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Research Article

Strong Convergence of the Iterative Methods for Hierarchical Fixed Point Problems of an Infinite Family of Strictly Nonself Pseudocontractions

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This paper deals with a new iterative algorithm for solving hierarchical fixed point problems of an infinite family of pseudocontractions in Hilbert spaces by $y_n = \beta_n S x_n + (1-\beta_n)x_n$, $x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)\sum_{i=1}^{\infty}\mu_i^{(n)} T_i y_n]$, and $\forall n \geq 0$, where $T_i : C \mapsto H$ is a nonself k_i-strictly pseudocontraction. Under certain approximate conditions, the sequence $\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$, which solves some variational inequality. The results here improve and extend some recent results.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. A mapping $f : C \mapsto H$ is called a contraction with coefficient γ if there exits a constant $\gamma \in [0,1)$ such that

$$\|f(x) - f(y)\| \leq \gamma\|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

A mapping $T : C \mapsto C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

A mapping $T : C \mapsto H$ is called k-strictly pseudocontraction if there exits a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

Write $F(T)$ as the set of fixed points of T, that is, $F(T) = \{x \in C, Tx = x\}$. In 2000, Moudafi [1] introduced an iterative scheme for nonexpansive mappings

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.4}$$

where f be a contraction on H and the sequence $\{x_n\}$ started with arbitrary initial $x_0 \in H$. In 2004, Xu [2] proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T under certain conditions on the parameters, which also solves the variational inequality

$$\langle (I-f)x^*, x-x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{1.5}$$

Recently, some authors studied the problems of fixed points of nonexpansive mappings with strongly positive operators, Lipschitizian, strongly monotone operators, and extragradient methods, and many convergence results were obtained (such as, see [3–9]).

In 2008, Yao et al. [10] introduced the following iterative scheme:

$$\begin{gathered} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad \forall n \geq 0, \end{gathered} \tag{1.6}$$

where f is a contraction on C and $T : C \mapsto C$ is nonexpansive mapping. In 2012, Song et al. [11] analyzed the following iterative algorithm:

$$\begin{gathered} x_0 = x \in C, \\ y_n = P_C\left[\beta_n x_n + (1-\beta_n)\sum_{i=1}^{\infty}\mu_i^{(n)}T_i x_n\right], \\ x_{n+1} = \alpha_n f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n F)y_n, \quad \forall n \geq 0, \end{gathered} \tag{1.7}$$

where T_i is a k_i-strictly pseudocontraction, $F : C \mapsto C$ is a lipschitzian and strongly monotone operator, $f : C \mapsto C$ is a contraction, and P_C is the metric projection from H onto C. Under certain conditions on the parameters, the sequence $\{x_n\}$ generated by (1.7) converges strongly to a fixed point of a countable family of k_i-strictly pseudocontraction, which is the solution of some variational inequality.

On the other hand, in 2010, Yao et al. [12] introduced the iterative algorithm for solving hierarchical fixed point of nonexpansive mappings and gave the following theorem.

Theorem YCL

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \mapsto H$ be a contraction with coefficient $\gamma \in [0,1)$. Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$;

(iii) $\lim_{n\to\infty}(|\alpha_{n+1}-\alpha_n|/\alpha_n) = 0$ and $\lim_{n\to\infty}(|\beta_{n+1}-\beta_n|/\beta_n) = 0$.

Then the sequence $\{x_n\}$ generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \beta_n S x_n + (1-\beta_n) x_n, \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + (1-\alpha_n) T y_n\right], \quad \forall n \geq 0, \end{aligned} \tag{1.8}$$

converges strongly to a point of $x^* \in H$, which is the unique solution of the variational inequality

$$x^* \in F(T), \quad \langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{1.9}$$

Motivated and inspired by the iterative schemes (1.7) and (1.8), we introduce and study the hybrid iterative algorithm for solving some hierarchical fixed point problem of infinite family of strictly nonself pseudocontractions:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \beta_n S x_n + (1-\beta_n) x_n, \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + (1-\alpha_n) \sum_{i=1}^{\infty} \mu_i^{(n)} T_i y_n\right], \quad \forall n \geq 0, \end{aligned} \tag{1.10}$$

where S, f, and P_C are the same in (1.8), $T_i : C \mapsto H$ is a nonself k_i-strictly pseudocontraction. Under certain conditions on the parameters, the sequence $\{x_n\}$ generated by (1.10) converges strongly to a common fixed point of infinite family of k_i-strictly pseudocontractions, which solves the variational inequality

$$x^* \in \bigcap_{i=1}^{\infty} F(T_i), \quad \langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i). \tag{1.11}$$

So, our results extend and improve some results of other authors (such as [10–12]) from self-mappings to nonself-mappings, from nonexpansive mappings to k_i-strictly pseudocontraction, and from one mapping to a infinite family mappings.

2. Preliminaries

In this section, we recall some basic facts that will be needed in the proof of the main results.

Lemma 2.1 (see [13] demiclosedness principle). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \mapsto C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$; in particular if $y = 0$, then $x \in F(T)$.*

Lemma 2.2 (see [9]). *Let $x \in H$ and $z \in C$ be any points. The following results hold:*

(1) *that $z = P_C x$ if and only if there holds the relation:*

$$\langle x - y, y - z \rangle \leq 0, \quad \forall y \in C; \tag{2.1}$$

(2) *that* $z = P_C x$ *if and only if there holds the relation:*

$$\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C; \tag{2.2}$$

(3) *there holds the ration:*

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.3}$$

Lemma 2.3 (see [14]). *For all* $x, y \in H$, *the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \tag{2.4}$$

Lemma 2.4 (see [3]). *Let* $f : C \mapsto H$ *be a contraction with coefficient* $\gamma \in [0, 1)$ *and let* $T : C \mapsto C$ *be a nonexpansive mapping. Then for all* $x, y \in C$:

(1) *the mapping* $(I - f)$ *is strongly monotone with coefficient* $(1 - \gamma)$, *that is,*

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \gamma)\|x - y\|^2, \tag{2.5}$$

(2) *the mapping* $(I - T)$ *is monotone:*

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq 0. \tag{2.6}$$

Lemma 2.5 (see [15]). *Let* H *be a Hilbert space and let* C *be a nonempty convex subset of* H. *Let* $T : C \mapsto H$ *be a* k*-strictly pseudocontractive mapping with* $F(T) \neq \emptyset$. *Then* $F(P_C T) = F(T)$.

Lemma 2.6 (see [16]). *Let* H *be a Hilbert space and let* C *be a nonempty convex subset of* H. *Let* $T : C \mapsto H$ *be a* k*-strictly pseudocontractive mapping. Define a mapping* $Jx = \delta x + (1 - \delta)Tx$ *for all* $x \in C$. *Then as* $\delta \in [k, 1)$, J *is a nonexpansive mapping such that* $F(J) = F(T)$.

Lemma 2.7 (see [11]). *Let* H *be a Hilbert space and let* C *be a nonempty convex subset of* H. *Assume that* $T_i : C \mapsto H$ *is a countable family of* k_i*-strictly pseudocontraction for some* $0 \leq k_i < 1$ *and* $\sup\{k_i : i \in N\} < 1$ *such that* $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Assume that* $\{\mu_i\}$ *is a positive sequence such that* $\sum_{i=1}^{\infty} \mu_i = 1$. *Then* $\sum_{i=1}^{\infty} \mu_i T_i : C \mapsto H$ *is a* k*-strictly pseudocontraction with coefficient* $k = \sup\{k_i : i \in N\}$ *and* $F(\sum_{i=1}^{\infty} \mu_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$.

Lemma 2.8 (see [17]). *Let* $\{\alpha_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:* $\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n$, *where* (1) $\{\gamma_n\} \subset (0, 1)$, $\Sigma_{n=1}^{\infty} \gamma_n = \infty$; (2) $\limsup_{n \to \infty}(\delta_n / \gamma_n) \leq 0$ *or* $\Sigma_{n=1}^{\infty} |\delta_n| < \infty$, *then* $\lim_{n \to \infty} \alpha_n = 0$.

3. Main Results

In this section, we prove some strong convergence results on the iterative algorithm for solving hierarchical fixed point problem.

Theorem 3.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $f : C \mapsto H$ be a (possibly nonself) contraction with coefficient $\gamma \in [0,1)$, and let $S : C \mapsto C$ be a nonexpansive mapping. Let $T_i : C \mapsto H$ be a countable family of k_i-strictly (possibly nonself) pseudocontraction with $0 \le k_i \le k < 1$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by (1.10) with $\{\alpha_n\}$, $\{\beta_n\}$ in $[0,1]$. Suppose for each n, $\sum_{i=1}^{\infty} \mu_i^{(n)} = 1$, for all n and $\mu_i^{(n)} \ge 0$, for all $i \in N$. Assume that the parameters satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$;

(iii) $\lim_{n\to\infty}(|\alpha_{n+1} - \alpha_n|/\alpha_n) = 0$, $\lim_{n\to\infty}(|\beta_{n+1} - \beta_n|/\alpha_n) = 0$, *and* $\lim_{n\to\infty}(\sum_{i=1}^{\infty} |\mu_i^{(n)} - \mu_i^{(n-1)}|/\alpha_n) = 0$;

(iv) $\lim_{n\to\infty} \sum_{i=1}^{\infty} |\mu_i^{(n)} - \mu_i| = 0$ *and* $\sum_{i=1}^{\infty} \mu_i = 1$ $(\mu_i > 0)$.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$, which solves the variational inequality

$$\langle (I - f)x^*, x - x* \rangle \ge 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i). \tag{3.1}$$

Proof. The proof is divided into four steps.
Step 1. We show that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

For each $n \ge 0$, write $B_n = \sum_{i=1}^{\infty} \mu_i^{(n)} T_i$ and by Lemma 2.7, we have B_n is a k-strictly pseudocontraction on C and $F(B_n) = \bigcap_{i=1}^{\infty} F(T_i)$, for all $n \in N$. Therefore, the iterative algorithm (1.10) can be written as

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + (1 - \alpha_n) B_n y_n\right], \quad \forall n \ge 0. \end{aligned} \tag{3.2}$$

By condition (ii), without loss of generality, we may assume $\beta_n \le \alpha_n$, for all $n \ge 0$. Take $p \in \bigcap_{i=1}^{\infty} F(T_i)$ and we estimate $\|B_n y_n - p\|$. For fixed approximate $\delta \in [k,1)$, define a mapping $Jx = \delta x + (1 - \delta) B_n x$ and by Lemma 2.6, J is a nonexpansive mapping and $\mathrm{Fix}(J) = \mathrm{Fix}(B_n)$. So

$$\begin{aligned} \|B_n y_n - p\| &= \left\| \frac{1}{1-\delta}(J y_n - \delta y_n) - p \right\| \\ &\le \frac{1}{1-\delta}(\|J y_n - J p\| + \delta \|y_n - p\|) \\ &\le \frac{1+\delta}{1-\delta}\|y_n - p\|. \end{aligned} \tag{3.3}$$

Together with (3.2) and (3.3), we get

$$\begin{aligned}
\|x_{n+1}-p\| &= \|P_C[\alpha_n f(x_n)+(1-\alpha_n)B_n y_n]-P_C p\| \\
&\le \|\alpha_n f(x_n)+(1-\alpha_n)B_n y_n - p\| \\
&\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|B_n y_n - p\| \\
&\le \alpha_n\gamma\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\frac{1+\delta}{1-\delta}\|y_n-p\| \\
&\le \alpha_n\gamma\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\frac{1+\delta}{1-\delta}\beta_n\|Sx_n-p\| \\
&\quad +(1-\alpha_n)\frac{1+\delta}{1-\delta}(1-\beta_n)\|x_n-p\| \\
&\le \alpha_n\gamma\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\frac{1+\delta}{1-\delta}\beta_n\|Sx_n-Sp\| \\
&\quad +(1-\alpha_n)\frac{1+\delta}{1-\delta}\beta_n\|Sp-p\|+(1-\alpha_n)\frac{1+\delta}{1-\delta}(1-\beta_n)\|x_n-p\| \\
&\le \alpha_n\gamma\|x_n-p\|+(1-\alpha_n)\frac{1+\delta}{1-\delta}\|x_n-p\|+\frac{1+\delta}{1-\delta}\alpha_n\|Sp-p\|+\alpha_n\|f(p)-p\| \\
&= \left[1-\alpha_n\left(\frac{1+\delta}{1-\delta}-\gamma\right)\right]\|x_n-p\| \\
&\quad +\alpha_n\left(\frac{1+\delta}{1-\delta}-\gamma\right)\left[\frac{1}{((1+\delta)/(1-\delta))-\gamma}\left(\frac{1+\delta}{1-\delta}\|Sp-p\|+\|f(p)-p\|\right)\right].
\end{aligned} \tag{3.4}$$

Therefore, we obtain

$$\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{1}{((1+\delta)/(1-\delta))-\gamma}\left(\frac{1+\delta}{1-\delta}\|Sp-p\|+\|f(p)-p\|\right)\right\}, \tag{3.5}$$

which gives the results that the sequence $\{x_n\}$ is bounded and so are $\{f(x_n)\}$, $\{y_n\}$, $\{B_n x_n\}$, $\{B_n y_n\}$.

Step 2. Now we show that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$. Let

$$v_n = \alpha_n f(x_n)+(1-\alpha_n)B_n y_n. \tag{3.6}$$

Next we estimate $\|x_{n+1}-x_n\|$. From (3.2),we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|P_C[v_n]-P_C[v_{n-1}]\| \\
&\le \|v_n-v_{n-1}\| \\
&= \|\alpha_n f(x_n)+(1-\alpha_n)B_n y_n-\alpha_{n-1}f(x_{n-1})-(1-\alpha_{n-1})B_{n-1}y_{n-1}\| \\
&= \|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})f(x_{n-1})+(1-\alpha_n)(B_n y_n-B_n y_{n-1}) \\
&\qquad +(1-\alpha_n)B_n y_{n-1}+(1-\alpha_{n-1})(B_n y_{n-1}-B_{n-1}y_{n-1})\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1}) - B_n y_{n-1}\| \\
&\quad + (1-\alpha_n)\frac{1+\delta}{1-\delta}\|y_n - y_{n-1}\| + (1-\alpha_{n-1})\sum_{i=1}^{\infty}\left|\mu_i^{(n)} - \mu_i^{(n-1)}\right| \|T_i y_{n-1}\| \\
&\leq \alpha_n \gamma \|x_n - x_{n-1}\| + (1-\alpha_n)\frac{1+\delta}{1-\delta}\|y_n - y_{n-1}\| \\
&\quad + M\left[|\alpha_n - \alpha_{n-1}| + \sum_{i=1}^{\infty}\left|\mu_i^{(n)} - \mu_i^{(n-1)}\right|\right],
\end{aligned}
\tag{3.7}
$$

where M is a constant such that

$$
\sup_{n\in N}\{\|f(x_{n-1}) - B_n y_{n-1}\| + \|S(x_{n-1}) - x_{n-1}\| + \|T_i y_{n-1}\|\} \leq M. \tag{3.8}
$$

From (3.2), we also obtain

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \|\beta_n S x_n + (1-\beta_n)x_n - \beta_{n-1} S x_{n-1} - (1-\beta_{n-1})x_{n-1}\| \\
&= \|\beta_n (S x_n - S x_{n-1}) + (\beta_n - \beta_{n-1}) S x_{n-1} \\
&\qquad + (1-\beta_n)(x_n - x_{n-1}) + (\beta_{n-1} - \beta_n)x_{n-1}\| \\
&= \|\beta_n (S x_n - S x_{n-1}) + (1-\beta_n)(x_n - x_{n-1}) + (\beta_n - \beta_{n-1})(S x_{n-1} - x_{n-1})\| \\
&\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| M.
\end{aligned}
\tag{3.9}
$$

Together with (3.7) and (3.9), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|v_n - v_{n-1}\| \\
&\leq \alpha_n \gamma \|x_n - x_{n-1}\| + (1-\alpha_n)\frac{1+\delta}{1-\delta}\|x_n - x_{n-1}\| + (1-\alpha_n)|\beta_n - \beta_{n-1}|\frac{1+\delta}{1-\delta}M \\
&\quad + |\alpha_n - \alpha_{n-1}| M + (1-\alpha_{n-1})\sum_{i=1}^{\infty}\left|\mu_i^{(n)} - \mu_i^{(n-1)}\right| M \\
&\leq \left[1 - \alpha_n\left(\frac{1+\delta}{1-\delta} - \gamma\right)\right]\|x_n - x_{n-1}\| \\
&\quad + M\left[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\frac{1+\delta}{1-\delta} + \sum_{i=1}^{\infty}\left|\mu_i^{(n)} - \mu_i^{(n-1)}\right|\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[1-\alpha_n\left(\frac{1+\delta}{1-\delta}-\gamma\right)\right]\|x_n-x_{n-1}\| + M\left(\alpha_n\left(\frac{1+\delta}{1-\delta}-\gamma\right)\right) \\
&\quad \times \left[\frac{1}{\alpha_n(((1+\delta)/(1-\delta))-\gamma)}\right. \\
&\qquad \left.\times\left(|\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}|\frac{1+\delta}{1-\delta} + \sum_{i=1}^{\infty}\left|\mu_i^{(n)}-\mu_i^{(n-1)}\right|\right)\right].
\end{aligned} \tag{3.10}
$$

By Lemma 2.8 and conditions (i)–(iii), we immediately get $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$.

Step 3. Next we prove that $\|x_n - P_C(\sum_{i=1}^{\infty}\mu_i T_i)x_n\| \to 0$ as $n \to \infty$.

Let $B = \sum_{i=1}^{\infty}\mu_i T_i$. By Lemma 2.7 and condition (iv), we get the results that $B : C \mapsto H$ is a k-strictly pseudocontraction with $F(B) = \bigcap_{i=1}^{\infty} F(T_i)$ and $B_n x \to Bx$ as $n \to \infty$, for any $x \in C$,

$$
\begin{aligned}
\|x_n - P_C B x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - P_C B x_n\| \\
&= \|x_n - x_{n+1}\| + \|P_C[v_n] - P_C B x_n\| \\
&\le \|x_n - x_{n+1}\| + \left\|\alpha_n f(x_n) + (1-\alpha_n)B_n y_n - B_n x_n\right\| + \|B_n x_n - B x_n\| \\
&= \|x_n - x_{n+1}\| + \left\|\alpha_n\left(f(x_n) - B_n x_n\right) + (1-\alpha_n)\left(B_n y_n - B_n x_n\right)\right\| \\
&\quad + \|B_n x_n - B x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - B_n x_n\right\| \\
&\quad + (1-\alpha_n)\frac{1+\delta}{1-\delta}\left\|y_n - x_n\right\| + \|B_n x_n - B x_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - B_n x_n\right\| \\
&\quad + (1-\alpha_n)\frac{1+\delta}{1-\delta}\beta_n\|S x_n - x_n\| + \|B_n x_n - B x_n\|.
\end{aligned} \tag{3.11}
$$

Because $\alpha_n \to 0$, $\beta_n \to 0$, $\|x_{n+1}-x_n\| \to 0$, and $B_n x \to Bx$, so we obtain $\|x_n - \sum_{i=1}^{\infty}\mu_i T_i x_n\| \to 0$ as $n \to \infty$.

Step 4. Now we show that $\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0$, where $x^* = P_{F(B)} f(x^*)$.

Since the sequence $\{x_n\}$ is bounded, we take s subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle f(x^*) - x^*, x_{n_k} - x^*\rangle$ and $x_{n_k} \rightharpoonup x'$. Notice that $\|x_n - P_C B x_n\| \to 0$ and by Lemmas 2.1 and 2.5, we have $x' \in \mathrm{Fix}(P_C B) = F(B) = \bigcap_{i=1}^{\infty} F(T_i)$. Then

$$
\lim_{k\to\infty}\left\langle f(x^*) - x^*, x_{n_k} - x^*\right\rangle = \left\langle f(x^*) - x^*, x' - x^*\right\rangle \le 0, \quad x' \in F(B). \tag{3.12}
$$

Now, by Lemma 2.2, we get $\langle P_C[v_n] - v_n, P_C[v_n] - x^*\rangle \le 0$. Therefore, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle P_C[v_n] - x^*, x_{n+1} - x^*\rangle \\
&= \langle P_C[v_n] - v_n, x_{n+1} - x^*\rangle + \langle v_n - x^*, x_{n+1} - x^*\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle P_C[v_n] - v_n, P_C[v_n] - x^* \rangle + \langle v_n - x^*, x_{n+1} - x^* \rangle \\
&\le \langle v_n - x^*, x_{n+1} - x^* \rangle \\
&= \langle \alpha_n f(x_n) + (1-\alpha_n) B_n y_n - x^*, x_{n+1} - x^* \rangle \\
&= \alpha_n \langle (f(x_n) - f(x^*)), x_{n+1} - x^* \rangle + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\quad + (1-\alpha_n) \langle B_n y_n - x^*, x_{n+1} - x^* \rangle \\
&\le \alpha_n \gamma \|x_n - x^*\| \cdot \|x_{n+1} - x^*\| + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\quad + (1-\alpha_n) \langle B_n [\beta_n S x_n + (1-\beta_n) x_n] - B_n x^*, x_{n+1} - x^* \rangle \\
&\le \alpha_n \gamma \|x_n - x^*\| \cdot \|x_{n+1} - x^*\| + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\quad + (1-\alpha_n) \| \beta_n (S x_n - S x^*) + \beta_n (S x^* - x^*) \\
&\quad + (1-\beta_n)(x_n - x^*) \| \cdot \|x_{n+1} - x^*\| \\
&\le \alpha_n \gamma \|x_n - x^*\| \cdot \|x_{n+1} - x^*\| + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\quad + (1-\alpha_n) \|x_n - x^*\| \cdot \|x_{n+1} - x^*\| + (1-\alpha_n)\beta_n \|S x^* - x^*\| \cdot \|x_{n+1} - x^*\| \\
&= [1 - (1-\gamma)\alpha_n] \|x_n - x^*\| \cdot \|x_{n+1} - x^*\| + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\quad + (1-\alpha_n)\beta_n \|S x^* - x^*\| \cdot \|x_{n+1} - x^*\| \\
&\le \frac{1-(1-\gamma)\alpha_n}{2} \left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 \right) + \alpha_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\quad + (1-\alpha_n)\beta_n \|S x^* - x^*\| \cdot \|x_{n+1} - x^*\|.
\end{aligned}
\tag{3.13}
$$

Hence it follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \left[1 - \frac{2(1-\gamma)\alpha_n}{1+(1-\gamma)\alpha_n} \right] \|x_n - x^*\|^2 + \frac{2\alpha_n}{1+(1-\gamma)\alpha_n} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\quad + \frac{2(1-\alpha_n)\beta_n}{1+(1-\gamma)\alpha_n} \|S x^* - x^*\| \cdot \|x_{n+1} - x^*\| \\
&= \left[1 - \frac{2(1-\gamma)\alpha_n}{1+(1-\gamma)\alpha_n} \right] \|x_n - x^*\|^2 \\
&\quad + \frac{2(1-\gamma)\alpha_n}{1+(1-\gamma)\alpha_n} \left[\frac{1}{(1-\gamma)} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \right. \\
&\qquad \left. + \frac{(1-\alpha_n)\beta_n}{(1-\gamma)\alpha_n} \|S x^* - x^*\| \cdot \|x_{n+1} - x^*\| \right].
\end{aligned}
\tag{3.14}
$$

Now, by Lemma 2.8, conditions (i)–(iii), and

$$\limsup_{n\to\infty}\left[\frac{1}{(1-\gamma)}\langle f(x^*)-x^*,x_{n+1}-x^*\rangle+\frac{(1-\alpha_n)\beta_n}{(1-\gamma)\alpha_n}\|Sx^*-x^*\|\cdot\|x_{n+1}-x^*\|\right]\le 0, \tag{3.15}$$

we have $x_n \to x^* \in \bigcap_{i=1}^{\infty} F(T_i)$ as $n \to \infty$ and x^* also solves the variational inequality

$$\langle (I-f)x^*, x-x^*\rangle \ge 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i). \tag{3.16}$$

This completes the proof. □

From Theorem 3.1, if we take $S = I$ or $\beta_n = 0$, for all $n \in N$, we get the following corollary.

Corollary 3.2. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $f : C \mapsto H$ be a (possibly nonself) contraction with coefficient $\gamma \in [0,1)$ and let $T_i : C \mapsto H$ be a countable family of k_i-strictly (possibly nonself) pseudocontraction with $0 \le k_i < 1$ and $\sup\{k_i : i \in N\} < 1$ such that $\bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{gathered} x_0 = x \in C, \\ x_{n+1} = P_C\left[\alpha_n f(x_n) + (1-\alpha_n)\sum_{i=1}^{\infty}\mu_i^{(n)} T_i x_n\right], \quad \forall n \ge 0, \end{gathered} \tag{3.17}$$

with $\{\alpha_n\}$ in $[0,1]$. Suppose for each n, $\sum_{i=1}^{\infty}\mu_i^{(n)} = 1$, for all n and $\mu_i^{(n)} \ge 0$, for all $i \in N$. Assume that the parameters satisfied the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\lim_{n\to\infty}(|\alpha_{n+1}-\alpha_n|/\alpha_n) = 0$, *and* $\lim_{n\to\infty}((\sum_{i=1}^{\infty}|\mu_i^{(n)}-\mu_i^{(n-1)}|)/\alpha_n) = 0$;

(iii) $\lim_{n\to\infty}\sum_{i=1}^{\infty}|\mu_i^{(n)}-\mu_i| = 0$ *and* $\sum_{i=1}^{\infty}\mu_i = 1$ $(\mu_i > 0)$.

Then the the sequence $\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$, which solves the variational inequality

$$\langle (I-f)x^*, x-x^*\rangle \ge 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i). \tag{3.18}$$

Remark 3.3. Theorem 3.1 extends and improves Theorem YCL in the following way. The nonexpasnsive self-mapping $T : C \mapsto C$ is extended to a infinite family of nonself k_i-strictly pseudocontraction $T_i : C \mapsto H$. If we take $k_i = 0$, $i \in N$ in Theorem 3.1, then $B = \sum_{i=1}^{\infty}\mu_i T_i$ reduces to a nonexpasnsive (possibly nonself) mapping, thus Theorem 3.1 reduces to Theorem YCL.

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