Research Article

Viscosity Iterative Schemes for Finding Split Common Solutions of Variational Inequalities and Fixed Point Problems

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We introduce some new iterative schemes based on viscosity approximation method for finding a split common element of the solution set of a pair of simultaneous variational inequalities for inverse strongly monotone mappings in real Hilbert spaces with a family of infinitely nonexpansive mappings. Some strong convergence theorems are also given. Our results generalize and improve some well-known results in the literature and references therein.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let I be the identity mapping on H and C be a nonempty closed convex subset of H . Let $T : C \rightarrow H$ be a nonlinear operators. Then the canonical variational inequality problem for the operator T ((VIP) _{T} or (VIP), for short) is to find $u \in C$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in C. \quad (1.1)$$

We use the symbol $VI(C, T)$ to denote the solution set of (VIP), that is

$$VI(C, T) = \{u \in C : \langle Tu, v - u \rangle \geq 0, \forall v \in C\}. \quad (1.2)$$

(VIP) was extensively investigated and generalized to the vector variational inequality problems for single-valued or multivalued maps and contains optimization problems, quasi-variational inequality problems, equilibrium problems, fixed-point problems, complementary problems, bilevel problems, and semi-infinite problems as special cases and applications; see [1–6] and references therein.

Let $S, T : C \rightarrow H$ be two nonlinear operators. In [7], some authors have considered the following pair of simultaneous variational inequality problems for operators S and T ((PSVIP) $_{S,T}$, for short):

$$(\text{PSVIP})_{S,T} \text{ Find } u \in C \text{ such that } \langle Su, v - u \rangle \geq 0 \text{ and } \langle Tu, v - u \rangle \geq 0, \quad \forall v \in C. \quad (1.3)$$

An element $u \in C$ is a solution of $(\text{PSVIP})_{S,T}$ if and only if $u \in \text{VI}(C, S) \cap \text{VI}(C, T)$. Clearly, $(\text{PSVIP})_{S,T}$ reduces to (VIP) if $S = T$.

Example 1.1. Let $\mathbb{R}$ with usual inner product and let $a, b \in \mathbb{R}$ with $a < b$. Define two real-valued functions T_1, T_2 by $T_1x = x^2$, $T_2x = x^4$, for all $x \in [a, b]$. Then $T'_1x = 2x$, $T'_2x = 4x^3$ and there exists $x_0 \in [a, b]$ such that $T_1x_0 = \min_{x \in [a, b]} T_1x$ and $T_2x_0 = \min_{x \in [a, b]} T_2x$. If $x_0 \in (a, b)$, then $T_1x_0 = T_2x_0 = 0$; if $x_0 = a$, then $T'_1x_0 \geq 0$ and $T'_2x_0 \geq 0$; if $x_0 = b$, then $T'_1x_0 \leq 0$ and $T'_2x_0 \leq 0$. So we have

$$\langle T'_1x_0, x - x_0 \rangle = T'_1x_0(x - x_0) \geq 0, \quad \langle T'_2x_0, x - x_0 \rangle = T'_2x_0(x - x_0) \geq 0, \quad \forall x \in [a, b] \quad (1.4)$$

or $x_0 \in \text{VI}(C, T'_1) \cap \text{VI}(C, T'_2)$ which means that x_0 is the solution of $(\text{PSVIP})_{T'_1, T'_2}$.

Obviously, the problem $(\text{PSVIP})_{S,T}$ is considered in the same subset of the same space. But many cases, two variational inequality problems often lie in different subset of spaces. So, as a further development of the problem $(\text{PSVIP})_{S,T}$, Censor et al. [8] presented a split variational inequality problem. Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$ and $K \subset H_2$ two closed convex sets. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. $T : C \rightarrow H_1$ and $S : K \rightarrow H_2$ are two nonlinear operators. The split variational inequality problem for T and S ((SVIP) $_{T,S}$, for short) is defined as follows:

$$\begin{aligned} (\text{PSVIP})_{S,T} \text{ Find } p \in C \text{ such that } \langle Tp, v - p \rangle \geq 0, \quad \forall v \in C, \\ \text{and } u := Ap \in K \text{ solves } \langle Su, w - u \rangle \geq 0, \quad \forall w \in K. \end{aligned} \quad (1.5)$$

It is well known to find a solution of (VIP) or a common element of the solution set of (VIP) and a fixed point of nonlinear operators, which has been studied by many authors (see [9–16]) using all kinds of auxiliary techniques and formulations. In 2005, Iiduka and Takahashi [9] established the following iteration scheme: let $x_1 \in H$ be arbitrary, define

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S_1 P_C(x_n - \lambda_n T x_n), \quad (1.6)$$

where S_1 is a nonexpansive mapping. They proved that the sequence $\{x_n\}$ defined by (1.6) strongly converge to $x^* \in F(S_1) \cap \text{VI}(C, T)$, if the coefficient α_n, λ_n satisfy the following conditions:

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty. \quad (1.7)$$

In 2007, Chen et al. [10] studied the following iterative process:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S_1 P_C(x_n - \lambda_n T x_n), \quad (1.8)$$

where S_1 is a nonexpansive mapping. If $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, then they proved that $\{x_n\}$ converges strongly to $q \in F(S_1) \cap \text{VI}(C, T)$, which solves the variational inequality:

$$\langle f q - q, p - q \rangle \leq 0, \quad \forall p \in F(S_1) \cap \text{VI}(C, T). \quad (1.9)$$

For some split common solution problems, they have been studied by some authors; see [17, 18] and therein references. In this paper, we continue to study the (SVIP) and introduce some new iterative schemes based on viscosity approximation method for finding a common element of the fixed points set of nonexpansive mappings and the split solution set of a pair of variational inequalities for inverse strongly monotone mappings in real Hilbert spaces. Our results are new development of finding a common element of fixed point of nonlinear operators and variational inequality problems.

2. Preliminaries

In this paper, we use symbols $\rightarrow$ and $\rightharpoonup$ to denote strong and weak convergence, respectively. A Banach space $(X, \|\cdot\|)$ is said to satisfy *Opial's condition*, if for each sequence $\{x_n\}$ in X with $x_n \rightharpoonup x \in X$, we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|, \quad \forall y \in X, y \neq x. \quad (2.1)$$

It is well known that each Hilbert space satisfies Opial's condition; see, for example, [19]. Let $T : X \rightarrow X$ be a mapping. In this paper, the set of fixed points of T is denoted by $F(T)$.

A set-valued mapping $T_1 : H \rightarrow 2^H$ is said to be *monotone*, if for all $x, y \in H$, $f \in T_1 x$, and $g \in T_1 y$ imply that $\langle f - g, x - y \rangle \geq 0$. A monotone mapping $T_1 : H \rightarrow H$ is said to be *maximal*, if the graph $G(T_1)$ of T_1 is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T_1 is maximal, if and only if for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \geq 0$ for every $(y, g) \in G(T_1)$ implies that $f \in T_1 x$. Let $T : C \rightarrow H$ be a monotone mapping and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C\}$. Define

$$T_1 v = \begin{cases} T v + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \quad (2.2)$$

Then T_1 is maximal monotone and $\mathbf{0} \in T_1 v$ if and only if $v \in \text{VI}(C, T)$, where $\mathbf{0}$ is the zero vector of H ; see, for example, [9, 20, 21] for more details.

For any $x \in H$, there exists a unique nearest point in C , denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the *projection operator* (or *metric projection*) from H onto C .

Let H_1 and H_2 be two Hilbert spaces. Let $A : H_1 \rightarrow H_2$ and $B : H_2 \rightarrow H_1$ be two bounded linear operators. B is called the adjoint operator (or adjoint) of A , if for all $z \in H_1$, $w \in H_2$, B satisfies $\langle Az, w \rangle = \langle z, Bw \rangle$. It is known that the adjoint operator of a bounded linear operator on a Hilbert space always exists and is bounded linear and unique. Moreover, it is not hard to show that if B is an adjoint operator of A , then $\|A\| = \|B\|$.

A mapping $T : C \rightarrow C$ is said to be

- (1) v -expansive if there exists a constant $v > 0$ such that $\|Tx - Ty\| \geq v\|x - y\|$ for all $x, y \in C$. In particular, if $v = 1$, then T is called expansive.
- (2) v -strongly monotone if there exists a constant $v > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq v\|x - y\|^2, \quad \forall x, y \in C. \quad (2.3)$$

Clearly, any v -strongly monotone mapping is v -expansive.

- (3) u -inverse strongly monotone if there exists a constant $u > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq u\|Tx - Ty\|^2, \quad \forall x, y \in C. \quad (2.4)$$

- (4) Relaxed u -cocoercive if there exists a constant $u > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-u)\|Tx - Ty\|^2, \quad \forall x, y \in C. \quad (2.5)$$

- (5) Relaxed (u, v) -cocoercive if there exists constants $u, v > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-u)\|Tx - Ty\|^2 + v\|x - y\|^2, \quad \forall x, y \in C. \quad (2.6)$$

Especially, if $u = 0$, then T is v -strongly monotone. So this class of mapping is more general than the class of strongly monotone mapping.

- (6) An α -Lipschitz mapping if there exists a constant $\alpha > 0$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$. In particular, if $0 < \alpha < 1$ ($\alpha = 1$, resp.), then T is called a contraction (a nonexpansive mapping, resp.)

Remark ST (see [9]). If T is v -strongly monotone and μ -Lipschitz continuous, that is, $\|Tx - Ty\| \leq \mu\|x - y\|$ for all $x, y \in C$, then T is (v/μ^2) -inverse strongly monotone.

Example 2.1. Let $Tx = -2x$, for all $x \in \mathbb{R}$. Then it is easy to see that for any $x, y \in \mathbb{R}$,

$$\langle Tx - Ty, x - y \rangle = -2|x - y|^2 \geq (-1)|Tx - Ty|^2 + |x - y|^2. \quad (2.7)$$

Hence T is a relaxed $(1, 1)$ -cocoercive mapping, but T is not a strongly monotone mapping.

Now, let $\{T_i\}_{i \in \mathbb{N}}$ be a family of infinitely nonexpansive mappings. In [22], a mapping W_n is defined by the following:

$$\begin{aligned}
 U_{n,n+1} &= I, \\
 U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\
 U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\
 &\vdots \\
 U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\
 U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\
 &\vdots \\
 U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\
 W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I,
 \end{aligned} \tag{2.8}$$

where $\{\lambda_i\}_{i \in \mathbb{N}} \subset [0, 1]$. Such a mapping W_n is called the W -mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$.

The following properties for a W -mapping are well known.

Theorem 2.2 (see [22, 23]). *Let C be a nonempty closed convex subset of a Hilbert space E , let $T_1, T_2, \dots$ be a family of infinitely nonexpansive mappings from C into itself such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty, and let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 < \lambda_i \leq b < 1$ for any $i \in \mathbb{N}$. Then the following statements hold:*

- (1) W_n is a nonexpansive mapping and $F(W_n) = \bigcap_{i=1}^n F(T_i)$.
- (2) For each $x \in C$ and for each positive integer k , the limit $\lim_{n \rightarrow \infty} U_{n,k}x$ exists.
- (3) The mapping $W : C \rightarrow C$ defined by $Wx := \lim_{n \rightarrow \infty} W_n x = \lim_{n \rightarrow \infty} U_{n,1}x$, $x \in C$, is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$ and it is called the W -mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$.

Theorem 2.3 (see [23]). *Let C be a nonempty closed convex subset of a Hilbert space H , $T_1, T_2, \dots$ be nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) = \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \leq b < 1$ for any $i \in \mathbb{N}$. If K is any bounded subset of C , then*

$$\lim_{n \rightarrow \infty} \sup_{x \in K} \|Wx - W_n x\| = 0. \tag{2.9}$$

In particular, if $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence in C , then $\lim_{n \rightarrow \infty} \|Wx_n - W_n x_n\| = 0$.

The following results are crucial in this paper.

Lemma 2.4 (see [19]). *For a given $z \in H$, $x \in C$ satisfies the inequality $\langle x - z, y - x \rangle \geq 0$, for all $y \in C$ if and only if $x = P_C(z)$, where P_C is a projection operator from H onto C .*

It is well known that the projection operator P_C is nonexpansive and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H. \quad (2.10)$$

Lemma 2.5 (see [9]). *The element $u \in C$ is a solution of $(VIP)_T$ if and only if $u \in C$ satisfies the relation $u = P_C(u - \rho Tu)$, where P_C is the projection operator, $\rho > 0$ is a constant.*

Lemma 2.6 (see [24]). *Let $\{a_n\}$ be a nonnegative real sequence satisfying the following condition:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n b_n, \quad \forall n \geq n_0, \quad (2.11)$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence in $\mathbf{R}$ such that

$$(i) \sum_{n=0}^{\infty} \lambda_n = \infty;$$

$$(ii) \limsup_{n \rightarrow \infty} b_n \leq 0 \text{ or } \sum_{n=0}^{\infty} \lambda_n b_n \text{ is convergent.}$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.7 (see [25]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all integers $n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.*

Lemma 2.8 (see [26]). *Let E be a real Banach space and $J : E \rightarrow 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$ the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y). \quad (2.12)$$

Epecially, when $E = H$, then $J = I$. So, from Lemma 2.8, one has

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H. \quad (2.13)$$

The following result is simple, but it is very useful in this paper.

Lemma 2.9. *Let $\{a_n\}, \{b_n\}$ be two nonnegative real sequences. If $\lim_{n \rightarrow \infty} a_n = 0$, then $\liminf_{n \rightarrow \infty} (a_n + b_n) = \liminf_{n \rightarrow \infty} b_n$.*

3. Main Results

In this section, we construct an iteration scheme including a pair of mappings $T : C \rightarrow H_1$ and $S : K \rightarrow H_2$ which are u -inverse strongly monotone to solve the split variational inequality problem. For the purpose we first give the following Lemmas.

Lemma 3.1 (see page 3 in [9]). *Let $T : C \rightarrow H$ be a u -inverse strongly monotone mapping. Then $I - \lambda T$ is nonexpansive for any $\lambda \in [0, 2u]$.*

Example 3.2. Let $Tx = 3x$ for all $x \in \mathbb{R}$ and $u = 1/6$. Since

$$\langle Tx - Ty, x - y \rangle = 3|x - y|^2 \geq u|Tx - Ty|^2, \quad (3.1)$$

T is u -inversely monotone. Let $\lambda \in [0, 1/3] = [0, 2u]$. It is easy to see that

$$|(I - \lambda T)x - (I - \lambda T)y| = (1 - 3\lambda)|x - y| \leq |x - y|. \quad (3.2)$$

So $I - \lambda T$ is nonexpansive for all $\lambda \in [0, 2u]$.

Applying Lemma 3.1, we have the following important result.

Lemma 3.3. *Let $T, S : C \rightarrow H$ be two u -inverse strongly monotone mappings and $S_1 : C \rightarrow C$ be a nonexpansive mapping. Then for any given sequences $\{r_n\}$ and $\{s_n\}$ in $[0, 2u]$, $P_C(I - s_n T)$, $P_C(I - r_n S)$, $S_1 P_C(I - s_n T)$ and $S_1 P_C(I - r_n S)$ are all nonexpansive for all $n \in \mathbb{N}$.*

The following conclusion is immediate from Lemma 2.5.

Lemma 3.4. *The element $u \in C$ is a solution of $(SVIP)_{TS}$ if and only if $u \in C$ satisfies the relation*

$$u = P_C(u - \rho Tu), \quad Au = P_K(u - \rho SAu), \quad (3.3)$$

where P_C and P_K are the projection operators, $\rho > 0$ is a constant.

Theorem 3.5. *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1, K \subset H_2$ two nonempty closed convex sets. Let $T : C \rightarrow H_1$ and $S : K \rightarrow H_2$ be u -inversely monotone. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator with adjoint operator A^* . Let $f : C \rightarrow C$ be a contraction with contraction constant α . Let $\{T_i\}_{i \in \mathbb{N}}$ be a family of infinitely nonexpansive mappings of C into itself and U a nonexpansive mapping of K into itself such that $\Omega = \{p \in (\bigcap_{i=1}^{\infty} F(T_i)) \cap \text{VI}(C, T) : Ap \in F(U) \cap \text{VI}(K, S)\} \neq \emptyset$. Let ξ be a real number and $\{\lambda_i\}_{i \in \mathbb{N}}$ be a sequence of real numbers such that $0 < \lambda_i \leq \xi < 1$ for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let W_n be the W -mapping of C into itself generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily,} \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_K(I - \beta_n S)Ay_n, \\ z_n &= (1 - \theta)x_n + \theta W_n P_C(y_n + rA^*(Ul_n - Ay_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \quad (3.4)$$

where $r \in (0, 1/\|A\|^2)$ and $\theta \in (0, 1)$ are two constants and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in $(0, 1)$. If $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ further satisfy the following conditions:

- (C₁) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (C₂) $\{\beta_n\} \subset [a, b]$ and $\lim_{n \rightarrow \infty} |\beta_{n+1} - \beta_n| = 0$, where $0 < a, b < 2u$,

then the following statements hold:

- (a) there exists a unique $q \in \Omega$ such that $P_\Omega f(q) = q$;
- (b) $\{x_n\}$ converges strongly to q .

Proof. Let $p \in \Omega$. By Lemma 3.3, $P_C(I - \beta_n T)$, and $P_K(I - \beta_n S)$ are nonexpansive for all $n \in \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N} \cup \{0\}$, by (3.4) and Lemma 3.4, we obtain the following inequalities:

$$\begin{aligned} \|y_n - p\| &= \|P_C(I - \beta_n T)x_n - P_C(I - \beta_n T)p\| \leq \|x_n - p\|, \\ \|l_n - Ap\| &= \|P_K(I - \beta_n S)Ay_n - P_K(I - \beta_n S)Ap\| \leq \|Ay_n - Ap\|, \\ \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n) \|z_n - p\| \leq \alpha_n \|f(x_n) - p\| + \|z_n - p\|. \end{aligned} \quad (3.5)$$

Let $h_n = P_C(y_n + rA^*(Ul_n - Ay_n))$ for $n \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned} \|W_n h_n - p\|^2 &\leq \|h_n - p\|^2 = \|P_C(y_n + rA^*(Ul_n - Ay_n)) - p\|^2 \leq \|y_n + rA^*(Ul_n - Ay_n) - p\|^2 \\ &= \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 + 2r\langle y_n - p, A^*(Ul_n - Ay_n) \rangle \\ &\leq \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\ &\quad + 2r\langle A(y_n - p) + Ul_n - Ay_n - (Ul_n - Ay_n), Ul_n - Ay_n \rangle \\ &= \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\ &\quad + 2r\left\{\frac{1}{2}\|Ul_n - Ap\|^2 + \frac{1}{2}\|Ul_n - Ay_n\|^2 - \|Ay_n - Ap\|^2 - \|Ul_n - Ay_n\|^2\right\} \\ &\leq \|y_n - p\|^2 + r^2\|A^*\|^2\|Ul_n - Ay_n\|^2 - r\|Ul_n - Ay_n\|^2 \\ &= \|y_n - p\|^2 - r(1 - r\|A^*\|^2)\|Ul_n - Ay_n\|^2, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \|z_n - p\|^2 &= \|(1 - \theta)(x_n - p) + \theta(W_n h_n - p)\|^2 \\ &\leq (1 - \theta)\|x_n - p\|^2 + \theta\|W_n h_n - p\|^2 \\ &\leq (1 - \theta)\|x_n - p\|^2 + \theta\|h_n - p\|^2 \\ &\leq (1 - \theta)\|x_n - p\|^2 + \theta\|y_n - p\|^2 \\ &\leq \|x_n - p\|^2, \end{aligned} \quad (3.7)$$

for all $n \in \mathbb{N} \cup \{0\}$. Next, we will show that the conclusion is true by several steps.

Step 1. We show that all $\{x_n\}, \{y_n\}, \{z_n\}, \{Tx_n\}, \{SAy_n\}, \{l_n\}$, and $\{W_n h_n\}$ are bounded.

To prove it, it suffices to show $\{x_n\}$ is bounded. Let $p \in \Omega$. We claim that

$$\|x_n - p\| \leq \mathcal{L} := \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \alpha} \right\} \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (3.8)$$

Indeed, it is obvious that (3.8) is true for $n = 0$. Assume that (3.7) is true for $n = k$, $k \in \mathbb{N}$. Since $\|y_k - p\| \leq \|x_k - p\|$ and $\|z_k - p\| \leq \|x_k - p\|$ by (3.5) and (3.7), it follows from (3.5) that

$$\begin{aligned} \|x_{k+1} - p\| &\leq \alpha_k \|f(x_k) - p\| + (1 - \alpha_k) \|z_k - p\| \\ &= \alpha_k \|f(x_k) - f(x_k) + f(x_k) - p\| + (1 - \alpha_k) \|z_k - p\| \\ &\leq \alpha_k \alpha \|x_k - p\| + \alpha_k \|f(p) - p\| + (1 - \alpha_k) \|x_k - p\| \\ &\leq (1 - \alpha_k(1 - \alpha)) \|x_k - p\| + \alpha_k \|f(p) - p\| \leq \mathcal{L}, \end{aligned} \quad (3.9)$$

which prove that (3.8) is true for $n = k + 1$. By induction, (3.8) holds for all $n \in \mathbb{N} \cup \{0\}$. Hence, by (3.8), we know that $\{x_n\}$ is bounded and so are $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{Tx_n\}$, $\{SAy_n\}$, $\{l_n\}$, $\{h_n\}$, $\{W_nh_n\}$, $\{l_n\}$, and $\{Ul_n\}$. This also means that there exists a bounded subset $C_1 \subset C$ such that

$$\{x_n\}, \{y_n\}, \{z_n\}, \{Tx_n\}, \{h_n\}, \{W_nh_n\} \subset C_1. \quad (3.10)$$

Step 2. Prove $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.
For each $n \in \mathbb{N} \cup \{0\}$, by Lemma 3.1,

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|P_C(I - \beta_{n+1}T)x_{n+1} - P_C(I - \beta_{n+1}T)x_n\| + \|P_C(I - \beta_{n+1}T)x_n - P_C(I - \beta_nT)x_n\| \\ &\leq \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|Tx_n\|. \end{aligned} \quad (3.11)$$

Similarly,

$$\begin{aligned} \|l_{n+1} - l_n\| &\leq \|P_K(I - \beta_{n+1}S)Ay_{n+1} - P_K(I - \beta_{n+1}S)Ay_n\| \\ &\quad + \|P_K(I - \beta_{n+1}S)Ay_n - P_K(I - \beta_nS)Ay_n\| \\ &\leq \|Ay_{n+1} - Ay_n\| + |\beta_{n+1} - \beta_n| \|SAy_n\|, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (3.12)$$

Since $h_n = P_C(y_n + rA^*(U l_n - Ay_n))$, $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\|h_{n+1} - h_n\|^2 &\leq \|y_{n+1} + rA^*(U l_{n+1} - Ay_{n+1}) - (y_n + rA^*(U l_n - Ay_n))\|^2 \\
&\leq \|y_{n+1} - y_n\|^2 + \|rA^*(U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n))\|^2 \\
&\quad + 2r\langle y_{n+1} - y_n, A^*(U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n)) \rangle \\
&\leq \|y_{n+1} - y_n\|^2 + r^2\|A^*\|^2\|U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n)\|^2 \\
&\quad + 2r\langle Ay_{n+1} - Ay_n + U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n), U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n) \rangle \\
&\quad - 2r\langle U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n), U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n) \rangle \\
&= \|y_{n+1} - y_n\|^2 + r^2\|A^*\|^2\|U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n)\|^2 \\
&\quad + 2r\left\{\frac{1}{2}\|U l_{n+1} - U l_n\|^2 + \frac{1}{2}\|U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n)\|^2 - \frac{1}{2}\|Ay_{n+1} - Ay_n\|^2\right\} \\
&\quad - 2r\|U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n)\|^2 \\
&= \|y_{n+1} - y_n\|^2 - r(1 - r\|A^*\|^2)\|U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n)\|^2 \\
&\quad + r\left\{\|U l_{n+1} - U l_n\|^2 - \|Ay_{n+1} - Ay_n\|^2\right\} \\
&= \|y_{n+1} - y_n\|^2 - r(1 - r\|A^*\|^2)\|U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n)\|^2 \\
&\quad + r|\beta_{n+1} - \beta_n|(\|U l_{n+1} - U l_n\| + \|Ay_{n+1} - Ay_n\|)\|SAy_n\| \\
&\leq \|x_{n+1} - x_n\|^2 + |\beta_{n+1} - \beta_n|\left(\|Tx_n\|\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|Tx_n\|^2\right) \\
&\quad - r(1 - r\|A^*\|^2)\|U l_{n+1} - Ay_{n+1} - (U l_n - Ay_n)\|^2 \\
&\quad + r|\beta_{n+1} - \beta_n|(\|U l_{n+1} - U l_n\| + \|Ay_{n+1} - Ay_n\|)\|SAy_n\| \\
&\leq \|x_{n+1} - x_n\|^2 + |\beta_{n+1} - \beta_n|M_1,
\end{aligned} \tag{3.13}$$

where M_1 is a constant such that

$$\left(\|Tx_n\|\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|Tx_n\|^2\right) + r(\|U l_{n+1} - U l_n\| + \|Ay_{n+1} - Ay_n\|)\|SAy_n\| \leq M_1 \tag{3.14}$$

for any $n \in \mathbb{N} \cup \{0\}$. Since $\{h_n\} \subset C_1$, for each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} \|W_{n+1}h_{n+1} - W_n h_n\| &\leq \|W_{n+1}h_{n+1} - W h_{n+1}\| + \|W h_{n+1} - W h_n\| + \|W h_n - W_n h_n\| \\ &\leq \sup_{x \in C_1} \|W_{n+1}x - Wx\| + \sup_{x \in C_1} \|Wx - W_n x\| + \|h_{n+1} - h_n\|, \end{aligned} \quad (3.15)$$

$$\|W_{n+1}h_{n+1} - W_n h_n\|^2 \leq \|h_{n+1} - h_n\|^2 + \omega_n, \quad (3.16)$$

where

$$\begin{aligned} \omega_n &= \left(\sup_{x \in C_1} \|W_{n+1}x - Wx\| + \sup_{x \in C_1} \|Wx - W_n x\| \right) \\ &\quad \times \left(\sup_{x \in C_1} \|W_{n+1}x - Wx\| + \sup_{x \in C_1} \|Wx - W_n x\| + 2\|h_{n+1} - h_n\| \right). \end{aligned} \quad (3.17)$$

So, we have

$$\begin{aligned} \|z_{n+1} - z_n\|^2 &\leq (1 - \theta)\|x_{n+1} - x_n\|^2 + \theta\|W_{n+1}h_{n+1} - W_n h_n\|^2 \\ &\leq (1 - \theta)\|x_{n+1} - x_n\|^2 + \theta\|h_{n+1} - h_n\|^2 + \omega_n \\ &\leq (1 - \theta)\|x_{n+1} - x_n\|^2 + \theta\|x_{n+1} - x_n\|^2 + |\beta_{n+1} - \beta_n| M_1 + \omega_n \\ &= \|x_{n+1} - x_n\|^2 + |\beta_{n+1} - \beta_n| M_1 + \omega_n, \end{aligned} \quad (3.18)$$

for any $n \in \mathbb{N} \cup \{0\}$.

Choose a sequence $\{\bar{y}_n\}$ such that $x_{n+1} = \gamma_n \bar{y}_n + (1 - \gamma_n)x_n$, where $\gamma_n = 1 - (1 - \theta)(1 - \alpha_n)$, then we have

$$\bar{y}_n = \frac{\alpha_n f(x_n) + (1 - \alpha_n)\theta W_n h_n}{\gamma_n}, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (3.19)$$

It follows that

$$\begin{aligned} \|\bar{y}_{n+1} - \bar{y}_n\| &\leq \frac{\alpha_n}{\gamma_n} \|f(x_n)\| + \frac{\alpha_{n+1}}{\gamma_{n+1}} \|f(x_{n+1})\| \\ &\quad + \frac{(1 - \alpha_{n+1})\theta}{\gamma_{n+1}} \|W_{n+1}h_{n+1} - W_n h_n\| + \left| \frac{(1 - \alpha_{n+1})\theta}{\gamma_{n+1}} - \frac{(1 - \alpha_n)\theta}{\gamma_n} \right| \|W_n h_n\| \\ &\leq (\alpha_n + \alpha_{n+1})M_3 + \frac{(1 - \alpha_{n+1})\theta}{\gamma_{n+1}} \|W_{n+1}h_{n+1} - W_n h_n\| + \left| \frac{\alpha_n - \alpha_{n+1}}{\gamma_{n+1}\gamma_n} \right| \theta \|W_n h_n\| \\ &\leq 2(\alpha_n + \alpha_{n+1})M_3 + \frac{(1 - \alpha_{n+1})\theta}{\gamma_{n+1}} \|W_{n+1}h_{n+1} - W_n h_n\|, \end{aligned} \quad (3.20)$$

where M_3 is a constant such that $\sup_{n \in \mathbb{N} \cup \{0\}} \{\|f(x_n)/\gamma_n\|, \|W_n h_n\|\} \leq M_3$. From (3.20), (3.16), and (3.13) we have

$$\begin{aligned}
\|\bar{y}_{n+1} - \bar{y}_n\|^2 &\leq 4(\alpha_{n+1} + \alpha_n)^2 M_3^2 + \frac{(1 - \alpha_{n+1})^2 \theta^2}{\gamma_{n+1}^2} \|W_{n+1} h_{n+1} - W_n h_n\|^2 \\
&\quad + 4(\alpha_n + \alpha_{n+1}) M_3 \frac{(1 - \alpha_{n+1}) \theta}{\gamma_{n+1}} \|W_{n+1} h_{n+1} - W_n h_n\| \\
&\leq 4(\alpha_{n+1} + \alpha_n)^2 M_3^2 + \frac{(1 - \alpha_{n+1})^2 \theta^2}{\gamma_{n+1}^2} \|h_{n+1} - h_n\|^2 + \frac{(1 - \alpha_{n+1})^2 \theta^2}{\gamma_{n+1}^2} \omega_n \\
&\quad + 4(\alpha_n + \alpha_{n+1}) M_3 \frac{(1 - \alpha_{n+1}) \theta}{\gamma_{n+1}} \|W_{n+1} h_{n+1} - W_n h_n\| \\
&\leq 4(\alpha_{n+1} + \alpha_n)^2 M_3^2 + \frac{(1 - \alpha_{n+1})^2 \theta^2}{\gamma_{n+1}^2} \|x_{n+1} - x_n\|^2 + \frac{(1 - \alpha_{n+1})^2 \theta^2}{\gamma_{n+1}^2} |\beta_{n+1} - \beta_n| M_1 \\
&\quad + \frac{(1 - \alpha_{n+1})^2 \theta^2}{\gamma_{n+1}^2} \omega_n + 4(\alpha_n + \alpha_{n+1}) M_3 \frac{(1 - \alpha_{n+1}) \theta}{\gamma_{n+1}} \|W_{n+1} h_{n+1} - W_n h_n\|.
\end{aligned} \tag{3.21}$$

Applying the condition (C2), it follows from (3.21) that

$$\limsup_{n \rightarrow \infty} \left\{ \|\bar{y}_{n+1} - \bar{y}_n\|^2 - \|x_{n+1} - x_n\|^2 \right\} = 0, \tag{3.22}$$

which implies

$$\limsup_{n \rightarrow \infty} \{ \|\bar{y}_{n+1} - \bar{y}_n\| - \|x_{n+1} - x_n\| \} = 0. \tag{3.23}$$

Applying Lemma 2.7, we obtain $\lim_{n \rightarrow \infty} \|\bar{y}_n - x_n\| \rightarrow 0$ which implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \gamma_n \|\bar{y}_n - x_n\| = 0. \tag{3.24}$$

Step 3. Prove $\lim_{n \rightarrow \infty} \|Tx_n - Tp\| = \lim_{n \rightarrow \infty} \|SAy_n - Sap\| = 0$.

For any $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(I - \beta_n T)x_n - P_C(I - \beta_n T)p\|^2 \\
&\leq \|(I - \beta_n T)x_n - (I - \beta_n T)p\|^2 \\
&= \|x_n - p\|^2 - 2\beta_n \langle x_n - p, Tx_n - Tp \rangle + \beta_n^2 \|Tx_n - Tp\|^2 \\
&\leq \|x_n - p\|^2 - \beta_n (2u - \beta_n) \|Tx_n - Tp\|^2.
\end{aligned} \tag{3.25}$$

Similarly,

$$\begin{aligned}\|Ul_n - Ap\|^2 &\leq \|l_n - Ap\|^2 = \|P_K(I - \beta_n S)Ay_n - Ap\|^2 \\ &\leq \|Ay_n - Ap\|^2 - \beta_n(2u - \beta_n)\|SAy_n - SAp\|^2.\end{aligned}\quad (3.26)$$

From (3.5) again, we have

$$\|x_{n+1} - p\|^2 \leq (\alpha_n \|f(x_n) - p\| + \|z_n - p\|)^2 \leq \alpha_n M_4 + \|z_n - p\|^2, \quad (3.27)$$

where M_4 is a constant such that $\sup_{n \in \mathbb{N} \cup \{0\}} \{\alpha_n \|f(x_n) - p\|^2 + 2\|f(x_n) - p\|\|z_n - p\|\} \leq M_4$. It follows that

$$\begin{aligned}0 < \theta\beta_n(2u - \beta_n)\|Tx_n - Tp\|^2 &\leq \theta\|x_n - p\|^2 - \theta\|y_n - p\|^2 \quad (\text{by (3.24)}) \\ &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \quad (\text{by (3.7)}) \\ &\leq \|x_n - p\|^2 + \alpha_n M_4 - \|x_{n+1} - p\|^2 \quad (\text{by (3.26)}) \\ &\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n M_4 \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned}\quad (3.28)$$

which yields that $\lim_{n \rightarrow \infty} \|Tx_n - Tp\| = 0$ (by the condition $0 < a \leq \beta_n \leq b < 2u$).

For any $n \in \mathbb{N} \cup \{0\}$, by (3.6), (3.7), and (3.27), we have

$$\begin{aligned}\theta r(1 - r\|A^*\|^2)\|Ul_n - Ay_n\|^2 &\leq \theta\|y_n - p\|^2 - \theta\|W_n h_n - p\|^2 \\ &\leq \theta\|x_n - p\|^2 - \theta\|z_n - p\|^2 \\ &\leq \theta\|x_n - p\|^2 + \alpha_n M_4 - \|x_{n+1} - p\|^2 \\ &= (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n M_4.\end{aligned}\quad (3.29)$$

So,

$$\lim_{n \rightarrow \infty} \|Ul_n - Ay_n\| = 0. \quad (3.30)$$

From (3.26) and (3.30) again, we have

$$\begin{aligned}0 < \beta_n(2u - \beta_n)\|SAy_n - SAp\|^2 &\leq \|Ay_n - Ap\|^2 - \|Ul_n - Ap\|^2 \\ &= (\|Ay_n - Ap\| + \|Ul_n - Ap\|)(\|Ay_n - Ap\| - \|Ul_n - Ap\|) \\ &\leq (\|Ay_n - Ap\| + \|Ap\|)\|Ul_n - Ay_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned}\quad (3.31)$$

which implies that $\lim_{n \rightarrow \infty} \|SAy_n - Sap\| = 0$ (by the condition $0 < a \leq \beta_n \leq b < 2u$).

On the other hand, since

$$\begin{aligned}
 \|l_n - Ap\|^2 &= \|P_K(I - \beta_n S)Ay_n - P_K(I - \beta_n S)Ap\|^2 \\
 &\leq \langle (I - \beta_n S)Ay_n - (I - \beta_n S)Ap, l_n - Ap \rangle \\
 &= \frac{1}{2} \left\{ \|(I - \beta_n S)Ay_n - (I - \beta_n S)Ap\|^2 \right. \\
 &\quad \left. + \|l_n - Ap\|^2 - \|l_n - Ay_n - \beta_n(SAy_n - Sap)\|^2 \right\} \\
 &\leq \frac{1}{2} \left\{ \|Ay_n - Ap\|^2 + \|l_n - Ap\|^2 - \|l_n - Ay_n\|^2 \right. \\
 &\quad \left. + 2\beta_n \langle l_n - Ay_n, SAy_n - Sap \rangle - \beta_n^2 \|SAy_n - Sap\|^2 \right\} \\
 &\leq \frac{1}{2} \left\{ \|Ay_n - Ap\|^2 + \|l_n - Ap\|^2 - \|l_n - Ay_n\|^2 + 2\beta_n \|l_n - Ay_n\| \|SAy_n - Sap\| \right\},
 \end{aligned} \tag{3.32}$$

we get

$$\|l_n - Ap\|^2 \leq \|Ay_n - Ap\|^2 - \|l_n - Ay_n\|^2 + 2\beta_n \|l_n - Ay_n\| \|SAy_n - Sap\|, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.33}$$

By (3.6) and (3.33), we have

$$\begin{aligned}
 \|W_n h_n - p\|^2 &\leq \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\
 &\quad + 2r \left\{ \frac{1}{2} \|Ul_n - Ap\|^2 + \frac{1}{2} \|Ul_n - Ay_n\|^2 - \|Ay_n - Ap\|^2 - \|Ul_n - Ay_n\|^2 \right\} \\
 &\leq \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\
 &\quad + 2r \left\{ \frac{1}{2} \|l_n - Ap\|^2 + \frac{1}{2} \|Ul_n - Ay_n\|^2 - \|Ay_n - Ap\|^2 - \|Ul_n - Ay_n\|^2 \right\} \\
 &\leq \|y_n - p\|^2 + \|rA^*(Ul_n - Ay_n)\|^2 \\
 &\quad + 2r \left\{ -\frac{1}{2} \|l_n - Ay_n\|^2 + \beta_n \|l_n - Ay_n\| \|SAy_n - Sap\| - \frac{1}{2} \|Ul_n - Ay_n\|^2 \right\} \\
 &\leq \|y_n - p\|^2 + r^2 \|A^*\|^2 \|Ul_n - Ay_n\|^2 - r \|Ul_n - Ay_n\|^2 - r \|l_n - Ay_n\|^2 \\
 &\quad + 2r\beta_n \|l_n - Ay_n\| \|SAy_n - Sap\| \\
 &= \|y_n - p\|^2 + r(1 - r\|A^*\|^2) \|Ul_n - Ay_n\|^2 - r \|l_n - Ay_n\|^2 \\
 &\quad + 2r\beta_n \|l_n - Ay_n\| \|SAy_n - Sap\| \\
 &\leq \|y_n - p\|^2 - r \|l_n - Ay_n\|^2 + 2r\beta_n \|l_n - Ay_n\| \|SAy_n - Sap\|,
 \end{aligned} \tag{3.34}$$

Using (3.7), (3.27), and (3.34), we obtain

$$\begin{aligned}
\theta r \|l_n - Ay_n\|^2 &\leq \theta \|y_n - p\|^2 - \theta \|W_n h_n - p\|^2 + 2r\theta\beta_n \|l_n - Ay_n\| \|SAy_n - Sap\| \\
&\leq \theta \|x_n - p\|^2 - \theta \|z_n - p\|^2 + 2r\theta\beta_n \|l_n - Ay_n\| \|SAy_n - Sap\| \\
&\leq \theta \|x_n - p\|^2 + \alpha_n M_4 - \|x_{n+1} - p\|^2 + 2r\theta\beta_n \|l_n - Ay_n\| \|SAy_n - Sap\| \\
&= (\|x_n - p\| + \|x_{n+1} - p\|) \|x_{n+1} - x_n\| + \alpha_n M_4 + 2r\theta\beta_n \|l_n - Ay_n\| \|SAy_n - Sap\| \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{3.35}$$

which implies

$$\lim_{n \rightarrow \infty} \|l_n - Ay_n\| = 0. \tag{3.36}$$

According to (3.30) and (3.36), we derive that

$$\lim_{n \rightarrow \infty} \|Ul_n - l_n\| = 0. \tag{3.37}$$

Step 4. Prove $\lim_{n \rightarrow \infty} \|x_n - z_n\| = \lim_{n \rightarrow \infty} \|y_n - x_n\| = \lim_{n \rightarrow \infty} \|h_n - x_n\| = 0$.
Since

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0, \quad \lim_{n \rightarrow \infty} \|x_{n+1} - z_n\| = \lim_{n \rightarrow \infty} \alpha_n \|f(x_n) - z_n\| = 0, \tag{3.38}$$

we have $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$. For any $n \in \mathbb{N} \cup \{0\}$, since

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(I - \beta_n T)x_n - P_C(I - \beta_n T)p\|^2 \\
&\leq \langle (I - \beta_n T)x_n - (I - \beta_n T)p, y_n - p \rangle \\
&= \frac{1}{2} \left\{ \|(I - \beta_n T)x_n - (I - \beta_n T)p\|^2 + \|y_n - p\|^2 - \|x_n - y_n - \beta_n(Tx_n - Tp)\|^2 \right\} \\
&\leq \frac{1}{2} \left\{ \|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 + 2\beta_n \langle x_n - y_n, Tx_n - Tp \rangle - \beta_n^2 \|Tx_n - Tp\|^2 \right\} \\
&\leq \frac{1}{2} \left\{ \|x_n - p\|^2 + \|y_n - p\|^2 - \|x_n - y_n\|^2 + 2\beta_n \|x_n - y_n\| \|Tx_n - Tp\| \right\},
\end{aligned} \tag{3.39}$$

we get

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\beta_n \|x_n - y_n\| \|Tx_n - Tp\|. \tag{3.40}$$

It follows from (3.27), (3.7), and (3.40) that

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \alpha_n M_4 + \|z_n - p\|^2 \\
 &\leq \alpha_n M_4 + (1 - \theta) \|x_n - p\|^2 + \theta \|y_n - p\|^2 \\
 &\leq \alpha_n M_4 + \|x_n - p\|^2 - \theta \|x_n - y_n\|^2 + 2\theta \beta_n \|x_n - y_n\| \|Tx_n - Tp\|,
 \end{aligned} \tag{3.41}$$

which yields that

$$\begin{aligned}
 \theta \|x_n - y_n\|^2 &\leq \alpha_n M_4 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\beta_n \|x_n - y_n\| \|Tx_n - Tp\| \\
 &\leq \alpha_n M_4 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_{n+1} - x_n\| + 2\beta_n \|x_n - y_n\| \|Tx_n - Tp\|,
 \end{aligned} \tag{3.42}$$

By Steps 1–3 and $\lim_{n \rightarrow \infty} \alpha_n = 0$, it follows from (3.42) that $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Since $\|h_n - y_n\| \leq \|\gamma A^*\| \|l_n - Ay_n\|$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n \rightarrow \infty} \|h_n - y_n\| = 0$. Using it with $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$, we get $\lim_{n \rightarrow \infty} \|x_n - h_n\| = 0$.

Step 5. Prove $\lim_{n \rightarrow \infty} \|W_n x_n - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|Wx_n - x_n\| = 0$.

Indeed, since

$$\|W_n h_n - x_n\| = \frac{1}{\theta} \|z_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.43}$$

we have

$$\|W_n x_n - x_n\| \leq \|W_n x_n - W_n h_n\| + \|W_n h_n - x_n\| \leq \|x_n - h_n\| + \|W_n h_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.44}$$

By Theorem 2.3, $\lim_{n \rightarrow \infty} \|Wx_n - W_n x_n\| = 0$. Since

$$\|Wx_n - x_n\| \leq \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\| \quad \forall n, \tag{3.45}$$

we obtain $\lim_{n \rightarrow \infty} \|Wx_n - x_n\| = 0$.

Step 6. There exists a unique $q \in \Omega \subset H$ such that $P_\Omega f(q) = q$.

Indeed, for any $x, y \in H$,

$$\|P_\Omega f(x) - P_\Omega f(y)\| \leq \|f(x) - f(y)\| \leq \alpha \|x - y\|. \tag{3.46}$$

Since $\alpha \in [0, 1)$, $P_\Omega f$ is a contraction on H . Applying Banach contraction principle, there exists a unique $q \in H$ such that $q = P_\Omega f(q) \in \Omega$.

Step 7. Prove $\limsup_{n \rightarrow \infty} \langle fq - q, x_n - q \rangle \leq 0$.

For this purpose, we may choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle fq - q, x_n - q \rangle = \lim_{i \rightarrow \infty} \langle fq - q, x_{n_i} - q \rangle. \tag{3.47}$$

Since $\{x_{n_i}\}$ is a bounded sequence, there exists a subsequence of $\{x_{n_i}\}$, which is still denoted by $\{x_{n_i}\}$, such that $x_{n_i} \rightharpoonup x^* \in H$. Therefore, we have

$$\limsup_{i \rightarrow \infty} \langle f q - q, x_{n_i} - q \rangle = \langle f q - q, x^* - q \rangle. \quad (3.48)$$

Next we prove $x^* \in \Omega$.

(a) $Wx^* = x^*$. In fact, if $Wx^* \neq x^*$, then we have

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|x_{n_i} - x^*\| &< \liminf_{i \rightarrow \infty} \|x_{n_i} - Wx^*\| \\ &\leq \liminf_{i \rightarrow \infty} (\|x_{n_i} - Wx_{n_i}\| + \|Wx_{n_i} - Wx^*\|) \\ &\leq \liminf_{i \rightarrow \infty} (\|x_{n_i} - Wx_{n_i}\| + \|x_{n_i} - x^*\|) \\ &= \liminf_{i \rightarrow \infty} \|x_{n_i} - x^*\| \quad (\text{by Step 5 and Lemma 2.9}). \end{aligned} \quad (3.49)$$

This is a contradiction. Hence, $Wx^* = x^*$, which implies that $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$ by Theorem 2.2.

(b) Prove $x^* \in \text{VI}(C, T)$. Since $x_{n_i} \rightharpoonup x^*$ and $\lim_{i \rightarrow \infty} \|x_{n_i} - y_{n_i}\| = 0$, we have $y_{n_i} \rightharpoonup x^*$. Let

$$T_1 x = \begin{cases} Tx + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \quad (3.50)$$

Since T is u -inversely monotone, T is monotone and hence T_1 is a maximal monotone mapping. For any given $(x, z) \in G(T_1)$, since $z - Tx \in N_C x$ and $y_n \in C$, by the definition of N_C , we have

$$\langle x - y_n, z - Tx \rangle \geq 0 \quad \forall n. \quad (3.51)$$

On the other hand, since $y_n = P_C(I - \beta_n T)x_n$, we have

$$\langle x - y_n, y_n - (x_n - \beta_n T x_n) \rangle \geq 0. \quad (3.52)$$

In particular,

$$\left\langle x - y_n, \frac{y_n - x_n}{\beta_n} + T x_n \right\rangle \geq 0 \quad \forall n. \quad (3.53)$$

From (3.51) and (3.53) we have

$$\begin{aligned}
\langle x - y_{n_i}, z \rangle &\geq \langle x - y_{n_i}, Tx \rangle \\
&\geq \langle x - y_{n_i}, Tx \rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\beta_{n_i}} + Tx_{n_i} \right\rangle \\
&= \langle x - y_{n_i}, Tx - Ty_{n_i} \rangle + \langle x - y_{n_i}, Ty_{n_i} - Tx_{n_i} \rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\beta_{n_i}} \right\rangle \\
&\geq \langle x - y_{n_i}, Ty_{n_i} - Tx_{n_i} \rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\beta_{n_i}} \right\rangle,
\end{aligned} \tag{3.54}$$

which implies

$$\langle x - x^*, z \rangle = \lim_{i \rightarrow \infty} \langle x - y_{n_i}, z \rangle \geq 0. \tag{3.55}$$

This shows $0 \in T_1 x^*$, that is, $x^* \in \text{VI}(C, T)$.

(c) Prove $Ax^* \in \text{VI}(K, S)$ and $Ax^* \in F(U)$. Let

$$T_2 x = \begin{cases} Sx + N_K x, & x \in K, \\ \emptyset, & x \notin K. \end{cases} \tag{3.56}$$

Then T_2 is a maximal monotone mapping. For any given $(x, z) \in G(T_2)$, since $z - Sx \in N_K x$ and $l_n \in K$, by the definition of N_K we have

$$\langle x - l_n, z - Sx \rangle \geq 0 \quad \forall n. \tag{3.57}$$

On the other hand, since $l_n = P_K(I - \beta_n S)Ay_n$, we have

$$\langle x - l_n, l_n - (Ay_n - \beta_n S Ay_n) \rangle \geq 0, \tag{3.58}$$

and hence

$$\left\langle x - l_n, \frac{l_n - Ay_n}{\beta_n} + S Ay_n \right\rangle \geq 0 \quad \forall n. \tag{3.59}$$

Since $x_{n_i} \rightharpoonup x^*$ and $\lim_{n \rightarrow \infty} \|x_{n_i} - y_{n_i}\| = 0$ and $\lim_{n \rightarrow \infty} \|l_{n_i} - Ay_{n_i}\| = 0$, we have $l_{n_i} \rightharpoonup Ax^*$. From (3.57) and (3.59) we have

$$\begin{aligned}
\langle x - l_{n_i}, z \rangle &\geq \langle x - l_{n_i}, Sx \rangle \\
&\geq \langle x - l_{n_i}, Sx - Sl_{n_i} \rangle + \langle x - l_{n_i}, Sl_{n_i} - S Ay_{n_i} \rangle - \left\langle x - l_{n_i}, \frac{l_{n_i} - Ay_{n_i}}{\beta_{n_i}} \right\rangle \\
&\geq \langle x - l_{n_i}, l_{n_i} - Ay_{n_i} \rangle - \left\langle x - l_{n_i}, \frac{l_{n_i} - Ay_{n_i}}{\beta_{n_i}} \right\rangle.
\end{aligned} \tag{3.60}$$

So

$$\langle x - Ax^*, z \rangle = \lim_{i \rightarrow \infty} \langle x - l_{n_i}, z \rangle \geq 0. \quad (3.61)$$

This shows $0 \in T_2 Ax^*$, that is, $Ax^* \in \text{VI}(K, S)$. In addition, by (3.36), (3.37), and Opial's condition, we can prove easily $Ax^* \in F(U)$.

By (a), (b), and (c), $x^* \in \Omega$ is proved. Hence

$$\limsup_{n \rightarrow \infty} \langle fq - q, x_n - q \rangle = \langle fq - q, x^* - q \rangle \leq 0. \quad (3.62)$$

Step 8. Prove $\{x_n\}$ converges strongly to $q = P_\Omega f(q) \in \Omega$.

In fact, by Step 7 we have

$$\limsup_{n \rightarrow \infty} \langle fq - q, x_n - q \rangle \leq 0. \quad (3.63)$$

It follows from (3.5) and Lemma 2.8 that

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n f(x_n) + (1 - \alpha_n)z_n - q\|^2 \\ &\leq (1 - \alpha_n)^2 \|z_n - q\|^2 + 2\alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle f(x_n) - f(q) + f(q) - q, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \alpha \|x_n - q\| \|x_{n+1} - q\| + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\ &\leq (1 - \alpha_n(2 - 2\alpha)) \|x_n - q\|^2 + \alpha_n^2 \|x_n - q\|^2 + 2\alpha_n \alpha \|x_n - q\| \|x_{n+1} - x_n\| \\ &\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (3.64)$$

Let $\lambda_n = \alpha_n(2 - 2\alpha)$, $n \in \mathbb{N} \cup \{0\}$. Then, for any $n \in \mathbb{N} \cup \{0\}$, we obtain

$$\begin{aligned} \alpha_n^2 \|x_n - q\|^2 &= \frac{\lambda_n \alpha_n}{2 - 2\alpha} \|x_n - q\|^2, \\ 2\alpha_n \alpha \|x_n - q\| \|x_{n+1} - x_n\| &= \frac{\lambda_n \alpha}{1 - \alpha} \|x_n - q\| \|x_{n+1} - x_n\|, \\ 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle &= \frac{\lambda_n}{1 - \alpha} \langle f(q) - q, x_{n+1} - q \rangle. \end{aligned} \quad (3.65)$$

Set

$$b_n = \frac{\alpha_n}{2 - 2\alpha} \|x_n - q\|^2 + \frac{\alpha}{1 - \alpha} \|x_n - q\| \|x_{n+1} - x_n\| + \frac{1}{1 - \alpha} \langle f(q) - q, x_{n+1} - q \rangle. \quad (3.66)$$

The condition (C1) and the boundedness of $\{x_n\}$ ensure $\lim_{n \rightarrow \infty} (\alpha_n / (2 - 2\alpha)) \|x_n - q\|^2 = 0$, Step 2 ensure $\lim_{n \rightarrow \infty} (1 / (1 - \alpha)) \alpha \|x_n - q\| \|x_{n+1} - x_n\| = 0$, and Step 7 ensure $\limsup_{n \rightarrow \infty} (1 / (1 - \alpha)) \langle f(q) - q, x_{n+1} - q \rangle \leq 0$, so $\limsup_{n \rightarrow \infty} b_n \leq 0$. Applying Lemma 2.6 and the inequality

$$\|x_{n+1} - q\|^2 \leq (1 - \lambda_n) \|x_n - q\|^2 + \lambda_n b_n, \quad (3.67)$$

we obtain that $\lim_{n \rightarrow \infty} \|x_n - q\| = 0$ which means that the sequence $\{x_n\}$ strongly converge to q . This completes the proof of the Theorem 3.5. $\square$

The following convergence theorems can be established by applying Theorem 3.5 with $U = I$.

Corollary 3.6. *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1, K \subset H_2$ two nonempty closed convex sets. Let $T : C \rightarrow H_1$ and $S : K \rightarrow H_2$ be u -inversely monotone. $A : H_1 \rightarrow H_2$ is a bounded linear operator with adjoint operator A^* . Let $f : C \rightarrow C$ be a contraction with contraction constant α . Let $\{T_i\}_{i \in \mathbb{N}}$ be a family of infinitely nonexpansive mappings of C into itself such that $\Omega = \{p \in \bigcap_{i=1}^{\infty} F(T_i) \cap \text{VI}(C, T) : Ap \in \text{VI}(K, S)\} \neq \emptyset$. Let ξ be a real number and $\{\lambda_i\}_{i \in \mathbb{N}}$ be a sequence of real numbers such that $0 < \lambda_i \leq \xi < 1$, for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let W_n be the W -mapping of C into itself generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily,} \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_K(I - \beta_n S)Ay_n, \\ z_n &= (1 - \theta)x_n + \theta W_n P_C(y_n + rA^*(l_n - Ay_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \quad (3.68)$$

where $r \in (0, 1/\|A\|^2)$ and $\theta \in (0, 1)$ are two constants and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in $(0, 1)$. If $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ further satisfy the following conditions:

- (C₁) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C₂) $\{\beta_n\} \subset [a, b]$ and $\lim_{n \rightarrow \infty} |\beta_{n+1} - \beta_n| = 0$, where $0 < a, b < 2u$;

then the following statements hold:

- (a) there exists a unique $q \in \Omega$ such that $P_{\Omega} f(q) = q$;
- (b) $\{x_n\}$ converges strongly to q .

If $T_i = I$ for all $i \in \mathbb{N}$ in Theorem 3.5, then we have the following result.

Corollary 3.7. *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1, K \subset H_2$ two nonempty closed convex sets. Let $T : C \rightarrow H_1$ and $S : K \rightarrow H_2$ be u -inversely monotone. $A : H_1 \rightarrow H_2$ is a bounded linear operator with adjoint operator A^* . Let $f : C \rightarrow C$ be a contraction with contraction constant α . Let U be a nonexpansive mapping of K into itself such that $\Omega = \{p \in \text{VI}(C, T) : Ap \in F(U) \cap \text{VI}(K, S)\} \neq \emptyset$. Let ξ be a real number and $\{\lambda_i\}_{i \in \mathbb{N}}$ be a sequence of real numbers such that*

$0 < \lambda_i \leq \xi < 1$, for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let W_n be the W -mapping of C into itself generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily,} \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_K(I - \beta_n S)Ay_n, \\ z_n &= (1 - \theta)x_n + \theta P_C(y_n + rA^*(Ul_n - Ay_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{3.69}$$

where $r \in (0, 1/\|A\|^2)$ and $\theta \in (0, 1)$ are two constants and $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are two sequences in $(0, 1)$. If $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ further satisfy the following conditions:

(C₁) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

(C₂) $\{\beta_n\} \subset [a, b]$ and $\lim_{n \rightarrow \infty} |\beta_{n+1} - \beta_n| = 0$, where $0 < a, b < 2u$;

then the following statements hold:

(a) there exists a unique $q \in \Omega$ such that $P_\Omega f(q) = q$;

(b) $\{x_n\}$ converges strongly to q .

In Theorem 3.5, if $T_i = I$ for all $i \in \mathbb{N}$ and $U = I$, then we obtain Corollary 3.8.

Corollary 3.8. Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1, K \subset H_2$ two nonempty closed convex sets. Let $T : C \rightarrow H_1$ and $S : K \rightarrow H_2$ be u -inversely monotone. $A : H_1 \rightarrow H_2$ is a bounded linear operator with adjoint operator A^* . Let $f : C \rightarrow C$ be a contraction with contraction constant α . Let $\{T_i\}_{i \in \mathbb{N}}$ be a family of infinitely nonexpansive mappings of C into itself and U a nonexpansive mapping of K into itself such that $\Omega = \{p \in \text{VI}(C, T) : Ap \in \text{VI}(K, S)\} \neq \emptyset$. Let ξ be a real number and $\{\lambda_i\}_{i \in \mathbb{N}}$ be a sequence of real numbers such that $0 < \lambda_i \leq \xi < 1$, for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let W_n be the W -mapping of C into itself generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily,} \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_K(I - \beta_n S)Ay_n, \\ z_n &= (1 - \theta)x_n + \theta P_C(y_n + rA^*(l_n - Ay_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{3.70}$$

where $r \in (0, 1/\|A\|^2)$ and $\theta \in (0, 1)$ are two constants and $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are two sequences in $(0, 1)$. If $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ further satisfy the following conditions:

(C₁) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

(C₂) $\{\beta_n\} \subset [a, b]$ and $\lim_{n \rightarrow \infty} |\beta_{n+1} - \beta_n| = 0$, where $0 < a, b < 2u$;

then the following statements hold:

- (a) there exists a unique $q \in \Omega$ such that $P_\Omega f(q) = q$;
- (b) $\{x_n\}$ converges strongly to q .

In Theorem 3.5, if $H_1 = H_2$, $C = K$ and $A = A^* = I$, then we have Corollary 3.9.

Corollary 3.9. Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T : C \rightarrow H$ and $S : C \rightarrow H$ be u -inversely monotone. Let $f : C \rightarrow C$ be a contraction with contraction constant α . Let $U : C \rightarrow C$ be a nonexpansive mapping and $\{T_i\}_{i \in \mathbb{N}} : C \rightarrow C$ a family of infinitely nonexpansive mappings such that $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \cap F(U) \cap VI(C, T) \cap VI(K, S) \neq \emptyset$. Let ξ be a real number and $\{\lambda_i\}_{i \in \mathbb{N}}$ be a sequence of real numbers such that $0 < \lambda_i \leq \xi < 1$, for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let W_n be the W -mapping of C into itself generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily,} \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_C(I - \beta_n S)y_n, \\ z_n &= (1 - \theta)x_n + \theta W_n(y_n + r(Ul_n - y_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{3.71}$$

where $r \in (0, 1)$ and $\theta \in (0, 1)$ are two constants and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences in $(0, 1)$. If $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ further satisfy the following conditions:

- (C₁) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (C₂) $\{\beta_n\} \subset [a, b]$ and $\lim_{n \rightarrow \infty} |\beta_{n+1} - \beta_n| = 0$, where $0 < a, b < 2u$;

then the following statements hold:

- (a) there exists a unique $q \in \Omega$ such that $P_\Omega f(q) = q$;
- (b) $\{x_n\}$ converges strongly to q .

In Theorem 3.5, if $H_1 = H_2$, $C = K$, $A = A^* = I$, and $U = I$, then we get the following result.

Corollary 3.10. Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T : C \rightarrow H$ and $S : C \rightarrow H$ be u -inversely monotone. Let $f : C \rightarrow C$ be a contraction with contraction constant α . Let $\{T_i\}_{i \in \mathbb{N}} : C \rightarrow C$ be a family of infinitely nonexpansive mappings such that $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, T) \cap VI(K, S) \neq \emptyset$. Let ξ be a real number and $\{\lambda_i\}_{i \in \mathbb{N}}$ be a sequence of real numbers such that $0 < \lambda_i \leq \xi < 1$, for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let W_n be the W -mapping of C into itself generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily,} \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_C(I - \beta_n S)y_n, \\ z_n &= (1 - \theta)x_n + \theta W_n(y_n + r(l_n - y_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{3.72}$$

where $r \in (0, 1)$ and $\theta \in (0, 1)$ are two constants and $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are two sequences in $(0, 1)$. If $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ further satisfy the following conditions:

(C₁) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

(C₂) $\{\beta_n\} \subset [a, b]$ and $\lim_{n \rightarrow \infty} |\beta_{n+1} - \beta_n| = 0$, where $0 < a, b < 2u$;

then the following statements hold:

- (a) there exists a unique $q \in \Omega$ such that $P_\Omega f(q) = q$;
- (b) $\{x_n\}$ converges strongly to q .

Finally, if we let $H_1 = H_2$, $C = K$, $A = A^* = I$, $T_i = I$, for all $i \in \mathbb{N}$ and $U = I$ in Theorem 3.5, then the following result can be established.

Corollary 3.11. Let H be a real Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T : C \rightarrow H$ and $S : C \rightarrow H$ be u -inversely monotone. Let $f : C \rightarrow C$ be a contraction with contraction constant α . Suppose that $\Omega = \text{VI}(C, T) \cap \text{VI}(K, S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{aligned} x_0 &= x \in C \quad \text{chosen arbitrarily,} \\ y_n &= P_C(I - \beta_n T)x_n, \\ l_n &= P_C(I - \beta_n S)y_n, \\ z_n &= (1 - \theta)x_n + \theta(y_n + r(l_n - y_n)), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)z_n, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{3.73}$$

where $r \in (0, 1)$ and $\theta \in (0, 1)$ are two constants and $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are two sequences in $(0, 1)$. If $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ further satisfy the following conditions:

(C₁) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

(C₂) $\{\beta_n\} \subset [a, b]$ and $\lim_{n \rightarrow \infty} |\beta_{n+1} - \beta_n| = 0$, where $0 < a, b < 2u$;

then the following statements hold:

- (a) there exists a unique $q \in \Omega$ such that $P_\Omega f(q) = q$;
- (b) $\{x_n\}$ converges strongly to q .

Remark 3.12. (a) In [11, 16], the authors gave some algorithms for (u, v) -cocoercive and μ -Lipschitz continuous operator and obtain some strongly convergence theorems; see [11, Theorems 2.1 and 2.2] and [16, Corollary 3.3]. However, the (u, v) -cocoercive and μ -Lipschitz continuous operator considered by [11, 16] is actually a strongly monotone and μ -Lipschitz continuous operator. Then, by Remark ST, such operators studied in [11, 16] are u -inverse strongly monotone. Hence our results obtained in this paper conclude some results in [11, 16] as special cases.

(b) Our results are different from the main results in [9–11, 16] and references therein.

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