_{s_n}(0)$ converges. Letting $t = s_n \to 0^+$ in (4.23), we get (4.41). Taking pullback by $E^{-1}(x) = \log_2 x$ in (4.39), (4.40), and (4.41), we have

$$\begin{aligned} \left\|u - A(\log_2 x) - c_1\right\| &\leq 4\epsilon, \\ \left\|v - A(\log_2 x) - c_2\right\| &\leq 4\epsilon, \\ \left\|w - A(\log_2 x) - c_3\right\| &\leq 4\epsilon. \end{aligned} \tag{4.43}$$

Finally we show that the solution $A$ of the Cauchy equation (4.35) has the form $A(x) = ax$ for some $a \in \mathbb{C}$. Recall that $g$ is the supremum limit of a collection of continuous functions $\tilde{v} * \delta_t$, $0 < t < 1$. Thus, if we let $g = g_1 + ig_2$, then both $g_1$ and $g_2$ are Lebesgue measurable functions. Now, as we see in the proof of Hyers-Ulam stability Theorem [2], the function $A$ is given by

$$A(x) = \lim_{n \to \infty} 2^{-n} g(2^n x). \tag{4.44}$$

Thus, let $A = A_1 + iA_2$. Then $A_1$ and $A_2$ are Lebesgue measurable functions as limits of sequences of Lebesgue measurable functions. It is well known that every Lebesgue measurable solution $A$ of the Cauchy functional equation (4.35) has the form $A(x) = cx$ for some $c \in \mathbb{C}$. This completes the proof. □

As a consequence of the above result we have the following.

**Corollary 4.2.** *Let $f, g, h : \mathbb{R}_+ \to \mathbb{C}$, $j = 1, 2, 3$, be locally integrable functions satisfying*

$$\left\| f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) \right\|_{L^\infty} \leq \epsilon. \tag{4.45}$$

*Then there exist $a, c_1, c_2, c_3 \in \mathbb{C}$ such that*

$$\begin{aligned} \|f(x) - c_1 - a \ln x\|_{L^\infty} &\leq 4\epsilon, \\ \|g(x) - c_2 - a \ln x\|_{L^\infty} &\leq 4\epsilon, \\ \|h(x) - c_3 - a \ln x\|_{L^\infty} &\leq 4\epsilon. \end{aligned} \tag{4.46}$$

*Proof.* Every locally integrable function $f$ defines a distribution via the equation

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)dx, \quad \varphi \in C_c^\infty(\mathbb{R}_+). \tag{4.47}$$

Viewing $f, g, h$ as distributions, the inequality (4.45) implies

$$\|f \circ S - g \circ P - h \circ R\| \leq \epsilon. \tag{4.48}$$

By Theorem 4.1, we have

$$\begin{aligned} \left| \int (f(x) - c_1 - a \ln x)\varphi(x)dx \right| &\leq 4\epsilon \|\varphi\|_{L^1}, \\ \left| \int (g(x) - c_2 - a \ln x)\varphi(x)dx \right| &\leq 4\epsilon \|\varphi\|_{L^1}, \\ \left| \int (h(x) - c_3 - a \ln x)\varphi(x)dx \right| &\leq 4\epsilon \|\varphi\|_{L^1} \end{aligned} \tag{4.49}$$

for all $\varphi \in C_c^\infty(\mathbb{R}_+)$. Viewing $C_c^\infty$ as a subspace of $L^1$ (dense subspace) and using the Hahn-Banach theorem we obtain that the inequalities (4.49) hold for all $\varphi \in L^1$. Now since $L^\infty = (L^1)'$ we get the inequalities (4.46). This completes the proof. □

As a direct consequence of the above result we solve the functional equation

$$f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) = 0 \tag{4.50}$$

in $L^\infty$-sense, that is, we obtain the following.

**Corollary 4.3.** *Let $f, g, h : \mathbb{R}_+ \to \mathbb{C}$, $j = 1, 2, 3$, be locally integrable functions satisfying*

$$\left\| f(x+y) - g(xy) - h\left(\frac{1}{x} + \frac{1}{y}\right) \right\|_{L^\infty} = 0. \tag{4.51}$$

*Then there exist* $a, c_1, c_2, c_3 \in \mathbb{C}$ *such that*

$$\begin{aligned} \|f(x) - c_1 - a\ln x\|_{L^\infty} &= 0, \\ \|g(x) - c_2 - a\ln x\|_{L^\infty} &= 0, \\ \|h(x) - c_3 - a\ln x\|_{L^\infty} &= 0. \end{aligned} \tag{4.52}$$

Finally, we discuss the locally integrable solution $f, g, h : \mathbb{R}_+ \to \mathbb{C}$ of (4.50) (c.f. [6]).

**Corollary 4.4.** *Every locally integrable solution* $f, g, h : \mathbb{R}_+ \to \mathbb{C}$ *of* (4.50) *has the form*

$$f(x) = c_1 + c_2 + a\ln x, \tag{4.53}$$

$$g(x) = c_1 + a\ln x, \tag{4.54}$$

$$h(x) = c_2 + a\ln x \tag{4.55}$$

*for some* $a, c_1, c_2 \in \mathbb{C}$.

*Proof.* It follows from Corollary 4.3 that the equalities (4.53), (4.54), and (4.55) hold in almost everywhere sense, that is, there exists a subset $\Omega \subset \mathbb{R}_+$ with Lebesgue measure $m(\Omega^c) = 0$ such that the equalities (4.53), (4.54), and (4.55) hold for all $x \in \Omega$. For given $x > 0$, let $p, q : (0, x) \to \mathbb{R}$ by $p(t) = 1/t + (1/(x-t))$, $q(t) = t(x-t)$. Since $m[(p^{-1}(\Omega) \cap q^{-1}(\Omega))^c] = m[p^{-1}(\Omega^c) \cup q^{-1}(\Omega^c)] = 0$, we can choose $y \in p^{-1}(\Omega) \cap q^{-1}(\Omega)$. Let $z = x - y$. Then $y + z = x$ and $yz = y(x-y) = q(y) \in \Omega$, $1/y + 1/z = p(y) \in \Omega$. Thus we can write

$$\begin{aligned} f(x) &= g(yz) + h\left(\frac{1}{y} + \frac{1}{z}\right) \\ &= c_1 + a\ln(yz) + c_2 + a\ln\left(\frac{1}{y} + \frac{1}{z}\right) \\ &= c_1 + c_2 + a\ln(y+z) = c_1 + c_2 + a\ln x, \end{aligned} \tag{4.56}$$

which gives (4.53). For given $x > 0$, let $p : \mathbb{R}_+ \to \mathbb{R}$ by $p(t) = 1/t + t/x$. Then we have $p^{-1}(\Omega) \neq \emptyset$. Choose $y \in p^{-1}(\Omega)$ and let $z = x/y$. Then $yz = x$, $1/y + 1/z \in \Omega$. Thus, using (4.53), we can write

$$\begin{aligned} g(x) &= f(y+z) - h\left(\frac{1}{y} + \frac{1}{z}\right) \\ &= c_1 + c_2 + a\ln(y+z) - c_2 - a\ln\left(\frac{1}{y} + \frac{1}{z}\right) \\ &= c_1 + a\ln(yz) = c_1 + a\ln x, \end{aligned} \tag{4.57}$$

which gives (4.54). Finally, the equality (4.55) follows from (4.50), (4.53), and (4.54). This completes the proof. □

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